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ON THE STRUCTURE OF THE SPACE OF CUSP FORMS FOR A SEMISIMPLE GROUP OVER A NUMBER FIELD

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Abstract. Let G be a semisimple algebraic group defined over a number field k. We study unramified irreducible components of irreducible automorphic cuspidal representations in the space of cusp forms $\mathcal{A}_{cusp}(G(k)\setminus G(\mathbb{A}))$ using the action of an unramified Hecke algebra on compactly supported cuspidal Poincaré series.

1. Introduction

The existence of cusp forms is a fundamental problem in the modern theory of automorphic forms ([1], [18], [14], [6], [7], [17], [15])). We study this problem in the following setup. Let G be a semisimple algebraic group defined over a number field k. We write $\mathbb{A}$ for the ring of adèles of k. In ([11], Theorem 1-1) we studied the existence of irreducible automorphic cuspidal representations having their ramified local components belonging to prescribed Bernstein classes (see below). In the present paper we study unramified local components of representations constructed in ([11], Theorem 1-1). To explain our results, we introduce some notation.

We write V_f (resp., V_∞) for the set of finite (resp., Archimedean) places. For $v \in V_\infty \cup V_f$, we write k_v for the completion of k at v; if $v \in V_f$, then we let $\mathcal{O}_v$ be the ring of integers of k_v. Let $G_\infty = \prod_{v\in V_\infty} G(k_v)$. This is a semisimple Lie group with finite center. It follows from ([16], 3.9.1) that a group G is unramified over k_v for almost all $v \in V_f$. In this case G is defined over $\mathcal{O}_v$ and $K_v \stackrel{\text{def}}{=} G(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup of $G(k_v)$.

If T is a finite set of places in V_f such that G is unramified over k_v, then we write $\mathcal{H}_T$ for the algebra under convolution consisting of all $\prod_{v\in T} K_v$–bi–invariant smooth compactly supported complex functions on $\prod_{v\in T} G(k_v)$. It is well–known that the Satake isomorphism can be used to identify this algebra with the algebra of regular functions on a certain complex affine algebraic variety (see [4]; see also [13], Section 1). We denote this variety by $Spec_{max}\ \mathcal{H}_T$.

Let $v \in V_f$. Then there exists a well–known notion of Bernstein classes for $G(k_v)$ (see for example [11], Section 5). A Bernstein class $\mathfrak{M}_v$ is determined by the pair (M_v, ρ_v), where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is an (irreducible) supercuspidal representation of M_v. We say that $\pi_v \in \operatorname{Irr} G(k_v)$ belongs to $\mathfrak{M}_v$ if π_v

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is a subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v\rho_v)$, for some unramified character χ_v of M_v. Here P_v is an arbitrary parabolic subgroup of $G(k_v)$ containing M_v as a Levi subgroup.

The main result of this paper is the following theorem:

Theorem 1-1. *Let S be a finite set of places containing V_∞ such that G is unramified over k_v for all $v \in V_f - S$. For every $v \in S - V_\infty$, let $\mathfrak{M}_v$ be a Bernstein class of $G(k_v)$. Let $v_0 \in V_f - S$ be a place. We write $\mathfrak{C}_{v_0,S}$ for the set of equivalence classes of irreducible automorphic cuspidal representations $\pi = \pi_\infty \otimes \bigotimes_{v\in V_f} \pi_v$ such that π_v belongs to the class $\mathfrak{M}_v$ for $v \in S - V_\infty$, and π_v is unramified for $V_f - S - \{v_0\}$. Then, for a non–empty finite subset T of $V_f - S - \{v_0\}$, the set consisting of all Satake parameters of unramified representations $\bigotimes_{v\in T} \pi_v$ when π ranges over $\mathfrak{C}_{v_0,S}$, is Zariski dense in $Spec_{max}\ \mathcal{H}_T$. Moreover, we can choose infinitely many $\pi \in \mathfrak{C}_{v_0,S}$ such that the representations $\bigotimes_{v\in T} \pi_v$ are mutually non–equivalent and every $\bigotimes_{v\in T} \pi_v$ is isomorphic to the corresponding full unramified principal series (determined by its Satake parameter* [4]*).*

The proof of Theorem 1-1 is given in Section 3. It is based on a spectral decomposition of Poincaré series for compactly supported cuspidal functions in $L^2(G(k) \setminus G(\mathbb{A}))$, a method that was already successfully applied to study cusp forms in [11] and [12], combined with a method of reducing the problem to S–arithmetic cuspidal forms [13], Theorem 0-3 and Theorem 0-4. The idea for the studying of $\mathcal{H}_T$–modules of Poincaré series to obtain results such as Theorem 1-1 is taken from [13], but the main problem here is to find the correct form of a cuspidal compactly supported Poincaré series in order to prove Theorem 1-1 (see Lemma 3-3). Lemma 3-3 is proved in Section 4.

2. Preliminary results

In this section we fix the notation used in this paper. We let G be a semisimple algebraic group defined over a number field k. We write V_f (resp., V_∞) for the set of finite (resp., Archimedean) places. For $v \in V \stackrel{\text{def}}{=} V_\infty \cup V_f$, we write k_v for the completion of k at v. If $v \in V_f$, then we let $\mathcal{O}_v$ denote the ring of integers of k_v. Let $\mathbb{A}$ be the ring of adèles of k. For almost all places of k, G is defined over $\mathcal{O}_v$ ([16], 3.9.1). We write $K_v \stackrel{\text{def}}{=} G(\mathcal{O}_v)$. The group of adèlic points $G(\mathbb{A}) = \prod_v' G(k_v)$ is a restricted product over all places of k of the groups $G(k_v)$: $g = (g_v)_{v\in V} \in G(\mathbb{A})$ if and only if $g_v \in K_v$ for almost all v. The group $G(\mathbb{A})$ is a locally compact group and $G(k)$ is embedded diagonally as a discrete subgroup of $G(\mathbb{A})$.

