ON THE DISTRIBUTION AND INTERLACING OF THE ZEROS OF STIELTJES POLYNOMIALS

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Abstract. Polynomial solutions to the generalized Lamé equation, the Stieltjes polynomials, and the associated Van Vleck polynomials have been studied since the 1830’s, beginning with Lamé in his studies of the Laplace equation on an ellipsoid, and in an ever widening variety of applications since. In this paper we show how the zeros of Stieltjes polynomials are distributed and present two new interlacing theorems. We arrange the Stieltjes polynomials according to their Van Vleck zeros and show, firstly, that the zeros of successive Stieltjes polynomials of the same degree interlace, and secondly, that the zeros of certain Stieltjes polynomials of successive degrees interlace.

1. Introduction and Main Results

Let $\alpha_1 < \cdots < \alpha_p$ be any $p$ distinct real numbers, and let $\rho_1, \ldots, \rho_p$ be positive numbers. The generalized Lamé equation is the second order ODE given by

$$S''(x) + \sum_{j=1}^{p} \frac{\rho_j}{x - \alpha_j} S'(x) = \frac{V(x)}{A(x)} S(x),$$

where $A(x) = \prod_{j=1}^{p} (x - \alpha_j)$ and $V(x)$ is a polynomial of degree $p - 2$. A result of Stieltjes, known as the Heine-Stieltjes Theorem, says that there exist exactly $\binom{k+p-2}{k}$ polynomials $V(x)$ for which (1) has a polynomial solution $S$ of degree $k$. These polynomial solutions are often called Stieltjes or Heine-Stieltjes polynomials, and the corresponding polynomials $V(x)$ are known as Van Vleck polynomials.

In this paper we will consider the case when $p = 3$, in which case (1) is a Heun equation and the Heine-Stieltjes Theorem says that there are $k + 1$ values of $\nu$ for which (1) has a polynomial solution of degree $k$ with $V(x) = \mu(x - \nu)$. We refer to these values of $\nu$ as the Van Vleck zeros of order $k$.

The equation (1) was studied by Lamé in the 1830’s in the special case, $\rho_j = 1/2$, in connection with the separation of variables in the Laplace equation using elliptical coordinates. [25, Ch. 23]. The equation has since found a strikingly wide variety

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1 Usually the term generalized Lamé equation refers to such equations in which the $\alpha_j$’s are allowed to be complex. But here we only consider the real case.
of other applications, from electrostatics \[10\] \[13\] \[8\] \[17\] to completely quantum integrable systems such as the quantum C. Neumann oscillators, the asymmetric top and the geodesic flow on an ellipsoid \[1\] \[9\] \[5\] \[11\]. In particular, the zeros of the Stieltjes polynomials may be nicely interpreted as the equilibrium positions of \(k\) unit charges in a logarithmic potential in which at each position \(\alpha_j\) in the complex plane is fixed a charge of magnitude \(\rho_j\).

Much is known about the properties of Stieltjes and Van Vleck polynomials in the case when the degree of the Stieltjes polynomial is fixed. A few important facts are that the zeros of any Stieltjes polynomial are simple, lie inside the interval \((\alpha_1, \alpha_3)\), and none of them can equal \(\alpha_2\) or its corresponding Van Vleck zero. Similarly, for fixed \(k\), the Van Vleck zeros are distinct and also lie within \((\alpha_1, \alpha_3)\). The proofs of these results can be found in \[24\] \[23\] \[18\] \[19\].

