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EMBEDDINGS OF k-CONNECTED n-MANIFOLDS INTO $\mathbb{R}^{2n-k-1}$

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Abstract. We obtain estimations for isotopy classes of embeddings of closed k-connected n-manifolds into $\mathbb{R}^{2n-k-1}$ for $n \geq 2k+6$ and $k \geq 0$. This is done in terms of an exact sequence involving the Whitney invariants and an explicitly constructed action of $H_{k+1}(N;\mathbb{Z}_2)$ on the set of embeddings. The proof involves a reduction to the classification of embeddings of a punctured manifold and uses the *parametric connected sum* of embeddings.

Corollary. *Suppose that N is a closed almost parallelizable k-connected n-manifold and $n \geq 2k+6 \geq 8$. Then the set of isotopy classes of embeddings $N \to \mathbb{R}^{2n-k-1}$ is in 1–1 correspondence with $H_{k+2}(N;\mathbb{Z}_2)$ for $n-k = 4s+1$.*

1. Introduction

This paper is on the classical Knotting Problem: *for an n-manifold N and a number m describe the set $\mathrm{E}^m(N)$ of isotopy classes of embeddings $N \to \mathbb{R}^m$.* For recent surveys, see [RS99], [Sk08], [HCEC]; whenever possible we refer to these surveys, not to original papers.

Denote CAT = DIFF (smooth) or PL (piecewise linear). If the category is omitted, then a statement is correct (or a definition is given) for both categories.

By $\mathbb{Z}_{(k)}$ we denote $\mathbb{Z}$ for k even and $\mathbb{Z}_2$ for k odd.

The Haefliger-Zeeman Unknotting Theorem states that *for a closed k-connected orientable n-manifold N, each two embeddings $N \to \mathbb{R}^m$ are isotopic for $m \geq 2n-k+1$ and $n \geq 2k+2$* [Sk08, Theorem 2.8.b].

The classification of embeddings of N into $\mathbb{R}^{2n-k}$ is as follows:[1] *for a closed k-connected orientable n-manifold N, $n \geq 2k+4$ and $k \geq 0$ there is a 1–1 correspondence (defined in Definition 1.3 below) $W_{2n-k} : \mathrm{E}^{2n-k}(N) \to H_{k+1}(N;\mathbb{Z}_{(n-k-1)})$.*

The classification of embeddings of N into $\mathbb{R}^{2n-k-1}$ which was known for $N = S^{k+1} \times S^{n-k-1}$: *$\mathrm{E}^{2n-k-1}(S^{k+1} \times S^{n-k-1})$ is in 1–1 corespondence with*[2]

- $\mathbb{Z} \oplus \mathbb{Z}_2$ for n even and to $\mathbb{Z}_2$ for n odd, provided $k = 0$ and $n \geq 6$;

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[1]This is a classical result of Haefliger-Hirsch (in the smooth category) and Weber-Hudson-Bausum-Vrabec (in the PL category) [Sk08, Theorem 2.13].

[2]See [Sk08, Theorem 3.9, and table before Theorem 3.10], where $KT^m_{p,q} = \mathrm{E}^m(S^p \times S^q)$. This result holds also for $n = 2k+5$ in the PL category. There is an isomorphism not only a 1–1 correspondence, for the 'parametric connected sum' group structure on the set of embeddings defined in [Sk06, §2], [Sk08, §3]; cf. [Sk08, §3.4]. See a description of generators and relations of $\mathrm{E}^{2n-k-1}(S^{k+1} \times S^{n-k-1})$ in [Sk, §3].

• $\mathbb{Z}_4$, 0, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2$ according to $n-k \equiv 0,1,2,3 \mod 4$, provided $n \geq 2k+6 \geq 8$.

Now we state the main result, then describe which parts of it are new. After that we state an open problem and define maps used in the statement.

An n-manifold N is called *p-parallelizable* if each embedding $S^p \to N$ extends to an embedding $S^p \times D^{n-p} \to N$.[3] If the coefficients of a homology group are omitted, then they are $\mathbb{Z}$. For a group G, denote by $G * \mathbb{Z}_2$ the set of elements of order at most 2 in G.

Main Theorem 1.1. (a) *Let N be a closed k-connected n-manifold. Suppose that $k \geq 1$, $n \geq 2k+6$ and N embeds*[4] *into $\mathbb{R}^{2n-k-1}$. For $n-k$ odd, assume that N is $(k+2)$-parallelizable. Then there is an exact sequence of sets*[5] *with an action b:*

$$H_{k+1}(N;\mathbb{Z}_2) \xrightarrow{b} \mathrm{E}^{2n-k-1}(N) \begin{cases} \xrightarrow{W \times W'} H_{k+2}(N) \times H_{k+1}(N;\mathbb{Z}_2) \to 0 & n-k \text{ even}, \\ \xrightarrow{W} H_{k+2}(N;\mathbb{Z}_2) \to 0 & n-k \text{ odd}. \end{cases}$$

(b) *Under the assumptions of* (a) *for $n-k = 4s+1$ there is a* 1–1 *correspondence*

$$W : \mathrm{E}^{2n-k-1}(N) \to H_{k+2}(N;\mathbb{Z}_2).$$

In this paper N is a closed connected n-manifold. Denote $N_0 := N - \operatorname{Int} B^n$, where $B^n \subset N$ is a codimension 0 ball. Consider the coefficient exact sequence

$$H_{k+2}(N) \xrightarrow{2} H_{k+2}(N) \xrightarrow{\rho_2} H_{k+2}(N;\mathbb{Z}_2) \xrightarrow{\beta} H_{k+1}(N) \xrightarrow{2} H_{k+1}(N).$$

Here 2 is the multiplication by 2, ρ_2 is the reduction modulo 2 and β is the Bockstein homomorphism.