For a finite subset $S \subset V$, we let

$$G_S = \prod_{v\in S} G(k_v).$$

In addition, if S contains all Archimedean places V_∞, then we let $G^S = \prod_{v\notin S}' G(k_v)$. Then

$$G(\mathbb{A}) = G_S \times G^S. \tag{2-1}$$

We let $G_\infty = G_{V_\infty}$ and $G(\mathbb{A}_f) = G^{V_\infty}$.

The group G_∞ is a semisimple Lie group. It might not be connected, but it has a finite center. The group $G(\mathbb{A}_f)$ is a totally disconnected group. Let $K_\infty \subset G_\infty$ be a maximal compact subgroup. Let $\mathfrak{g}_\infty = Lie(G_\infty)$ be the (real) Lie algebra of

G_∞. Let $\mathcal{U}(\mathfrak{g}_\infty)$ be the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{Z}(\mathfrak{g}_\infty)$ be the center of $\mathcal{U}(\mathfrak{g}_\infty)$.

We say that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ is smooth if $f(\cdot, g_f) \in C^\infty(G_\infty)$ for all $g_f \in G(\mathbb{A}_f)$ and if there exists an open compact subgroup $L \subset G(\mathbb{A}_f)$ such that $f(g_\infty, g_f \cdot l) = f(g_\infty, g_f)$ for all $(g_\infty, g_f) \in G_\infty \times G(\mathbb{A}_f)$ and $l \in L$. Here we consider f as a function of two variables, $f(g) = f(g_\infty, g_f)$, where $g = (g_\infty, g_f)$. We write $C^\infty(G(\mathbb{A}))$ for the vector space of all smooth functions on $G(\mathbb{A})$. We let $C_c^\infty(G(\mathbb{A}))$ be the space of all smooth compactly supported functions on $G(\mathbb{A})$. It is easy to show that $C_c^\infty(G(\mathbb{A}))$ is a span of the functions $f_\infty \otimes \bigotimes_{v \in V_f} f_v$, where $f_\infty \in C_c^\infty(G_\infty)$, $f_v \in C_c^\infty(G(k_v))$ $(v \in V_f)$, and $f_v = char_{K_v}$ for almost all v.

By definition, we let $C^\infty(G(k) \setminus G(\mathbb{A})) \subset C^\infty(G(\mathbb{A}))$ be the subspace consisting of all functions $f \in C^\infty(G(\mathbb{A}))$ such that $f(\gamma \cdot g) = f(g)$ for all $\gamma \in G(k)$ and $g \in G(\mathbb{A})$. The space $C_c^\infty(G(k) \setminus G(\mathbb{A}))$ is a subset of $C^\infty(G(k) \setminus G(\mathbb{A}))$ consisting of functions compactly supported modulo $G(k)$.

An automorphic form is a function in $C^\infty(G(k) \setminus G(\mathbb{A}))$ which is K_∞–finite on the right, $Z(\mathfrak{g}_\infty)$–finite and which satisfies a certain growth condition (see [3], 4.2). The space of all automorphic forms we denote by $\mathcal{A}(G(k) \setminus G(\mathbb{A}))$. The subspace of cuspidal automorphic forms we denote by $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$. By definition, $f \in \mathcal{A}(G(k) \setminus G(\mathbb{A}))$ is a cuspidal automorphic form if

$$\int_{U_P(k)\setminus U_P(\mathbb{A})} f(ng)dn = 0 \quad (\text{for all } g \in G(\mathbb{A})), \tag{2-2}$$

for all proper k–parabolic subgroups P of G. In this paper we write U_P for the unipotent radical of a k–parabolic subgroup P of G. In general, we say that a locally integrable function $f : G(k) \setminus G(\mathbb{A}) \to \mathbb{C}$ is a cuspidal function if it satisfies (2-2) for almost all $g \in G(\mathbb{A})$. The space of cuspidal automorphic forms is denoted by $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$.

The topological space $G(k) \setminus G(\mathbb{A})$ has a finite volume $G(\mathbb{A})$–invariant measure:

$$\int_{G(k)\setminus G(\mathbb{A})} P(f)(g)dg = \int_{G(\mathbb{A})} f(g)dg \quad (f \in C_c^\infty(G(\mathbb{A}))), \tag{2-3}$$

where *the adèlic compactly supported Poincaré series* $P(f)$ is defined as follows:

$$P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g) \in C_c^\infty(G(k) \setminus G(\mathbb{A})). \tag{2-4}$$

The measure introduced in (2-3) enables us to introduce the Hilbert space $L^2(G(k) \setminus G(\mathbb{A}))$ and its closed subspaces $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ consisting of all cuspidal functions in $L^2(G(k) \setminus G(\mathbb{A}))$. Both of them are unitary representations of $G(\mathbb{A})$. Moreover, the space $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ can be decomposed into a direct sum of irreducible unitary representations of $G(\mathbb{A})$, each occurring with a finite multiplicity (see [5]).

Let S be a finite set of places of V containing V_∞. Then we write $G(\mathbb{A})$ using the decomposition (2-1). Let $L \subset G^S$ be an open compact subgroup. Let

$$\Gamma = \Gamma_L = G(k) \cap L \subset G^S, \tag{2-5}$$

where we identify $G(k)$ with its image under the diagonal embedding into G^S. Identifying Γ with its image under another diagonal embedding $G(k) \hookrightarrow G_S$, it can be considered as a discrete subgroup of G_S. It is called an S–arithmetic subgroup

of $G(k)$. It is well known that we can fix a finite volume G_S–invariant measure on $\Gamma \setminus G_S$ by

$$\int_{\Gamma\setminus G_S} P_L(f_S)(g_S)dg_S = \int_{G_S} f_S(g_S)dg_S \tag{2-6}$$

for $f_S \in C_c^\infty(G_S)$, where *the compactly supported Poincaré series (for* Γ*)* is defined as follows:

$$P_L(f_S)(g_S) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma} f_S(\gamma \cdot g_S). \tag{2-7}$$

The function $P_L(f_S)$ belongs to the space $C_c^\infty(\Gamma \setminus G_S)$.