Recently we showed in \[6\] that the Van Vleck zeros of successive orders interlace. That is, if the Van Vleck zeros of order \(k\) are written in increasing order as \(\nu^{(k)}_1 < \nu^{(k)}_2 < \cdots < \nu^{(k)}_{k+1}\), then
\[
\alpha_1 < \nu^{(k+1)}_i < \nu^{(k)}_i < \nu^{(k+1)}_{i+1} < \alpha_3, \quad i = 1, \ldots, k+1.
\]

Much of the research in the past several years has focused on the asymptotic properties of the zeros of Stieltjes and Van Vleck polynomials as the degree of the corresponding Stieltjes polynomials tends toward infinity \[4\] \[16\] \[14\] \[15\] \[20\] \[21\]. Interlacing theorems such as the one above are interesting, not only for what they tell us about the zeros for a finite degree, but also because they help us to understand such asymptotic limits. They are “classical” results in the sense of being statements about finite degree polynomials, and they are also a bridge to connect other classical results with asymptotic limits. Our results below are further steps in this direction.

In this paper we present a theorem on the distribution of the Stieltjes zeros and two additional interlacing theorems. First we note that the leading coefficient \(\mu\) of \(V(x)\) is determined by the degree \(k\) of the Stieltjes polynomials by substitution and identification of the powers as
\[
\mu = \mu_k = k(1 + \rho_1 + \rho_2 + \rho_3).
\]
For each positive integer \(k\) we label the \(k+1\) Stieltjes polynomials of degree \(k\) according to their Van Vleck zeros as \(S^{(k)}_j(x)\). That is, \(S^{(k)}_j(x)\) is the polynomial of degree \(k\) that satisfies
\[
\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x - \alpha_j} \frac{d}{dx} - \frac{\mu_k \left(x - \nu^{(k)}_j\right)}{A(x)} S^{(k)}_j(x) = 0.
\]

Our first result is a strengthening of a classical result of Stieltjes. Let \(I\) be the closed interval bounded by \(\alpha_2\) and the Van Vleck zero \(\nu^{(k)}_j\). Then we have the following.

**Theorem 1.** There are exactly \(j - 1\) zeros of \(S^{(k)}_j\) in the interval \((\alpha_1, \alpha_2) \setminus I\) and \(k - j + 1\) zeros of \(S^{(k)}_j\) in the interval \((\alpha_2, \alpha_3) \setminus I\).

The following two theorems establish that the zeros of successive Stieltjes polynomials of the same degree interlace and that the zeros of certain Stieltjes polynomials of successive degrees interlace. The interlacing properties are illustrated in Figure [1].
Theorem 2. The zeros of \( S_j^{(k)} \) and \( S_{j+1}^{(k)} \) interlace; between any two consecutive zeros of \( S_j^{(k)} \) there is exactly one zero of \( S_{j+1}^{(k)} \). Moreover, the smallest zero of \( S_{j+1}^{(k)} \) is less than the smallest zero of \( S_j^{(k)} \). That is, let \( x_{i,j}^{(k)}, i = 1, \ldots, k \), be the zeros of the Stieltjes polynomial \( S_j^{(k)} \), arranged in increasing order. Then

\[
\alpha_1 < x_{1,j+1}^{(k)} < x_{2,j+1}^{(k)} < \cdots < x_{k,j+1}^{(k)} < \alpha_3,
\]

where \( \alpha_1 < x_{i,j}^{(k)} < x_{i+1,j}^{(k)} < \alpha_3, \) and

\[
\alpha_1 < x_{i,j+1}^{(k)} < x_{i,j}^{(k)} < x_{i+1,j+1}^{(k)} < \alpha_3, \quad i = 1, \ldots, k.
\]

Theorem 3. The zeros of \( S_j^{(k)} \) and \( S_{l}^{(k+1)} \) interlace if and only if \( l = j \) or \( l = j+1 \).

That is,

\[
\alpha_1 < x_{i,j+1}^{(k+1)} < x_{i,j}^{(k+1)} < x_{i+1,j+1}^{(k+1)} < \alpha_3, \quad i = 1, \ldots, k.
\]

If \( l \) is neither \( j \) nor \( j+1 \), then the zeros of \( S_j^{(k)} \) and \( S_{l}^{(k+1)} \) do not interlace.