Main Theorem 1.1. (c) *Let N be a closed connected orientable n-manifold.*[6] *If n is odd, assume that N is spin and the Hurewicz homomorphism $\pi_2(N) \to H_2(N)$ is epimorphic. For $n \geq 6$ (and for $n = 5$ in the PL category) there is an exact sequence of sets with an action b:*

$$H_1(N;\mathbb{Z}_2) \xrightarrow{b} \mathrm{E}^{2n-1}(N) \xrightarrow{W \times r} \begin{cases} H_2(N) \times \mathrm{E}^{2n-1}(N_0) \to 0 & n \text{ even}, \\ H_2(N;\mathbb{Z}_2) \times \mathrm{E}^{2n-1}(N_0) \xrightarrow{a} H_1(N) & n \text{ odd}. \end{cases}$$

Here r is the restriction-induced map and $a(x,f) := W_0'(f) - \beta(x)$.

In Main Theorem 1.1(c) the right-hand exactness implies that $\operatorname{im}(W \times r)$ is in 1–1 correspondence with $\operatorname{im} \rho_2 \times (2W_0')^{-1}(0)$.

For $N = S^{k+1} \times S^{n-k-1}$ Main Theorem 1.1 is covered by the known result cited before the formulation (this result does not follow from Main Theorem 1.1). Main Theorem 1.1(c) is new. Main Theorem 1.1(a,b) is new for $k = 1$. For $k \geq 2$ the new part of Main Theorem 1.1(a,b) is a direct geometric description of maps b, W, W'; the exact sequences could apparently be obtained using homotopy classification of

[3] Note that 1-parallelizability is equivalent to orientability and 2-parallelizability is equivalent to being a spin manifold. A reader who is bothered by new terms can replace in this paper the p-parallelizability by the almost parallelizability.

[4] The embeddability into $\mathbb{R}^{2n-k-1}$ is equivalent to $\overline{W}_{n-k-1}(N) = 0$, where $\overline{W}_{n-k-1}(N)$ is the normal Stiefel-Whitney class [Sk08, §2, Pr07, 11.3].

[5] The right-hand term can be represented by a formula valid for both odd and even $n-k$: $H_{k+2}(N) \otimes \mathbb{Z}_{(n-k)} \times H_{k+1}(N;\mathbb{Z}_{(n-k-1)}) * \mathbb{Z}_2$. The validity for $n-k$ even is obvious and for $n-k$ odd follows by the Universal Coefficients Formula.

[6] Each orientable n-manifold embeds into $\mathbb{R}^{2n-1}$ [Sk08, Theorem 2.4.a].

maps from an $(n-k-1)$-polyhedron to an $(n-k-2)$-connected space and the following result [BG71, Corollary 1.3]: *if N is a closed k-connected orientable n-manifold embeddable into $\mathbb{R}^m$, $m \geq 2n-2k+1$ and $2m \geq 3n+4$, then there is a 1–1 correspondence $\mathrm{E}^m(N) \to [N_0; V_{M,M+n-m+1}]$, where M is large.* Cf. [Sk, §3.g].

There exists a reduction of the classification of embeddings $N \to \mathbb{R}^{2n-k-1}$ to an equivariant homotopy problem [Sk08, §5]. However, an explicit solution of that problem is hard to obtain. Our result is explicit enough e.g. to yield the following corollary: *under the assumptions of Main Theorem* 1.1(a) *the set $\mathrm{E}^{2n-k-1}(N)$ is infinite if and only if $n-k$ is even and $H_{k+2}(N)$ is infinite.*

Our proof is *not* a generalization of the classical arguments as in [BG71, Corollary 1.3] or [Sk08, §8]. Our proof is direct geometric and is a generalization of the Haefliger-Hirsch-Hudson-Vrabec argument for the proof of the bijectivity of W_{2n-k} [Sk08, Theorem 2.13]. Our classification involves the explicit construction of all embeddings from a given embedding; see Remark 2.10.

A classification of $\mathrm{E}^{2n-1}(N)$ is announced in [Ya83] (it was probably meant for $n \geq 6$). Although no details are available via Google Scholar, in [Ya83] important preliminaries were set. Main Theorem 1.1(c) could be useful because the set $\mathrm{E}^{2n-1}(N_0)$ is apparently easier to describe explicitly than $\mathrm{E}^{2n-1}(N)$ (e.g. using methods of [Ya83] or [Sa99], cf. [Sk, §3, the Deleted Product Lemma]). For $n=4$ cf. [BH70, Sk05, Sk06, CS08].

Open Problem 1.2. (a) *Find the preimages of b.* (*See a discussion in* [Sk, §3.e].)

(b) *Describe the set $\mathrm{E}^{2n-1}(N_0)$; cf.* [Sk, §3, the Deleted Product Lemma].

(c) *For $n \geq 2k+6$ there is a group structure on $\mathrm{E}^{2n-k-1}(N)$ (and for $n \geq 4$ on $\mathrm{E}^{2n-1}(N_0)$) defined via the Haefliger-Wu α-invariant* [Sk08, §5]. *Are the maps from Main Theorem* 1.1 *homomorphisms? If yes, solve the extension problem.*

Definitions 1.3 of the Whitney invariants W, W_{2n-k}, W' and W'_0. We present definitions for $n \geq 2k+5$, N orientable and in the smooth category. The definition in the PL category is analogous [Sk08, §2.4].[7] Fix orientations on N and on $\mathbb{R}^{2n-k-1}$. Take embeddings $f, f_0 : N \to \mathbb{R}^{2n-k-1}$.

The *self-intersection set* of a map $H\colon X \to Y$ is $\Sigma(H) = \{x \in X \mid \#H^{-1}Hx \geq 1\}$.