Similarly as before, we define the notion of Γ–cuspidality by letting $U_{P,S} = \prod_{v\in S} U_P(k_v)$, and integrating over $\Gamma \cap U_{P,S} \setminus U_{P,S}$, for any proper k–parabolic subgroup P of G. We let $C_{c,cusp}^\infty(\Gamma \setminus G_S)$ be the subspace of Γ–cuspidal functions in $C_c^\infty(\Gamma \setminus G_S)$. In this paper we use the following analogue of ([11], Lemma 3-3):

Lemma 2-8. *Let $L \subset G^S$ be an open compact subgroup and let $\psi : U_P(k)\backslash U_P(\mathbb{A}) \to \mathbb{C}$ be a continuous function which is right–invariant under the open compact subgroup $L \cap U_P^S$. Then, we have the following formula:*

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} \psi(u)du = vol_{U_P^S}(L \cap U_P^S) \cdot \int_{\Gamma_L\cap U_{P,S}\backslash U_{P,S}} \psi(u_S)du_S.$$

The next lemma will be used several times in this paper.

Lemma 2-9. *Let $L \subset G^S$ be an open compact subgroup and let $\varphi_S \in C_c^\infty(G_S)$. Then, $P(\varphi_S \otimes char_L)$ is cuspidal if and only if $P_{\Gamma_L}(\varphi_S)$ is Γ_L–cuspidal.*

Proof. We can write $G^S = G(k) \cdot C \cdot L$, for some finite set C of minimal cardinality [2]. (Here we identify $G(k)$ with its image in G^S under the diagonal embedding.) We may assume that $1 \in C$.

Let P be a proper k–parabolic subgroup of G. Let $g^S = \gamma cl \in G^S = G(k) \cdot C \cdot L$. Then, we have the following:

$$\begin{aligned}
&\int_{U_P(k)\backslash U_P(\mathbb{A})} P(\varphi_S \otimes char_L)(u(g_S, g^S))du \\
&= \int_{U_P(k)\backslash U_P(\mathbb{A})} P(\varphi_S \otimes char_L)(u(g_S, \gamma cl))du \\
&= \int_{U_P(k)\backslash U_P(\mathbb{A})} P(\varphi_S \otimes char_L)(u(g_S, \gamma c))du \\
&= \int_{\gamma^{-1}U_P(k)\gamma\backslash\gamma^{-1}U_P(\mathbb{A})\gamma} P(\varphi_S \otimes char_L)(\gamma u\gamma^{-1}(g_S, \gamma c))du \\
&= \int_{\gamma^{-1}U_P(k)\gamma\backslash\gamma^{-1}U_P(\mathbb{A})\gamma} P(\varphi_S \otimes char_L)(u(\gamma^{-1}g_S, c))du.
\end{aligned}$$

Thus, $P(\varphi_S \otimes char_L)$ is cuspidal on $G(\mathbb{A})$ if and only if

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} P(\varphi_S \otimes char_L)(u(g_S, c))du = 0 \quad (g_S \in G_S,\ c \in C), \tag{2-10}$$

for every proper k–parabolic subgroup P of G. Since the function $u \in U_P(\mathbb{A}) \mapsto P(\varphi_S \otimes char_L)(u(g_S, c))$ is right–invariant under $U_P^S \cap cLc^{-1}$, Lemma 2-8 shows

that the integral on the left–hand side of (2-10) can be computed as follows:

$$\begin{aligned} & vol_{U_P^S}(U_P^S \cap cLc^{-1})^{-1} \cdot \int_{U_P(k)\backslash U_P(\mathbb{A})} P(\varphi_S \otimes char_L)(u(g_S, c))du \\ &= \int_{\Gamma_{cLc^{-1}} \cap U_{P,S} \backslash U_{P,S}} P(\varphi_S \otimes char_L)(u_S g_S, c) du_S \\ &= \int_{\Gamma_{cLc^{-1}} \cap U_{P,S} \backslash U_{P,S}} \left(\sum_{\gamma \in G(k)} \varphi_S(\gamma u_S g_S) char_L(\gamma c) \right) du_S. \end{aligned} \tag{2-11}$$

In order to evaluate the last integral in (2-11), we note that the inner sum is non–zero only if $c \in G(k)L$. Since C is assumed to be minimal subject to $G^S = G(k) \cdot C \cdot L$ and $1 \in C$, we obtain $c = 1$. Thus, if $c \neq 1$, then the integral in (2-11) is zero, while for $c = 1$ it becomes

$$\int_{\Gamma_L \cap U_{P,S} \backslash U_{P,S}} \left(\sum_{\gamma \in \Gamma_L} \varphi_\infty(\gamma u_S g_S) \right) du_S = \int_{\Gamma_L \cap U_{P,S} \backslash U_{P,S}} P_{\Gamma_L}(\varphi_S)(u_S g_S) du_S.$$

This proves the lemma. □

3. The proof of Theorem 1-1

Using the notation introduced in the statement of Theorem 1-1, we put $S' = S \cup \{v_0\}$. Let $K^{S'} \subset G^{S'}$ be the open compact subgroup defined by

$$K^{S'} = \prod_{v \notin S'} K_v. \tag{3-1}$$

We also put $G_T = \prod_{v \in T} G(k_v)$ and $K_T = \prod_{v \in T} K_v$. We consider the commutative algebra $\mathcal{H}_T$ described in the introduction.

It is well–known (see [4]) that the set of equivalence classes of irreducible unramified representations (i.e., having a non–zero K_T–fixed vector) and $Spec_{max}\ \mathcal{H}_T$ are in a one–to–one correspondence defined in the following way: for an unramified irreducible representation π_T of G_T, the algebra $\mathcal{H}_T$ acts on the one–dimensional space $\pi_T^{K_T}$ by a character $\omega_{\pi_T} \in Spec_{max}\ \mathcal{H}_T$.