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**Figure 1.** Zeros of Stieltjes and Van Vleck polynomials. On row \( j \) are shown *: zeros of \( S_j^{(k)} \); +: zeros of \( S_{j+1}^{(k)} \); O: Van Vleck zero \( \nu_j^{(k)} \); X: Van Vleck zero \( \nu_{j+1}^{(k+1)} \), for \( k = 6 \). Values of the parameters are \((\alpha_1, \alpha_2, \alpha_3) = (-1, 0, 2), (\rho_1, \rho_2, \rho_3) = (1, 2, 1/3)\).

In the following section we prove these results. Following this, we conclude with a section of remarks and comments on open problems. We show how these results can be combined with known asymptotic properties of Stieltjes and Van Vleck polynomials. In particular, we construct sequences of Van Vleck zeros to converge to any number in \([\alpha_1, \alpha_3]\). We also comment on the possible existence of orthogonal sequences of Stieltjes polynomials. This is a natural question to ask, for several reasons. For one thing, we see from Theorem 3 that any sequence of Stieltjes polynomials \( (S_{j_k}^{(k)}) \) with \( j_{k+1} = j_k \) or \( j_k + 1 \) shares at least one thing in common with orthogonal polynomials, namely the well known fact that the zeros of orthogonal polynomials of successive degrees interlace. We will also show that certain of the sequences \( (S_{j_k}^{(k)}) \) have the same asymptotic zero distribution as sequences of orthogonal polynomials.

The relationship between orthogonal polynomials and second order differential equations has been for many years a rich source of results and problems. Bochner
classified all orthogonal polynomials satisfying an eigenvalue problem of the type
\[ LP_n = \lambda_n P_n, \]
where \( L \) is a second order differential operator. (See [12] for a recent generalization
of these results.) In the Bochner problem, however, the operator \( L \) does not de-
pend on the degree of the polynomial solution, and there is exactly one polynomial
solution for each degree. This is not the case for the equation (3) considered in this
paper, which is instead an eigenvalue problem of the type
\[ L_n P_n = \lambda_n P_n, \]
where now \( L_n \) depends on the degree of the polynomial solutions, and there is more
than one solution for each \( n \). Determining the existence of orthogonal sequences
for (7) is thus much more difficult than for (6).

2. Proofs of main results

For the proofs of the theorems we will need the following lemma.

Lemma 1. There is a zero of \( S_{j+1}^{(k)} \) between every two zeros of \( S_j^{(k)} \) in the inter-
val \((\alpha_1, \alpha_2)\), and between every zero of \( S_j^{(k)} \) in \((\alpha_1, \alpha_2)\) and either of the singular
points \( \alpha_1, \alpha_2 \). Likewise, there is a zero of \( S_j^{(k)} \) between every two zeros of \( S_{j+1}^{(k)} \) in
the interval \((\alpha_2, \alpha_3)\), and between every zero of \( S_{j+1}^{(k)} \) in \((\alpha_2, \alpha_3)\) and either of the
singular points \( \alpha_2, \alpha_3 \).

Proof. For this proof, since we are dealing with Stieltjes polynomials of fixed degree
\( k \), we will omit the superscripts and simply write \( S_j = S_j^{(k)} \) and \( S_{j+1} = S_{j+1}^{(k)} \). Now,
the Stieltjes polynomials \( S_j \) and \( S_{j+1} \) satisfy
\[
\frac{d^2}{dx^2} + \sum_{j=1}^{3} \rho_j \frac{d}{dx} - \frac{\mu_k (x - \nu_j^{(k)})}{A(x)} \right] S_j(x) = 0, \\
\frac{d^2}{dx^2} + \sum_{j=1}^{3} \rho_j \frac{d}{dx} - \frac{\mu_k (x - \nu_{j+1}^{(k)})}{A(x)} \right] S_{j+1}(x) = 0.
\]
Define the integrating factor
\[ J(x) = \prod_{j=1}^{3} |x - \alpha_j|^{\rho_j}. \]
Then
\[ J'(x) = J(x) \sum_{j=1}^{3} \frac{\rho_j}{x - \alpha_j}. \]
Upon multiplying (8) by \( S_{j+1} \) and (9) by \( S_j \), taking the difference of the result,
and then multiplying by \( J \), we obtain
\[
\frac{d}{dx} [J (S'_{j+1} S_j - S_{j+1} S'_j)] = QS_j S_{j+1},
\]
where
\[ Q(x) = (\nu_j - \nu_{j+1}) \frac{J(x)}{A(x)}. \]
Note that $Q < 0$ in $(\alpha_1, \alpha_2)$ and $Q > 0$ in $(\alpha_2, \alpha_3)$. Now, consider two consecutive zeros of $S_j$, $x_1$ and $x_2$, in the interval $(\alpha_1, \alpha_2)$. Then $S'_j$ must alternate signs at $x_1$ and $x_2$. Thus, the expression