Take a general position homotopy $H : N \times I \to \mathbb{R}^{2n-k-1} \times I$ between f_0 and f. Since $n \geq 2k+5$, by general position, $\Sigma(H)$ is a $(k+2)$-submanifold (not necessarily compact). The closure $\mathrm{Cl}\,\Sigma(H)$ is a closed $(k+2)$-submanifold. For $n-k$ is even it has a natural orientation.[8] Define the Whitney invariant

$$W : \mathrm{E}^{2n-k-1}(N) \to H_{k+2}(N \times I; \mathbb{Z}_{(n-k)}) \cong H_{k+2}(N; \mathbb{Z}_{(n-k)})$$

[7] In Main Theorem 1.1(a,b), $k \geq 1$, so N is orientable. For an equivalent definition see the Difference Lemma 2.3 below or [Sk081, §1].

[8] *Definition of the orientation* is analogous to [Sk08, §2.3, p. 263]. Take smooth triangulations T and T' of the domain and the range of H such that H is simplicial. Then $\mathrm{Cl}\,\Sigma(H)$ is a subcomplex of T. Take any oriented simplex $\sigma \subset \mathrm{Cl}\,\Sigma(H)$. Let us show how to decide whether the orientation of σ is right or to be changed. By general position there is a unique simplex τ of T such that $f\sigma = f\tau$. The orientation on σ induces an orientation on $f\sigma$ and then on τ. The orientations on σ and τ induce orientations on normal spaces in $N \times I$ to these simplices. These two orientations (in this order) together with the orientation on $f\sigma$ induce an orientation on $\mathbb{R}^{2n-k-1} \times I$. If this orientation agrees with the fixed orientation of $\mathbb{R}^{2n-k-1} \times I$, then the orientation of σ is right, otherwise it should be changed. Since $n-k$ is even, these orientations agree for adjacent simplices [Hu69, Lemma 11.4]. So they define an orientation of $\mathrm{Cl}\,\Sigma(H)$.

by $W(f) = W_{f_0}(f) := [\operatorname{Cl}\Sigma(H)]$. Analogously to [Sk08, §2.4], this is well-defined.

The Whitney invariant $W_{2n-k} : \mathrm{E}^{2n-k}(N) \to H_{k+1}(N; \mathbb{Z}_{(n-k-1)})$ is defined analogously to the above. The Whitney invariant $W' : \mathrm{E}^{2n-k-1}(N) \to H_{k+1}(N; \mathbb{Z}_{(n-k-1)})$ is defined as the composition of W_{2n-k} and the map $\mathrm{E}^{2n-k-1}(N) \to \mathrm{E}^{2n-k}(N)$ induced by the inclusion $\mathbb{R}^{2n-k-1} \to \mathbb{R}^{2n-k}$.

By [Vr89, Theorem 3.1] the map W' equals (up to sign for $n-k$ odd) the composition

$$\mathrm{E}^{2n-k-1}(N) \xrightarrow{r} \mathrm{E}^{2n-k-1}(N_0) \xrightarrow{W'_0} H_{k+1}(N; \mathbb{Z}_{(n-k-1)}).$$

Here r is the restriction map and W'_0 is defined as follows.

The *singular set* of a smooth map $H : X \to Y$ between manifolds is $S(H) := \{x \in X \ : \ d_x H \text{ is degenerate}\}$.

Take a general position homotopy $H : N \times I \to \mathbb{R}^{2n-k-1} \times I$ between f_0 and f. Since $n \geq 2k+3$, by general position, $\operatorname{Cl} S(H)$ is a closed $(k+1)$-submanifold. For $n-k$ odd it has a natural orientation.[9] Define $W'_0(f) = W'_{0,f_0}(f) := [S(H)]$. It is well-known that $W'_0(f)$ is indeed independent of H (for fixed f and f_0).[10]

For an embedding $f : N \to \mathbb{R}^{2n-k-1} \subset S^{2n-k-1}$ of a closed connected n-manifold N denote

- $C = C_f := S^{2n-k-1} - f \operatorname{Int} N_0$,
- by $AD = AD_{f,i} : H_i(N) \to H_{i+n-k-2}(C)$ the composition of Alexander and Poincaré isomorphisms,
- by $h = h_{f,i} : \pi_i(C) \to H_i(C)$ the Hurewicz homomorphism.

Definition 1.4 of the action $b = b_N$.[11] We give a definition for $n \geq 2k+6$ and $H_k(N) = 0$. Take an embedding $f : N \to \mathbb{R}^{2n-k-1}$ and $x \in H_{k+1}(N; \mathbb{Z}_2)$. Since $H_k(N) = 0$, there is $\overline{x} \in H_{k+1}(N)$ such that $\rho_2 \overline{x} = x$. By general position and Alexander duality, C is $(n-2)$-connected. Hence h_{n-1} is an isomorphism. Consider the composition $S^n \xrightarrow{\Sigma^{n-3}\eta} S^{n-1} \xrightarrow{h_{n-1}^{-1} AD\overline{x}} C$, where $\eta : S^3 \to S^2$ is the Hopf map.[12]

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[9] *Definition of the orientation.* Recall the notation from the previous footnote. Take a $(k+1)$-simplex $\alpha \subset S(H)$. By general position there are $(k+2)$-simplices $\sigma, \tau \subset \operatorname{Cl}\Sigma(H)$ such that $f\sigma = f\tau$ and $\sigma \cap \tau = \alpha$. Define the 'right' orientation of α to be the orientation induced by the 'right' orientation of σ. This is well-defined because the 'right' orientation of τ induces the same orientation of α. (Indeed, since $n-k$ is odd, normal spaces of σ and of τ in $N \times I$ are even-dimensional, so the 'right' orientations on σ and on τ induce the same orientation on $f\sigma$.)

[10] We use $W'_0 r$, not W' in the proof. Although we do not need this, note that $W'_0 r$ is a regular homotopy invariant; if $k = 0$ and n is even, then W'_0 factors through $H_{k+1}(N)$; for $n-k$ odd, $2W'_0(f)$ equals the normal Euler class e of f because $e = AD^{-1}[f(\partial N_0)] = 2W'_0(f)$ [Vr89, Addendum 2.2].