Let $\mathfrak{p}_{v_0}$ be the maximal ideal in the ring $\mathcal{O}_{v_0}$, and let U_{v_0} be the kernel of the reduction homomorphism $G(\mathcal{O}_{v_0}) \to G(\mathcal{O}_{v_0}/\mathfrak{p}_{v_0})$. Let $\mathfrak{L}$ be the family of open compact subgroups L^S of G^S of the following form:

$$L^S = K^{S'} \times L_{v_0}, \quad L_{v_0} \subset U_{v_0} \text{ is a variable group.} \tag{3-2}$$

If we let $\Gamma_{L^S} = G(k) \cap L$, where $G(k)$ is identified with a subgroup of G^S, then the image of Γ_{L^S} under the diagonal embedding of $G(k)$ into G_S is a discrete subgroup. We continue to call this discrete subgroup Γ_{L^S}. The following lemma is the key fact for the proof of Theorem 1-1. It is proved in Section 4 and the proof uses many fine parts of [11].

Lemma 3-3. *Assume that $\varphi_v \in C_c^\infty(G(k_v))$ $(v \in S - V_\infty)$ are non–zero functions. Then, there exists $\varphi_\infty \in C_c^\infty(G_\infty)$ such that for every open compact subgroup $L^S \in \mathfrak{L}$ the Poincaré series $P(\varphi_S \otimes char_{L^S})$ is a cuspidal function on $G(k) \setminus G(\mathbb{A})$ and non–zero when restricted to G_∞.*

Clearly, cuspidal functions $P(\varphi_S \otimes char_{L^S})$ introduced in Lemma 3-3 are all K_T–invariant. So, we may study the structure of the $\mathcal{H}_T$–module generated by $P(\varphi_S \otimes char_{L^S})$:

$$J(\varphi_S, L^S)^T \stackrel{\text{def}}{=} \mathcal{H}_T.P(\varphi_S \otimes char_{L^S}) \subset C^\infty_{c,cusp}(G(k) \setminus G(\mathbb{A}))^{K_T}, \quad L^S \in \mathfrak{L}. \tag{3-4}$$

We denote its annihilator as follows:

$$I(\varphi_S, L^S)^T \stackrel{\text{def}}{=} \text{Ann } J(\varphi_S, L^S)^T. \tag{3-5}$$

Since the module $J(\varphi_S, L^S)^T$ is generated by the image of the identity and the algebra $\mathcal{H}_T$ is commutative, we have the following isomorphism of $\mathcal{H}_T$–modules:

$$J(\varphi_S, L^S)^T \simeq \mathcal{H}_T / I(\varphi_S, L^S)^T.$$

We let U_{L^S} denote the closed $G(\mathbb{A})$–invariant subspace generated by $P(\varphi_S \otimes char_{L^S})$ in $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$. We can decompose U_{L^S} into the Hilbert direct sum of closed irreducible $G(\mathbb{A})$–invariant subspaces [5],

$$U_{L^S} = \hat{\bigoplus_j} \mathfrak{H}^j_{L^S}. \tag{3-6}$$

We write

$$\pi^{(j)}_{L^S} = \pi^{(j)}_{\infty, L^S} \otimes \bigotimes_{v \in V_f} \pi^{(j)}_{v, L^S} \tag{3-7}$$

for the $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module of $\mathfrak{H}^j_{L^S}$. Now, we are ready to prove the following lemma:

Lemma 3-8. *Under the above assumptions, we have the following:*

(i) $\bigotimes_{v \in T} \pi^{(j)}_{v, L^S}$ *contains a non–zero* K_T*–invariant vector; i.e.,* $\bigotimes_{v \in T} \pi^{(j)}_{v, L^S}$ *is unramified.*

(ii) *The Zariski closure in* $Spec_{max}\ \mathcal{H}_T$ *of the set of all*

$$\omega_{T, L^S, j} \stackrel{\text{def}}{=} \text{ the character of } \mathcal{H}_T \text{ associated to } \bigotimes_{v \in T} \pi^{(j)}_{v, L^S}$$

is exactly the algebraic set $\mathcal{V}(I(\varphi_S, L^S)^T)$ *associated to the ideal* $I(\varphi_S, L^S)^T$*. In fact,* $I(\varphi_S, L^S)^T$ *is the intersection of the kernels of all characters* $\omega_{T, L^S, j}$*. In particular,* $I(\varphi_S, L^S)^T$ *is a radical ideal. (This notion is recalled in the course of the proof.)*

(iii) *Let us select a decreasing sequence of open compact subgroups of* $G(k_{v_0})$*:* $U_{v_0} \supset L_{v_0,1} \supset L_{v_0,2} \supset \cdots$ *such that* $\bigcap_{n \geq 1} L_{v_0,n} = \{1\}$*. Let* $L^S_n = K^{S'} \times L_{v_0,n}$*. Then the union* $\bigcup_{n \geq 1} \mathcal{V}(I(\varphi_S, L^S_n)^T)$ *is Zariski dense in* $Spec_{max}\ \mathcal{H}_T$*.*

Proof. We prove (i). First, we can decompose according to (3-6):

$$P(\varphi_S \otimes char_{L^S}) = \sum_j \psi_j, \quad \psi_j \in \mathfrak{H}^j_{L^S}. \tag{3-9}$$