$$J \left( S'_{j+1}S_j - S_{j+1}S'_j \right)_{x=x_2} = J \left( S_{j+1}S'_j \right)_{x=x_1}$$

has the same sign as $S_jS_{j+1}$ in $(x_1, x_2)$. But (11) implies that this expression is negative if $S_jS_{j+1} > 0$ in $(x_1, x_2)$ and positive if $S_jS_{j+1} < 0$ in $(x_1, x_2)$. Thus it must be that $S_{j+1}$ changes sign in $(x_1, x_2)$, and we have shown that between any two consecutive zeros of $S_j$ in $(\alpha_1, \alpha_2)$, there is a zero of $S_{j+1}$.

Now let $x_1$ be the smallest zero of $S_j$ in $(\alpha_1, \alpha_2)$. Then, since $J(\alpha_1) = 0$,

$$J \left( S'_{j+1}S_j - S_{j+1}S'_j \right)_{x=x_1} = -J(x_1)S_{j+1}(x_1)S'_j(x_1)$$

also has the same sign as $S_jS_{j+1}$ in $(\alpha_1, x_1)$ if $S_{j+1}$ does not change sign in this interval, again contradicting (11). A similar argument can be applied to the largest zero of $S_j$ in $(\alpha_1, \alpha_2)$ and $\alpha_2$, which establishes that between every zero of $S_j$ in $(\alpha_1, \alpha_2)$ and either of the singular points $\alpha_1, \alpha_2$, there is a zero of $S_{j+1}$.

This argument may be exactly repeated in the interval $(\alpha_2, \alpha_3)$, noting that $Q > 0$ in this interval. It follows that between any two consecutive zeros of $S_{j+1}$ in $(\alpha_2, \alpha_3)$, or between $\alpha_2$ and the smallest zero of $S_{j+1}$ in $(\alpha_2, \alpha_3)$, or between the largest zero of $S_{j+1}$ in $(\alpha_2, \alpha_3)$ and $\alpha_3$ there is a zero of $S_j$.

Proof of Theorem 1. First we show that there are no zeros of $S_j$ between $\alpha_2$ and its corresponding Van Vleck zero $\nu$. According to Shah [18] cf. Theorem 2], between any zero of $S_j$ and $\nu$ there is either a zero of $S'_j(x)$ or $\alpha_2$. Suppose $\nu > \alpha_2$ and that there is a zero $x_1$ of $S_j(x)$ between $\alpha_2$ and $\nu$. Since there is a zero of $S'_j(x)$ between $x_1$ and $\nu$, there must be a zero $x_2$ of $S_j(x)$ greater than $\nu$. We may assume that $x_1$ and $x_2$ are consecutive. But since the zeros of $S_j(x)$ are simple, there cannot be a zero of $S'_j(x)$ in both intervals $(x_1, \nu)$ and $(\nu, x_2)$, which is a contradiction. The case when $\nu < \alpha_2$ is similar.