[11] A reader who is not interested in explicit constructions can omit this definition and set $b(x)f := \psi_f b'(x)$, where b' and ψ_f are defined in §2.

[12] Since N is k-connected and $n \geq 2k+3$, we can represent $\overline{x}$ by an embedding $x' : S^{k+1} \to N$. If the restriction to $x'(S^{k+1})$ of the normal bundle $\nu_f : \partial C \to N$ is trivial, then the spheroid $h_{n-1}^{-1} AD\overline{x}$ can be constructed directly as follows; cf. [Sk081, end of §1]. Identify $X := \nu_f^{-1} x'(S^{k+1})$ with $S^{k+1} \times S^{n-k-2}$. Let us show how to make an embedded surgery of $S^{k+1} \times * \subset X$ to obtain an $(n-1)$-sphere $S^{n-1} \cong \Sigma \subset C$ whose inclusion into C represents $h_{n-1}^{-1} AD\overline{x}$. Take a vector field on $S^{k+1} \times *$ normal to X in $\mathbb{R}^{2n-k-1}$. Extend $S^{k+1} \times *$ along this vector field to a smooth map $\widetilde{x} : D^{k+2} \to S^{2n-k-1}$. Since $2n-k-1 > 2k+4$ and $n+k+2 < 2n-k-1$, by general position we may assume that $\widetilde{x}$ is a smooth embedding and $\widetilde{x}(\operatorname{Int} D^{k+2})$ misses $f(N) \cup X$. Denote $l := 2n-2k-3$. Since $n-k-1 > k+1$, we have $\pi_{k+1}(V_{l,n-k-2}) = 0$. Hence the standard framing of $S^{k+1} \times *$ in X extends to an l-framing on $\widetilde{x}(D^{k+2})$ in $\mathbb{R}^{2n-k-1}$. Thus $\widetilde{x}$ extends to

The connected sum in C of this composition with $f|_{B^n}$ is homotopic (relative to the boundary) to an embedding $x'' : B^n \to C$ by Theorem 2.5. Define $b(x)f$ to be f on N_0 and x'' on B^n.

This is well-defined (i.e. is independent of the choices of $\overline{x}$ and of x'') for $n \geq 2k+6$ and is an action by the equivalent definition given after the Construction 2.6 of ψ below.

2. Proof of Main Theorem 1.1

Main tools. The proof is based on the construction and application of the following commutative diagram:

$$\begin{tikzcd}
H_{k+1}(N;\mathbb{Z}_2) \arrow[r, "b"'] \arrow[d, "b'"] & & \\
\pi_n(C) \arrow[r, "\psi_f"'] \arrow[d, "AD^{-1}\circ h"] & r^{-1}r(f) \subset E^{2n-k-1}(N) \arrow[r, "r"'] \arrow[d, "W"] & E^{2n-k-1}(N_0) \arrow[d, "W_0'"] \\
H_{k+2}(N) \arrow[r, "\rho_{(n-k)}"'] \arrow[d] & H_{k+2}(N;\mathbb{Z}_{(n-k)}) \arrow[r, "\beta"'] & H_{k+1}(N;\mathbb{Z}_{(n-k-1)}) \\
0 & &
\end{tikzcd}$$

Here

- N is a closed homologically k-connected orientable n-manifold, $f : N \to \mathbb{R}^{2n-k-1}$ is an embedding, $n \geq 2k+6$ and $N_0 := N - \operatorname{Int} B^n$, where $B^n \subset N$ is a codimension 0 ball,
- C, AD, h are defined at the end of §1,
- W and W_0' are defined above in Definitions 1.3 of the Whitney invariants,
- r is the restriction-induced map,
- β is the Bockstein homomorphism defined only for $n-k$ odd,
- ψ_f is defined below in Construction 2.6 of ψ,
- $\rho_{(n-k)}$ is the identity for $n-k$ even and the reduction modulo 2 for $n-k$ odd,
- b is defined in §1 (and can be alternatively defined as $b := \psi_f b'$),
- b' is the composition

$$H_{k+1}(N;\mathbb{Z}_2) \overset{\cong}{\to} H_{n-1}(C;\mathbb{Z}_2) \overset{\cong}{\to} H_{n-1}(C) \otimes \mathbb{Z}_2 \overset{\cong}{\to} \pi_{n-1}(C) \otimes \pi_n(S^{n-1}) \to \pi_n(C)$$

of the Alexander duality, the coefficient isomorphism, tensor product of the Hurewicz and the Pontryagin isomorphisms, and the composition map.[13]

The proof of Main Theorem 1.1 in the next subsection shows how to apply this diagram. That proof uses statements of lemmas below, not their proofs.

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an embedding $\widehat{x} : D^{k+2} \times D^l \to C$ such that $\widehat{x}(\partial D^{k+2} \times D^l) \subset X$. Let

$$\Sigma := (X - \widehat{x}(\partial D^{k+2} \times \operatorname{Int} D^l)) \bigcup_{\widehat{x}(\partial D^{k+2} \times \partial D^l)} \widehat{x}(D^{k+2} \times \partial D^l) \cong S^{n-1}.$$

[13]The composition $\pi_{n-1}(C) \times \pi_n(S^{n-1}) \to \pi_n(C)$ is clearly linear in $\pi_n(S^{n-1})$; for $n \geq 4$ the composition is linear in $\pi_{n-1}(C)$ by [Po85, Lecture 4, Corollary on p. 167]. So the latter composition map is indeed well-defined.