Since U_{L^S} is generated by $P(\varphi_S \otimes char_{L^S})$, $\psi_j \neq 0$ and ψ_j is K^T–invariant for all j. This implies (i). Now, we prove (ii). Let $f \in \mathcal{H}_T$. Then we have the following:

$$f.P(\varphi_S \otimes char_{L^S}) = \sum_j f.\psi_j = \sum_j \omega_{T, L^S, j}(f) \psi_j, \tag{3-10}$$

since ψ_j is K^T–invariant in $\mathfrak{H}^j_{L^S}$ for all j. This implies that $f \in I(\varphi_S, L^S)^T$ if and only if $\omega_{T,L^S,j}(f) = 0$ for all j. Hence, we see that the ideal $I(\varphi_S, L^S)^T$ is a radical ideal (i.e., $f^m \in I(\varphi_S, L^S)^T$, for $f \in \mathcal{H}_T$ and $m > 0$, implies that $f \in I(\varphi_S, L^S)^T$). The remainder of (ii) follows from the usual "radical ideal–algebraic set" correspondence in algebraic geometry. This proves (ii). Finally, we prove (iii). The proof of (iii) can be done on the (simplified) lines of the proof of ([13], Theorem 2-8 (ii)). We give some hints and leave the details to the reader. We write every group $L^S \in \mathfrak{L}$ in the form $L^S = K_T \times L^S_1$, where $L^S_1 = K^{S'\cup T} \times L_{v_0}$ is an open compact subgroup of $G^{S\cup T}$ (see (3-2)). We define a discrete subgroup $\Gamma_{L^S_1}$ of $G_{S\cup T}$ in the standard way (see (2-5)). The function $P_{\Gamma_{L^S_1}}(\varphi_S \otimes char_{K_T})$ is $\Gamma_{L^S_1}$–cuspidal applying Lemmas 2-9 and 3-3. Direct computation shows that for $g_\infty \in G_\infty \subset G_{S\cup T}$, we have the following:

$$P_{\Gamma_{L^S_1}}(\varphi_S \otimes char_{K_T})(g_\infty) = P(\varphi_S \otimes char_{L^S})(g_\infty, 1).$$

Thus, by Lemma 3-3, $P_{\Gamma_{L^S_1}}(\varphi_S \otimes char_{K_T})$ is non–trivial. Then, the $\mathcal{H}_T$–submodule of $C_{c,cusp}(\Gamma_{L^S_1}\backslash G_{S\cup T})^{K_T}$ generated by $P_{\Gamma_{L^S_1}}(\varphi_S \otimes char_{K_T})$ is non–trivial. We denote this module by $J(\varphi_S, L^S_1)^T$. Let $I(\varphi_S, L^S_1)^T$ be its annihilator. Then, as in ([13], Lemma 2-20), we have the epimorphism of $\mathcal{H}_T$–modules,

$$J(\varphi_S, L^S)^T \to J(\varphi_S, L^S_1)^T \to 0,$$

and, consequently, we have

$$I(\varphi_S, L^S)^T \subset I(\varphi_S, L^S_1)^T. \tag{3-11}$$

We write $L^S_n = K_T \times (L_n)^S_1$ for $n \geq 1$. Then $(L_1)^S_1 \supset (L_2)^S_1 \supset \cdots$, and we have an epimorphic $\mathcal{H}_T$–module map,

$$J(\varphi_S, (L_{n+1})^S_1)^T \to J(\varphi_S, (L_n)^S_1)^T \to 0 \ \text{ for all } n \geq 1,$$

defined by

$$P_{(L_n)^S_1/(L_{n+1})^S_1}(f)(g_{S\cup T}) = \sum_{\gamma \in \Gamma_{(L_{n+1})^S_1}\backslash \Gamma_{(L_n)^S_1}} f(\gamma \cdot g_{S\cup T}).$$

This implies that $I(\varphi_S, (L_{n+1})^S_1)^T \subset I(\varphi_S, (L_n)^S_1)^T$ for all $n \geq 1$ ([13], Lemma 2-24). We show the following:

$$\bigcap_{n\geq 1} I(\varphi_S, (L_n)^S_1)^T = \{0\}. \tag{3-12}$$

First, identifying $G(k)$ with its image under the diagonal embedding in $G^{S\cup T}$, we have $\Gamma_{(L_n)^S_1} = G(k) \cap (L_n)^S_1$ (see (3-2)). Furthermore, this observation and the assumption $\bigcap_{n\geq 1} L_{v_0,n} = \{1\}$ show us that $\bigcap_{n\geq 1} \Gamma_{(L_n)^S_1} = \{1\}$ by considering the v_0–th component. Now, as in the proof ([13], Lemma 2-27), we show the claim in (3-12).

Now, (3-11) and (3-12) show that $\bigcap_{n\geq 1} I(\varphi_S, L^S_n)^T = \{0\}$. (See [13], Lemma 2-28). Since, by (ii), the ideals $I(\varphi_S, L^S_n)^T$ are radical, the following simple argument shows that the union $\bigcup_{n\geq 1} \mathcal{V}(I(\varphi_S, L^S_n)^T)$ is Zariski dense in $Spec_{max}\ \mathcal{H}_T$. If $f \in \mathcal{H}_T$ vanishes on that union, then by Nullstellensatz for each $n \geq 1$ there exists $m_n \in \mathbb{Z}_{>0}$ such that $f^{m_n} \in I(\varphi_S, L^S_n)^T$. Since $I(\varphi_S, L^S_n)^T$ is radical, we obtain $f \in I(\varphi_S, L^S_n)^T$ for all $n \geq 1$. Hence, $f \in \bigcap_{n\geq 1} I(\varphi_S, L^S_n)^T = \{0\}$. Thus, $f = 0$. This proves the claim. The proof of (iii) is complete. $\square$

Corollary 3-13. *Under the assumptions of Lemma 3-8 (iii), the union*

$$\bigcup_{n\geq 1}\{\omega_{T,L_n^S,j};\quad \text{all } j \text{ appearing in (3-9) for } L^S = L_n^S\}$$

is Zariski dense in $Spec_{max}\ \mathcal{H}_T$. Moreover, it contains a subset which is also Zariski dense in $Spec_{max}\ \mathcal{H}_T$ such that every unramified representation $\bigotimes_{v\in T}\pi_{v,L^S}^{(j)}$ in the corresponding set of unramified representations is isomorphic to the corresponding full unramified principal series of G_T.