The theorem will thus be proved once we have shown that there are $j - 1$ zeros of $S_j$ in the interval $(\alpha_1, \alpha_2)$. According to a classical result of Stieltjes [22, 25], every possible distribution of the zeros of the Stieltjes polynomials in the intervals $(\alpha_1, \alpha_2)$ and $(\alpha_2, \alpha_3)$ occurs. That is, for each integer $k$ and any integer $0 \leq m \leq k$, there is a Stieltjes polynomial of degree $k$ with $m$ zeros in $(\alpha_1, \alpha_2)$ and $k - m$ zeros in $(\alpha_2, \alpha_3)$. Thus it suffices to show that there are at least as many zeros of $S^{(k)}_{j+1}$ as there are of $S'_j$ in $(\alpha_1, \alpha_2)$. But this is an immediate consequence of Lemma 1.

Proof of Theorem 2. Combining Lemma 1 with Theorem 1, we see that between the $j - 1$ zeros of $S_j$ in $(\alpha_1, \alpha_2)$ there are $j - 2$ zeros of $S_{j+1}$. The other two zeros of $S_{j+1}$ in $(\alpha_1, \alpha_2)$ lie between $\alpha_1$ and the smallest zero of $S_j$ and between the largest zero of $S_j$ in $(\alpha_1, \alpha_2)$ and $\alpha_2$. Similarly, between the $k - j$ zeros of $S_{j+1}$ in $(\alpha_2, \alpha_3)$ there are $k - j - 1$ zeros of $S_j$, and the other two zeros of $S_j$ in $(\alpha_2, \alpha_3)$ lie between $\alpha_2$ and the smallest zero of $S_{j+1}$ in $(\alpha_2, \alpha_3)$ and between the largest zero of $S_{j+1}$ in $(\alpha_2, \alpha_3)$ and $\alpha_3$. We have thus accounted for all of the zeros of $S_j$ and $S_{j+1}$, and the interlacing is proved. \qed

We have actually proved a stronger statement than Theorem 1. We note this in the following corollary.
Corollary 1. Let $l > j$. Then between any two zeros of $S_j^{(k)}$ in $(\alpha_1, \alpha_2)$ there is a zero of $S_1^{(k)}$. Between any two zeros of $S_1^{(k)}$ in $(\alpha_2, \alpha_3)$ there is a zero of $S_j^{(k)}$.

Proof of Theorem. First we prove the “only if” part. In order for the zeros of $S_1^{(k+1)}$ and $S_j^{(k)}$ to interlace, it must be the case that the smallest zero of $S_1^{(k+1)}$ is smaller than the smallest zero of $S_j^{(k)}$, which is impossible if $i < j$, since then there would be fewer zeros of $S_1^{(k+1)}$ than of $S_j^{(k)}$ in the interval $(\alpha_1, \alpha_2)$. Similarly, if $i > j + 1$, then there are fewer zeros of $S_1^{(k+1)}$ than of $S_j^{(k)}$ in the interval $(\alpha_2, \alpha_3)$.

For the “if” part, as in the proof of Lemma we derive the following expression:

\[
\frac{d}{dx} \left[ J \left( \frac{dS_1^{(k+1)}}{dx} - S_j^{(k)} \frac{dS_j^{(k)}}{dx} \right) \right] = QS_j^{(k)}S_1^{(k+1)},
\]

where in this case,

\[
Q(x) = (\mu_{k+1} - \mu_k) \frac{J(x)}{A(x)} (x - \hat{\nu}_i), \quad \text{and}
\]

\[
\hat{\nu}_i = \frac{\mu_{k+1}\nu_i^{(k+1)} - \mu_k\nu_j^{(k)}}{\mu_{k+1} - \mu_k}.
\]

Note that implies that

\[
\hat{\nu}_j < \nu_j^{(k+1)} < \alpha_3 \quad \text{and} \quad \alpha_1 < \nu_{j+1}^{(k+1)} < \hat{\nu}_{j+1}.
\]