Complement Lemma 2.1. *Let N be a closed k-connected orientable n-manifold, $n \geq 4$ and $f : N \to \mathbb{R}^{2n-k-1}$ an embedding. Then the left column of the above diagram is exact.*

Proof. By general position and Alexander duality, C is $(n-2)$-connected. Since $n \geq 4$, by [Wh50] there is an exact sequence forming the first line of the following diagram:

$$\begin{array}{ccccccc} H_{n-1}(C;\mathbb{Z}_2) & \xrightarrow[b'\circ AD^{-1}]{} & \pi_n(C) & \xrightarrow[h]{} & H_n(C) & \longrightarrow & 0 \\ \uparrow{\scriptstyle AD\cong} & & & & \uparrow{\scriptstyle AD\cong} & & \\ H_{k+1}(N;\mathbb{Z}_2) & & & & H_{k+2}(N) & & \end{array}$$

Now the lemma follows by Alexander duality. □

The Whitney Invariant Lemma 2.2. *Let N be a closed k-connected orientable n-manifold embeddable into $\mathbb{R}^{2n-k-1}$.*

(W_0') The map W_0' is a 1–1 correspondence for $k \geq 1$;

the map W_0' is surjective for $k = 0$ and n even;

$\operatorname{im} W_0' \supset H_1(N) * \mathbb{Z}_2$ for $k = 0$ and n odd.

(r) Assume that $n \geq 2k+6$. The restriction-induced map $r : \mathrm{E}^{2n-k-1}(N) \to \mathrm{E}^{2n-k-1}(N_0)$ is surjective for $n-k$ even and $\operatorname{im} r = \ker(2W_0')$ for $n-k$ odd.

(β)[14] For $n-k$ odd, $W_0' r = \beta W$.

Part (W_0') for $k = 0$ follows by [Ya83, Main Theorem (i) and (iii)]. Part (W_0') for $k \geq 1$ and part (r) are proved in the PL category in [Vr89, Theorem 2.1, Theorem 2.4 and Corollary 3.2] and in the smooth category in [Ri70]. For the reader's convenience, the proofs of (W_0') and (r) are sketched below.

Sketch of the proof of (W_0'). Let Y be the set of regular homotopy classes of immersions $N_0 \to \mathbb{R}^{2n-k-1}$. Since N is k-connected, N_0 collapses to an $(n-k-1)$-polyhedron. So by general position the forgetful map $\mathrm{E}^{2n-k-1}(N_0) \to Y$ is surjective and, for $k \geq 1$, injective (see details e.g. in [Vr89, proof of Theorem 2.1] on p. 167). The map W_0' is a composition of the forgetful map and a certain map $Y \to H_{k+1}(N;\mathbb{Z}_{(n-k-1)})$ that is a 1–1 correspondence for $k \geq 1$ by the Smale-Hirsch (in the smooth category) or the Haefliger-Poenaru (in the PL category) classification of immersions.

Now assume that $k = 0$. Then $\operatorname{im} W_0' \supset \operatorname{im} W'$. By [Ya83, Main Theorem (i) and (iii)] W' is surjective for n even and $\operatorname{im} W' = \operatorname{im} \beta = H_1(N) * \mathbb{Z}_2$ for n odd. This implies the required result on $\operatorname{im} W_0'$.[15] □

Sketch of the proof of (r). Let $f : N_0 \to \mathbb{R}^{2n-k-1}$ be an embedding. If $n-k$ is even, then the *homology* class of $f(\partial N_0)$ in $H_{n-1}(C)$ is trivial [Vr89, proof of Theorem 2.4 and Addendum 2.2]. By general position and Alexander duality, C is $(n-2)$-connected. Hence h_{n-1} is an isomorphism. Therefore the *homotopy* class of

[14] This is analogous to the well-known relations $w_{2j+1} = \mathrm{Sq}^1 w_{2j}$ and $W_{2j+1} = \beta w_{2j}$ for the Stiefel-Whitney classes [Pr07, 11.3].

[15] For $k = 0$ the map W_0' is not injective. For $k = 0$ and n odd, $\operatorname{im} W_0'$ can be larger than $H_1(N) * \mathbb{Z}_2$ (because not all embeddings $D^1 \times S^{n-1} \to \mathbb{R}^{2n-1}$ extend to embeddings $S^1 \times S^{2n-1} \to \mathbb{R}^{2n-1}$). The result [Vr89, Corollary 3.1] holds for $k = 0$ and orientable N [Ya83, Main Theorem (2.i,iii)].

$f(\partial N_0)$ in $\pi_{n-1}(C)$ is trivial. Then by Theorem 2.5(a) f extends to an embedding $N \to \mathbb{R}^{2n-k-1}$. Thus r is surjective.

If $n-k$ is odd, then the homology class of $f(\partial N_0)$ equals $2AD(W_0'(f))$ [Vr89, proof of Theorem 2.4 and Addendum 2.2]. Thus $\operatorname{im} r = \ker(2W_0')$ analogously to the case when $n-k$ is even.[16] □

Proof of (β). Take a general position homotopy H between f_0 and f. Recall that in the Definition 1.3 of the Whitney invariants (including footnotes) we defined integer $(k+2)$- and $(k+1)$-chains $[\operatorname{Cl}\Sigma(H)]$ and $[\operatorname{Cl} S(H)]$ in $N \times I$ (simplicial chains in a certain smooth triangulation). The assumption that $n-k$ is even was only used to show that $\partial[\operatorname{Cl}\Sigma(H)] = 0$; the assumption that $n-k$ is odd was used to define $[\operatorname{Cl} S(H)]$.

Take two $(k+2)$-simplices $\sigma, \tau \subset \operatorname{Cl}\Sigma(H)$ intersecting by a $(k+1)$-simplex α. Clearly, for $\operatorname{Int}\alpha \subset \Sigma(H)$ (the 'right' orientations of σ and τ agree and) α appears in $\partial\sigma$ and in $\partial\tau$ with the opposite signs. Since $n-k$ is odd, for $\alpha \subset \operatorname{Cl} S(H)$ (the 'right' orientations of σ and τ disagree and) α appears in $\partial\sigma$ and in $\partial\tau$ with the same sign. This and $\operatorname{Cl}\Sigma(H) = \Sigma(H) \cup S(H)$ imply that $\partial[\operatorname{Cl}\Sigma(H)] = 2[S(H)]$ for $n-k$ odd.