Proof. The first claim follows directly from Lemma 3-8 (iii). The other claim has the same proof as ([13], Theorem 3-1). □

Now, we begin the proof of Theorem 1-1. Let $v \in S - V_\infty$. Then $G(k_v)$ acts by right translations on $C_c^\infty(G(k_v))$. We showed in ([11], Lemma 5-2 (i)) that $C_c^\infty(G(k_v))(\mathfrak{M}_v) \neq 0$. We select $\varphi_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$, $\varphi_v(1) \neq 0$. Next, for a K_∞–type δ, we write $d(\delta)$ and ξ_δ for the degree and character of δ, respectively. Since $P(\varphi_S \otimes char_{L^S})$ is non–zero, there exists δ such that its δ–isotypic component

$$\begin{aligned} &P(E_\delta(\varphi_\infty)\otimes \bigotimes_{v\in S-V_\infty}\varphi_v\otimes char_{L^S})(g)\\ &\quad = E_\delta(P(\varphi_S\otimes char_{L^S}))(g) = \int_{K_\infty} d(\delta)\overline{\xi_\delta(k)}P(\varphi_S\otimes char_{L^S})(gk)\ dk,\quad g\in G(\mathbb{A})\end{aligned}$$

is a non–zero function, where $E_\delta(\varphi_\infty) = \int_{K_\infty} d(\delta)\overline{\xi_\delta(k)}\varphi_\infty(g_\infty k)\ dk$ transforms as δ on the right. Now, in the settings of the paragraph before Lemma 3-8 and using (3-9), the spectral decomposition of $P(E_\delta(\varphi_\infty)\otimes\bigotimes_{v\in S-V_\infty}\varphi_v\otimes char_{L^S})$ is given by

$$P(E_\delta(\varphi_\infty)\otimes \bigotimes_{v\in S-V_\infty}\varphi_v\otimes char_{L^S}) = \sum_j E_\delta(\psi_j).$$

Using this, we see that $\pi_{v,L^S}^{(j)}$ belongs to the class of $\mathfrak{M}_v$ for all $v \in S - V_\infty$, applying the argument of the proof of ([11], Theorem 7-2 (ii)). It is clear that $\pi_{v,L^S}^{(j)}$ is unramified for $v \in V_f - S - \{v_0\}$. Now, Corollary 3-13 completes the proof of Theorem 1-1.

4. The Proof of Lemma 3-3

The proof of Lemma 3-3 reveals a new way of constructing cuspidal compactly supported Poincaré series. We continue with the notation introduced in Section 3 before the statement of Lemma 3-3.

First, using a convenient right translation, we may assume that $\varphi_v(1) \neq 0$ for all $v \in S - V_\infty$. We let $\varphi_{v_0} = char_{U_{v_0}}$.

Put $\Gamma_{S'} = G(k)\cap K^{S'}$ (see (3-1)), where we identify $G(k)$ with its image in $G^{S'}$ under the diagonal embedding. We can consider $\Gamma_{S'}$ as a discrete subgroup of $G_{S'}$ identifying it with the image under the diagonal embedding of $G(k)$ in $G_{S'}$.

We select open compact subgroups $L_v \subset G(k_v)$ for $v \in S - V_\infty$ such that φ_v is bi–invariant under L_v. We let

$$U = K^{S'} \times \prod_{v\in S-V_\infty} L_v \times U_{v_0}.$$

Then $\Gamma_U = G(k)\cap U$ can be considered as a discrete subgroup of G_∞.

Now, we construct a very special $f = f_\infty \otimes \bigotimes_{v\in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$ such that $P(f)$ is a cuspidal compactly supported Poincaré series. We let $f_v = char_{K_v}$ for $v \notin S$, and let $f_v = char_{L_v}$ for $v \in S - V_\infty$. We let f_{v_0}, $f_{v_0}(1) \neq 0$ be the matrix coefficient of a supercuspidal representation constructed in the following way. We select some supercuspidal irreducible representation ρ of a finite group $G(\mathcal{O}_{v_0}/\mathfrak{p}_{v_0})$. Then the reduction homomorphism $G(\mathcal{O}_{v_0}) \to G(\mathcal{O}_{v_0}/\mathfrak{p}_{v_0})$ enables us to consider ρ as an irreducible representation of $G(\mathcal{O}_{v_0})$. Now, ([8], Proposition on page 20; see also [9], Proposition 6.6), implies that the compactly induced representation $c-\mathrm{Ind}_{G(\mathcal{O}_{v_0})}^{G(k_{v_0})}(\rho)$ breaks into finitely many irreducible supercuspidal representations. Frobenius reciprocity implies that all of them contain ρ upon the restriction to $G(\mathcal{O}_{v_0})$. Hence they have a vector invariant under U_{v_0}. We select a matrix coefficient, f_{v_0}, $f_{v_0}(1) \neq 0$, of one of the irreducible subrepresentations of $c-\mathrm{Ind}_{G(\mathcal{O}_{v_0})}^{G(k_{v_0})}(\rho)$ which is bi–invariant under U_{v_0}.

Now, we apply ([11], Theorem 4-2) to construct a convenient Archimedean component f_∞. To explain this, we review the Iwasawa decomposition for G_∞. Let P_∞ be a minimal parabolic subgroup of G_∞. Let N_∞ be its unipotent radical, let A_∞ be the maximal split torus, and $M_\infty = Z_{K_\infty}(A_\infty)$. Then, we have the following diffeomorphism (the Iwasawa decomposition):

$$N_\infty \times A_\infty \times K_\infty \xrightarrow{(n,a,k)\mapsto n\cdot a\cdot k} G_\infty = N_\infty A_\infty K_\infty.$$

The Iwasawa decomposition implies that there exist unique C^∞–functions $a : G_\infty \to A_\infty$, $n : G_\infty \to N_\infty$, and $k : G_\infty \to K_\infty$ such that

$$g_\infty = n(g_\infty)\cdot a(g_\infty)\cdot k(g_\infty), \quad g_\infty \in G_\infty. \tag{4-1}$$