Suppose, for the moment, that $\alpha_1 < \hat{\nu}_j < \alpha_2$. Then, since $Q > 0$ in $(\hat{\nu}_j, \alpha_2)$, between every two consecutive zeros of $S_1^{(k+1)}$ in $(\hat{\nu}_j, \alpha_2)$ and between the largest zero of $S_j^{(k+1)}$ in $(\hat{\nu}_j, \alpha_2)$ and $\alpha_2$ there is a zero of $S_j^{(k)}$. Since $Q < 0$ in $(\alpha_1, \hat{\nu}_j)$, there is a zero of $S_j^{(k+1)}$ between every two zeros of $S_j^{(k)}$ in $(\alpha_1, \hat{\nu}_j)$ and between $\alpha_1$ and the smallest zero of $S_j^{(k)}$ in $(\alpha_1, \hat{\nu}_j)$. This accounts for all of the $j - 1$ zeros of $S_j^{(k)}$ and of $S_j^{(k+1)}$ in $(\alpha_1, \alpha_2)$. Similarly, since $Q < 0$ in $(\alpha_2, \alpha_3)$, between every two zeros of $S_j^{(k)}$ in $(\alpha_2, \alpha_3)$ and between every zero of $S_j^{(k)}$ in $(\alpha_2, \alpha_3)$ and either of the singular points $\alpha_2, \alpha_3$ there is a zero of $S_j^{(k+1)}$. This accounts for the $k$ zeros of $S_j^{(k)}$ and the $k + 1$ zeros of $S_j^{(k+1)}$, and the interlacing is proved in this case.

A nearly identical argument holds in the case when $\alpha_2 < \hat{\nu}_{j+1} < \alpha_3$ to show that the zeros of $S_1^{(k+1)}$ and $S_j^{(k)}$ interlace. The proof is thus completed by the following lemma.

Lemma 2. $\alpha_1 < \hat{\nu}_j < \alpha_2$ and $\alpha_2 < \hat{\nu}_{j+1} < \alpha_3$.

Proof. Suppose that $\hat{\nu}_j \leq \alpha_1$. Then it would be the case that $Q > 0$ in $(\alpha_1, \alpha_2)$ and by arguments similar to the above, between every two zeros of $S_j^{(k+1)}$ in $(\alpha_1, \alpha_2)$ and between every zero of $S_j^{(k+1)}$ in $(\alpha_1, \alpha_2)$ and either of the singular points $\alpha_1, \alpha_2$ there is a zero of $S_j^{(k)}$. But this would imply the existence of at least $j$ zeros of $S_j^{(k)}$ in $(\alpha_1, \alpha_2)$, which contradicts Theorem. Likewise, if $\hat{\nu}_j \geq \alpha_2$, a similar argument would imply the existence of at least $j$ zeros of $S_j^{(k+1)}$ in $(\alpha_1, \alpha_2)$, which is also a contradiction. This argument may be repeated to prove the second statement.
Moreover, let \((12)\) be a sequence of positive integers such that \(j_k/k \to \theta\). Some simple but tedious calculations show that the limit \(\lim_{k \to \infty} \nu^{(k)}_{j_k} = \nu\) of the corresponding Van Vleck zeros is determined by

\[
(12) \quad \frac{1}{\pi} \int_{\alpha_1}^{\min(\alpha_2, \nu)} \sqrt{\frac{\nu - x}{A(x)}} \, dx = \theta.
\]

Moreover, let \(I\) by the interval bounded by \(\alpha_2\) and \(\nu\). Then the asymptotic distribution of the Stieltjes polynomials \(S_{j_k}^{(k)}\) is given by

\[
(13) \quad \rho_S(x) = \begin{cases} 
\frac{1}{\pi} \sqrt{\frac{\nu - x}{A(x)}} & \text{if } x \in (\alpha_1, \alpha_3) \setminus I, \\
0 & \text{if } x \in I.
\end{cases}
\]

We note, in particular, that if \(\nu = \alpha_1, \alpha_2\) or \(\alpha_3\), then \(\rho_S\) is the so-called “arcsine distribution” supported on the intervals \((\alpha_2, \alpha_3)\), \((\alpha_1, \alpha_3)\) and \((\alpha_1, \alpha_2)\), respectively.