Then $\beta W(f) = [S(H)] = W_0' r(f)$. Here the first equality holds by definition of β, and the second equality holds by definition of W_0'. □

Difference Lemma 2.3. *Let N be a closed connected orientable n-manifold and $f, f' : N \to \mathbb{R}^{2n-k-1}$ embeddings coinciding on N_0. Then*[17]

$$W(f) - W(f') = \rho_{(n-k)} d(f', f), \quad \textit{where}$$
$$d(f', f) := AD_f^{-1} h_{f,n}[f'|_{B^n} \cup f|_{\overline{B}^n}] \in H_{k+2}(N).$$

Proof. Take a map $F : B^{n+1} \to \mathbb{R}^{2n-k-1}$ in general position with $f(N_0)$ and such that $F|_{\partial B^{n+1}} = f'|_{B^n} \cup f|_{\overline{B}^n}$. By Alexander duality $d(f, f')$ is the homology class carried by $f^{-1}F(\operatorname{Int} B^{n+1})$. There is a general position homotopy H between f and f' such that $\operatorname{pr}_N \operatorname{Cl}\Sigma(H) = f^{-1}F(\operatorname{Int} B^{n+1})$. For $n-k$ even observe that in this formula the signs of corrresponding simplices (in a certain smooth or PL triangulation of N) are the same. So the lemma follows. □

Construction Lemma 2.4. *Let N be a closed homologically k-connected orientable n-manifold, $f : N \to \mathbb{R}^{2n-k-1}$ an embedding and $n \geq 2k+6$. Then there is a map $\psi = \psi_f : \pi_n(C) \to r^{-1}r(f)$ such that*

(a) *$d(\psi(y), f) = AD^{-1}h_n(y)$ for each $y \in \pi_n(C)$.*
(b) *$W(\psi(y)) - W(f) = \rho_{(n-k)} AD^{-1} h_n(y)$ for each $y \in \pi_n(C)$.*
(c) *If $f = f'$ on N_0, then f' is isotopic to $\psi[f'|_{B^n} \cup f|_{\overline{B}^n}]$ relative to N_0.*
(d) *ψ is surjective.*
(e) *ψ defines an action.*

Theorem 2.5 [RS99, Theorem 3.2]**.** *Let N and M be n- and m-manifolds with boundary. Assume that $2m \geq 3n+4$.*

---

[16] For $n-k$ odd the inclusion $\operatorname{im} r \subset \ker(2W_0')$ of part (r) also follows by part (β) or by an analogue of the Boechat-Haefliger Lemma [Sk081, §2].

[17] In this formula $\overline{B}^n$ is B^n with reversed orientation; we have $C_f = C_{f'}$. The element $d(f', f)$ is an invariant of an isotopy (of f and f') relative to N_0.

(a) *If N is $(2n-m)$-connected and M is $(2n-m+1)$-connected, then any proper map $N \to M$ whose restriction to the boundary ∂N is an embedding is homotopic (relative to the boundary ∂N) to an embedding.*

(b) *If N is $(2n-m+1)$-connected and M is $(2n-m+2)$-connected, then any proper homotopy $N \times I \to M \times I$ fixed on the boundary ∂N is homotopic (relative to $\partial(N \times I)$) to an isotopy.*

Construction 2.6 of ψ (analogous to [Sk081], proof of the surjectivity of W in §5)**.** *Take $x \in \pi_n(C)$ represented by a map $x' : S^n \to C$. The connected sum $x' \# f|_{B^n}$ in C of x' with $f|_{B^n}$ is homotopic* rel ∂B^n *to a proper embedding $x'' : B^n \to C$ coinciding with f on ∂B^n, and x'' is uniquely defined by x up to isotopy* rel ∂B^n.[18] *Define $\psi(x)$ to be f on N_0 and x'' on B^n.*

For $x \in H_{k+1}(N; \mathbb{Z}_2)$ define $b(x)f := \psi_f b'(x)$. (Recall that b' is defined in the Complement Lemma 2.1*; this is clearly equivalent to the definition given in* §1*.)*

Proof of the Construction Lemma 2.4. Since $f = \psi(x)$ on N_0, we have $\psi(x) \in r^{-1}r(f)$.

Part (a) holds because $y = [\psi(y)|_{B^n} \cup f|_{\overline{B}^n}]$.

Part (a) and the Difference Lemma 2.3 imply (b).

Part (c) follows analogously to the uniqueness of x'' in the Construction 2.6 of ψ.

If $r(f_1) = r(f)$ for an embedding $f_1 : N \to \mathbb{R}^{2n-k-1}$, then f_1 is isotopic to an embedding f' such that $f = f'$ on N_0. Then by (c) $\psi[f'|_{B^n} \cup f|_{\overline{B}^n}]$ is isotopic rel N_0 to f' and hence to f_1. This implies (d).

Let us prove part (e). Take $x, y, x+y \in \pi_n(C)$ represented by maps $x', y', (x+y)' : S^n \to C$. We have that $x' \# (y' \# f|_{B^n})$ is homotopic rel ∂B^n to $(x' + y') \# f|_{B^n}$. Hence $\psi_f(x+y)$ is isotopic rel N_0 to $\psi_{\psi_f(y)}(x)$ analogously to the uniqueness of x'' in the Construction 2.6 of ψ. □

Proof of Main Theorem 1.1(a,c).

Proof of Main Theorem 1.1(a) *for $n-k$ even.* The map $W' = W'_0 r$ is surjective by the Whitney Invariant Lemma 2.2(r),(W'_0). Since $\rho_{(n-k)} = \mathrm{id}$ and h_n is epimorphic, by the Construction Lemma 2.4(b,d), $W \times W'$ is surjective.