As in the proof of ([11], Theorem 4-2), we can find neighborhoods of identities $U \subset N_\infty$ and $V \subset A_\infty$ such that

$$\Gamma_{S'} \cap \left[(UVK_\infty) \times \prod_{v\in S'-V_\infty} \mathrm{supp}\ (f_v)\right] = \Gamma_{S'} \cap \left[K_\infty \times \prod_{v\in S'-V_\infty} \mathrm{supp}\ (f_v)\right]. \tag{4-2}$$

We pick "a variable small" neighborhood W of the identity in K_∞ and assume that

$$\Gamma_{S'} \cap \left[(WW^{-1}) \times \prod_{v\in S'-V_\infty} \mathrm{supp}\ (f_v)\right] = \{1\}. \tag{4-3}$$

The function f_∞ is selected in the following way:

$$f_\infty(g_\infty) = \zeta(n(g_\infty))\eta(a(g_\infty))\xi(k(g_\infty)),$$

where $\zeta \in C_c^\infty(U)$, $\eta \in C_c^\infty(V)$, and $\xi \in C_c^\infty(W)$, $\zeta(1), \eta(1), \xi(1) \neq 0$. We have (see the proof of Theorem 4-2 in [11]) that

$$P(f)(g_\infty, 1) = \sum_{\gamma\in\Gamma_{S'}} \left(\prod_{v\in S'-V_\infty} f_v(\gamma)\right) f_\infty(\gamma\cdot g_\infty). \tag{4-4}$$

We require that

$$\sum_{\gamma\in\Gamma_{S'}} \left(\prod_{v\in S'-V_\infty} f_v(\gamma)\right)\cdot f_\infty(\gamma\cdot k_\infty) \neq 0$$

for some $k_\infty \in K_\infty$ to insure that $P(f)|_{G_\infty} \neq 0$. By (4-2), this is equivalent to

(4-5)
$$P(f)|_{G_\infty}(k_\infty) = \sum_{\gamma \in \Gamma_{S'} \cap K_\infty} \left(\prod_{v \in S' - V_\infty} f_v(\gamma) \right) \cdot \xi(\gamma \cdot k_\infty) \neq 0, \quad \text{for some } k_\infty \in K_\infty,$$

where in the summation we identify $\Gamma_{S'} \subset G(k)$ with a subgroup of G_∞. Although $\Gamma_{S'}$ is not a discrete subgroup of G_∞, the sum is finite if we "throw out" all γ's such that $\prod_{v \in S' - V_\infty} f_v(\gamma) = 0$. Applying (4-3), (4-5) is satisfied for every $\xi \in C_c^\infty(W)$ such that $\xi(1) \neq 0$.

The function $P(f)|_{G_\infty}$ is a Γ_U–cuspidal compactly supported Poincaré series for G_∞ (see ([11], Proposition 3-2)). Let $\varphi_\infty \in C_c^\infty(G_\infty)$ be such that

$$P_{\Gamma_U}(\varphi_\infty) = P(f)|_{G_\infty}. \tag{4-6}$$

Hence

$$P_{\Gamma_U}(\varphi_\infty) \text{ is } \Gamma_U\text{–cuspidal and non–zero.} \tag{4-7}$$

Put $\varphi_{S'} = \varphi_\infty \otimes \bigotimes_{v \in S' - V_\infty} \varphi_v$. Then, if we write $G(\mathbb{A}) = G_{S'} \times G^{S'}$ (see (2-1)), then we have the following:

$$P(\varphi_{S'} \otimes char_{K^{S'}})(g_\infty, 1) = \sum_{\gamma \in \Gamma_{S'}} \varphi_\infty(\gamma g_\infty) \prod_{v \in S' - V_\infty} \varphi_v(\gamma),$$
$$\text{for all } g_\infty \in G_\infty \text{ considering } G_\infty \subset G_{S'}.$$

For a fixed $g_\infty \in G_\infty$, this sum is actually finite since $\Gamma_{S'}$ is a discrete subgroup of $G_{S'}$. We transform the sum in a more convenient form:

$$P(\varphi_{S'} \otimes char_{K^{S'}})(g_\infty, 1) = \sum_{\gamma \in \Gamma_U \backslash \Gamma_{S'}} P_{\Gamma_U}(\varphi_\infty)(\gamma g_\infty) \prod_{v \in S' - V_\infty} \varphi_v(\gamma),$$

where we used our assumption that φ_v is bi–invariant under L_v for all $v \in S - V_\infty$ and φ_{v_0} is bi–invariant under U_{v_0}. We select the representatives in $\Gamma_{S'}$ for cosets in $\Gamma_U \backslash \Gamma_{S'}$ and let $1 = \gamma_1, \ldots, \gamma_l \in \Gamma_{S'}$ denote all of them satisfying

$$\prod_{v \in S' - V_\infty} \varphi_v(\gamma_i) \neq 0.$$

Then

$$P(\varphi_{S'} \otimes char_{K^{S'}})(g_\infty, 1) = \sum_{i=1}^{l} P_{\Gamma_U}(\varphi_\infty)(\gamma_i g_\infty) \prod_{v \in S' - V_\infty} \varphi_v(\gamma_i), \quad g_\infty \in G_\infty.$$

Thus, (4-4) and (4-6) imply that

$$\begin{aligned} &P(\varphi_{S'} \otimes char_{K^{S'}})(k_\infty, 1) \\ &= \sum_{i=1}^{l} \sum_{\gamma \in \Gamma_{S'}} \left(\prod_{v \in S' - V_\infty} f_v(\gamma) \varphi_v(\gamma_i) \right) f_\infty(\gamma \gamma_i k_\infty), \quad k_\infty \in K_\infty. \end{aligned} \tag{4-8}$$

Using (4-3), we see that $f_\infty(\gamma \gamma_i k_\infty) = 0$ unless $\gamma \gamma_i \in UVK_\infty$. If this is so, then we have

$$f_\infty(\gamma \gamma_i \cdot k_\infty) = \zeta(n(\gamma \gamma_i)) \eta(a(\gamma \gamma_i)) \xi(k(\gamma \gamma_i) k_\infty).$$