Equations (12) and (13) follow from the results of [16] and demonstrate that for any \(\nu \in [\alpha_1, \alpha_3]\) there exists a sequence of Van Vleck zeros that converges to \(\nu\). With Theorem 1 of the present paper we can provide a simple way to explicitly construct such a sequence. Since, by Theorem 1, \(j_k - 1\) zeros of \(S_{j_k}^{(k)}\) lie in \((\alpha_1, \alpha_2)\), if \(j_k/k \to \theta\), then the fraction of zeros of \(S_{j_k}^{(k)}\) in \((\alpha_1, \alpha_2)\) tends to \(\theta\). Combining the ordering of Theorem 1 with the asymptotic properties of [16], we have the following.

**Proposition 1.** (i) Let \((j_k)\) be a sequence of positive integers such that \(j_k/k \to \theta\). Then we have the limit of sequences of Van Vleck zeros:

\[
(14) \quad \lim_{k \to \infty} \nu^{(k)}_{j_k} = \nu,
\]

where \(\nu\) is determined by (12).

(ii) Given any \(\nu \in [\alpha_1, \alpha_3]\), if \(\theta\) is defined by (12) and \(j_k/k \to \theta\), then the sequence \(\nu^{(k)}_{j_k}\) converges to \(\nu\).

We may also calculate conditions under which the limiting distribution of Stieltjes zeros is supported throughout \((\alpha_1, \alpha_3)\). A necessary condition given by Theorem 1 is that \(\nu = \alpha_2\), and (13) tells us that this is also a sufficient condition. In order for the limiting distribution to be supported on \((\alpha_1, \alpha_3)\), the fraction of the zeros of \(S_{j_k}^{(k)}\) in \((\alpha_1, \alpha_2)\) must tend to \(\theta_c\), where \(\theta_c\) is calculated by setting \(\nu = \alpha_2\) in (12). A simple calculation shows that this is (see also [16] Prop. 1)

\[
(15) \quad \theta_c = \frac{2}{\pi} \sin^{-1} \sqrt{\frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1}}.
\]
Perhaps the most intriguing open question in the $p = 3$ case is whether there exist sequences of orthogonal Stieltjes polynomials, for, by Theorem 3, if the sequence $(j_k)$ satisfies $j_{k+1} = j_k$ or $j_k + 1$ (for example $j_k = \lceil k\theta + 1 \rceil$), then the sequence of Stieltjes polynomials $(S^{(k)}_{j_k})$ is an infinite sequence of polynomials with interlacing zeros and asymptotic zero distribution given by (13). Moreover, when $\nu$ in (13) is one of $\alpha_1, \alpha_2, \alpha_3$ or, equivalently, if $\theta = 0, 1$ or $\theta_e$, the sequence $(S^{(k)}_{\lceil k\theta + 1 \rceil})$ is an infinite sequence of polynomials with interlacing zeros and asymptotic zero distribution identical to the asymptotic zero distribution of a sequence of orthogonal polynomials.

However, orthogonality is a rather strict condition, and we conjecture that given any measure $\omega$, when $p > 2$ there is no sequence of Stieltjes polynomials that is orthogonal with respect to $\omega$. If it turns out that this conjecture is correct, it would be interesting to see what role is played by these special sequences of Stieltjes polynomials that share so many properties with orthogonal polynomials, which we might call “nearly orthogonal systems”.

Remarks. Some of the results of the current paper have recently been generalized by P. Brändén [7] to the case when $p > 3$. However, the proofs of these results require a large amount of additional information and numerous techniques.

References

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