By the Complement Lemma 2.1, $h_n b' = 0$. Hence by the Difference Lemma 2.3, $W(f) = W(\psi_f b'(x))$. Since r is a factor of W' and $r(f) = r(\psi_f b'(x))$, we have $W'(f) = W'(\psi_f b'(x))$.

Suppose that $W(f) = W(g)$ and $W'(f) = W'(g)$. Then $r(f) = r(g)$ by the Whitney Invariant Lemma 2.2 (W'_0) because $W' = W'_0 r$. Thus $g = \psi_f(y)$ for some $y \in \pi_n(C)$ by the Construction Lemma 2.4(d). By the Construction Lemma 2.4(b), $h_n(y) = 0$. Hence by the Complement Lemma 2.1, $y = b'(x)$ for some $x \in H_{k+1}(N; \mathbb{Z}_2)$. So $g = \psi_f b'(x) = b(x)f$. □

Proof of Main Theorem 1.1(c) *for n even.* By the Whitney Invariant Lemma 2.2(r) and the Construction Lemma 2.4(b,d) the map $r \times W$ is surjective.

Clearly, $r(f) = r(\psi_f b'(x))$. Analogously to the previous proof,

- $W(f) = W(\psi_f b'(x))$;
- if $W(f) = W(g)$ and $r(f) = r(g)$, then $g = \psi_f b'(x)$ for some $x \in H_1(N; \mathbb{Z}_2)$.

□

[18] This follows by Theorem 2.5 because by general position and Alexander duality C is $(n-2)$-connected, $2n - (2n-k-1) + 2 \le n-2$ and $2(2n-k-1) \ge 3n+4$.

Now we turn to the case when $n-k$ is odd. The proof of the following result is postponed.

Twisting Lemma 2.7. *Suppose that $n-k$ is odd, N is a closed connected $(k+2)$-parallelizable n-manifold and $n \geq 2k+6$. Assume that the Hurewicz homomorphism $\pi_{k+2}(N) \to H_{k+2}(N)$ is epimorphic (for $k \geq 1$ this follows from the k-connectedness). Then for each $x \in H_{k+2}(N)$ every embedding $f : N \to \mathbb{R}^{2n-k-1}$ is isotopic to an embedding $f' : N \to \mathbb{R}^{2n-k-1}$ such that $f = f'$ on N_0 and $d(f', f) = 2x \in H_{k+2}(N)$.*

Proof of Main Theorem 1.1(a) *for $n-k$ odd.* By the Whitney Invariant Lemma 2.2(r),(W_0'),(β) we have $\operatorname{im}(\beta W) = \operatorname{im}(W_0' r) = H_{k+1}(N) * \mathbb{Z}_2 = \operatorname{im}\beta$. Since h_n is surjective, by the Construction Lemma 2.4(b,d), $W(r^{-1}r(f)) = W(f) + \operatorname{im}\rho_2 = W(f) + \ker\beta$. Thus W is surjective.

Analogously to the case of $n-k$ even $W(f) = W(\psi_f b'(x))$ for each x. If $W(f) = W(g)$, then $W_0' r(f) = W_0' r(g)$ by the Whitney Invariant Lemma 2.2(β). Hence by the Whitney Invariant Lemma 2.2(W_0') we have $r(f) = r(g)$, i.e. g is isotopic to an embedding g_1 such that $g_1 = f$ on N_0. Since $W(g_1) = W(g) = W(f)$, by the Difference Lemma 2.3, $d(g_1, f)$ is even. Hence by the Twisting Lemma 2.7, g_1 is isotopic to an embedding g_2 such that $g_2 = f$ on N_0 and $d(g_2, f) = 0$. By the Construction Lemma 2.4(d) there is $y \in \pi_n(C)$ such that g_2 is isotopic to $\psi_f(y)$ relative to N_0. By the Construction Lemma 2.4(a), $AD^{-1}h_n(y) = d(\psi_f(y), f) = d(g_2, f) = 0$. Hence by the Complement Lemma 2.1, $y = b'z$ for some $z \in H_{k+1}(N; \mathbb{Z}_2)$. Thus g is isotopic to $\psi_f b'(z) = b(z)f$. □

Proof of Main Theorem 1.1(c) *for n odd.* If $W(f) = W(g)$ and $r(f) = r(g)$, then analogously to the proof of (a) for $n-k$ odd, $g = b(z)f$ for some $z \in H_1(N; \mathbb{Z}_2)$.

By the Whitney Invariant Lemma 2.2(β) we have $W_0' r = \beta W$, so $\operatorname{im}(W \times r) \subset \ker a$.

Let us prove that $\operatorname{im}(W \times r) \supset \ker a$. Take $x \in H_2(N; \mathbb{Z}_2)$ and $f : N_0 \to \mathbb{R}^{2n-1}$ such that $\beta(x) = W_0'(f)$. Then $2W_0'(f) = 2\beta(x) = 0$. Hence by the Whitney Invariant Lemma 2.2(r), f extends to an embedding $f_1 : N \to \mathbb{R}^{2n-1}$. By the Whitney Invariant Lemma 2.2(β) we have $\beta W(f_1) = W_0'(f) = \beta(x)$. Hence $W(f_1) - x = \rho_2 y'$ for some $y' \in H_2(N)$. Since $h_{f_1,n}$ is surjective, there is $y \in \pi_n(C_{f_1})$ such that $AD_{f_1} h_{f_1,n}(y) = y'$. Then by the Construction Lemma 2.4(b),

$$\psi_{f_1}(-y) = f_1 = f \quad \text{on} \quad N_0 \quad \text{and} \quad W(\psi_{f_1}(-y)) = W(f_1) - \rho_2 y' = x. \qquad \square$$

Parametric connected sum of embeddings. In this subsection we recall, with only minor modifications, some results of [Sk07], [PCS].