We remark that $\gamma\gamma_i \notin K_\infty$ implies that $n(\gamma\gamma_i) \neq 1$ or $a(\gamma\gamma_i) \neq 1$. Now, shrinking the supports of ζ and η, we can accomplish that $\zeta(n(\gamma\gamma_i))\eta(a(\gamma\gamma_i)) = 0$ for all those γ's which are not in K_∞. Now, the sum (4-8) is of the form

$$\begin{aligned} &P(\varphi_{S'} \otimes char_{K^{S'}})(k_\infty, 1) \\ &= \sum_{\delta \in \Gamma_{S'} \cap K_\infty} \xi(\delta k_\infty) \sum_{i=1}^{l} \left(\prod_{v \in S' - V_\infty} f_v(\delta\gamma_i^{-1})\varphi_v(\gamma_i) \right), \quad k_\infty \in K_\infty. \end{aligned} \tag{4-9}$$

The coefficient for the term $\delta = 1$ is given by

$$\sum_{i=1}^{l} \left(\prod_{v \in S' - V_\infty} f_v(\delta\gamma_i^{-1})\varphi_v(\gamma_i) \right) = \prod_{v \in S' - V_\infty} f_v(1)\varphi_v(1) \neq 0.$$

(Indeed, since $f_v(\gamma_i^{-1}) \neq 0$ implies $\gamma_i^{-1} \in L_v$, for $v \neq v_0$, and $\varphi_{v_0}(\gamma_i) \neq 0$ implies $\gamma_i \in U_{v_0}$, we have $\gamma_i \in \Gamma_U$. Hence $i = 1$ by the above selection of the representatives for $\Gamma_U \backslash \Gamma_{S'}$.)

Finally, we note that in (4-9) the sum runs over finitely many δ's independent of $k_\infty \in K_\infty$. Thus, if we shrink W (see (4-3)), then we can accomplish that $\delta W \cap W = \emptyset$ for $\delta \neq 1$. Now, for $k_\infty \in W$, (4-9) implies that

$$P(\varphi_{S'} \otimes char_{K^{S'}})(k_\infty, 1) = \left(\prod_{v \in S - V_\infty} f_v(1)\varphi_v(1) \right) \xi(k_\infty).$$

This shows that $P(\varphi_{S'} \otimes char_{K^{S'}})$ is non–zero when restricted to G_∞. Next, we show that it is cuspidal. The global Hecke algebra $C_c^\infty(G(\mathbb{A}_f))$ acts by convolution on $C_c^\infty(G(k)\backslash G(\mathbb{A}))$ and on the subspace of cuspidal functions as follows:

$$F.\psi(g) = \int_{G(\mathbb{A})} \psi(gh)F(h)dh = \psi \star F^\vee(g), \quad F \in C_c^\infty(G(\mathbb{A}_f)), \psi \in C_c^\infty(G(k)\backslash G(\mathbb{A})),$$

where $F^\vee(g) = F(g^{-1})$. A similar action exists on $C_c^\infty(G(\mathbb{A}_f))$, and the map $\varphi \mapsto P(\varphi)$ is equivariant. Since

$$\prod_{v \in S - V_\infty} vol(L_v) \cdot vol(U_{v_0}) \cdot \Big(\bigotimes_{v \in S' - V_\infty} \varphi_v^\vee \otimes \bigotimes char_{K^{S'}} \Big)$$

maps $\varphi_\infty \otimes \bigotimes_{v \in S - V_\infty} char_{L_v} \otimes \ char_{U_{v_0}} \otimes \ char_{K^{S'}}$ onto

$$\varphi_\infty \otimes \bigotimes_{v \in S' - V_\infty} \varphi_v \otimes \ char_{K^{S'}},$$

it is enough to show that $P(\varphi_\infty \otimes \bigotimes_{v \in S - V_\infty} char_{L_v} \otimes \ char_{U_{v_0}} \otimes \ char_{K^{S'}})$ is cuspidal. But this follows from Lemma 2-9 and (4-7). This proves the lemma for $L_{v_0} = U_{v_0}$.

In general, for $L_{v_0} \subset U_{v_0}$, we define discrete subgroups Γ_{L^S} and Γ_{U^S} of G_S using open compact subgroups $L^S = K^{S'} \times L_{v_0}$ and $U^S = K^{S'} \times U_{v_0}$ of G^S (see (3-2)). Then, using the notation from the first part of the proof, we have the following:

$$P(\varphi_{S'} \otimes char_{K^{S'}})(g_S, 1) = P_{\Gamma_{U^S}}(\varphi_S)(g_S) = \sum_{\gamma \in \Gamma_{L^S} \backslash \Gamma_{U_S}} P_{\Gamma_{L^S}}(\varphi_S)(\gamma g_S), \quad g_S \in G_S.$$

Hence $P_{\Gamma_{L^S}}(\varphi_S)$ is non–zero when restricted to G_∞. Since, by a direct computation, we find

$$P_{\Gamma_{L^S}}(\varphi_S)(g_S) = P(\varphi_S \otimes char_{L^S})(g_S, 1),$$

where we use the decomposition (2-1), we see that $P(\varphi_S \otimes char_{L^S})$ is non–zero when restricted to G_∞. It remains to show the cuspidality of $P(\varphi_S \otimes char_{L^S})$. As above, it is enough to show that $P(\varphi_\infty \otimes \bigotimes_{v\in S-V_\infty} char_{L_v} \otimes char_{L_{v_0}} \otimes char_{K^{S'}})$ is cuspidal. By Lemma 2-9, it is enough to show that $P_{\Gamma_L}(\varphi_\infty)$ is Γ_L–cuspidal, where $L = K^{S'} \times \prod_{v\in S-V_\infty} L_v \times L_{v_0}$. But $\Gamma_L \subset \Gamma_U$, (4-7) and ([12], Lemma 2-9) imply that.

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