Denote $D^k_\pm := \{(x_0, x_1, \dots, x_k) \in S^k \mid \pm x_0 \geq 0\}$. Identify D^p with D^p_+ and S^p with $D^p_+ \bigcup\limits_{\partial D^p_+ = \partial D^p_-} D^p_-$.

For $m \geq n+2$ denote by $t^m_{p,n-p}$ the CAT *standard embedding* that is the composition

$$S^p \times S^{n-p} \to \mathbb{R}^{p+1} \times \mathbb{R}^{n-p+1} \to \mathbb{R}^m \to S^m$$

of CAT standard embeddings.

Take an embedding $s : S^p \times D^{n-p} \to N$. A map $f : N \to S^m$ is called *s-standardized* if

- $f(N - \operatorname{im} s) \subset \operatorname{Int} D^m_+$ and
- $f \circ s : S^p \times D^{n-p} \to D^m_-$ is the restriction of the standard embedding.

A map $F : N \times I \to S^m \times I$ such that $F|_{N\times j} : N \times j \to S^m \times j$ is s-standardized (for $j = 0, 1$) is called *s-standardized* if

• $F((N - \operatorname{im} s) \times I) \subset \operatorname{Int} D^m_+ \times I$ and

• $F \circ (s \times \operatorname{id} I) : S^p \times D^{n-p} \times \{t\} \to D^m_- \times \{t\}$ is the restriction of the standard embedding for each $t \in I$.

Standardization Lemma 2.8 ([Sk07, Standardization Lemma])**.** *Let N be an n-manifold N and $s : S^p \times D^{n-p} \to N$ an embedding. For $m \geq n + p + 3$,*

• *each embedding $N \to S^m$ is isotopic to an s-standardized embedding, and*

• *each concordance between s-standardized embeddings is isotopic relative to the ends to an s-standardized concordance.*

Denote $T^{p,n-p} := S^p \times S^{n-p}$. Let $i : S^p \times D^{n-p} \to T^{p,n-p}$ be the standard inclusion. Recall that $N_0 = N - \operatorname{Int} B^n$, where $B^n \subset N$ is a codimension 0 ball.

Summation Lemma 2.9. *Assume that $m \geq n + p + 3$, N is a closed connected n-manifold and $f : N \to S^m$, $g : T^{p,n-p} \to S^m$ are embeddings.*

(a) By the Standardization Lemma 2.8 we can make concordances and assume that f and g are s-standardized and i-standardized, respectively. Then an embedding

$$f\#_s g : N \to S^m \text{ is well-defined by } (f\#_s g)(a) = \begin{cases} f(a) & a \notin \operatorname{im} s, \\ R_m g(x, R_{n-p} y) & a = s(x, y), \end{cases}$$

where R_k is the symmetry of S^k with respect to the hyperplane $x_1 = x_2 = 0$.

(b) If $g = t^m_{p,n-p}$ on $(T^{p,n-p})_0$, then $f\#_s g = f$ on N_0.

(c) If $m = 2n - p + 1$ and $g = t^m_{p,n-p}$ on $(T^{p,n-p})_0$, then $d(f\#_s g, f) = d(g, t^m_{p,n-p})[s|_{S^p\times 0}] \in H_p(N)$, where $p < n/2$ and $d(g, t^m_{p,n-p}) \in H_p(T^{p,n-p})$ is considered as an integer.

Proof. The argument for (a) is easy and similar to [Sk06, Sk07]. In order to prove that $f\#_s g$ is well-defined we need to show that the concordance class of $f\#_s g$ depends only on concordance classes of f and g but not on the chosen standardizations of f and g. Take concordances

$$F : N \times I \to S^m \times I \quad \text{and} \quad G : T^{p,n-p} \times I \to S^m \times I$$

between different standardizations of f and of g. By the 'concordance' part of the Standardization Lemma 2.8 we can take concordances relative to the ends and assume that F and G are s-standardized and i-standardized, respectively. Define a concordance

$$F\#_s G : N \times I \to S^m \times I \quad \text{by} \quad (F\#_s G)(a, t) = \begin{cases} F(a, t) & a \notin \operatorname{im} s, \\ R_m G(x, R_{n-p} y, t) & a = s(x, y). \end{cases}$$

If F is a concordance from f_0 to f_1 and G is a concordance from g_0 to g_1, then $F\#_s G$ is a concordance from $f_0\#_s g_0$ to $f_1\#_s g_1$.

Parts (b) and (c) are clear. □

Applications of the parametric connected summation.

Proof of the Twisting Lemma 2.7. Denote $t = t^{2n-k-1}_{k+2,n-k-2}$ and $t_1 := \psi_t(h^{-1}_{t,n} A D_t(2))$ for the generator $1 \in H_{k+2}(T^{k+2,n-k-2}) \cong \mathbb{Z}$ (the map $h_{t,n}$ is an isomorphism). Since $n - k$ is odd, by the Construction Lemma 2.4(b),

$$W(t_1) - W(t) = \rho_{(1)}(2) = 0 \in H_{k+2}(T^{k+2,n-k-2}; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Hence t_1 is isotopic to t by the bijectivity of W_{2n-k} [Sk08, Theorem 2.13].

Since the Hurewicz homomorphism $\pi_{k+2}(N) \to H_{k+2}(N)$ is epimorphic and $n \geq 2k+5$, by general position there is an embedding $S^{k+2} \to N$ realizing x. Since N is $(k+2)$-parallelizable, this embedding extends to an embedding $\overline{x} : S^{k+2} \times D^{n-k-2} \to N$.

Since $2n-k-1 \geq n+k+2+3$, embeddings $f = f\#_{\overline{x}}t$ and $f' := f\#_{\overline{x}}t_1$ are well-defined and are isotopic by the Summation Lemma 2.9(a). We have $d(f\#_{\overline{x}}t_1, f) = d(t_1, t)x = 2x$ by the Construction Lemma 2.4(a) and the Summation Lemma 2.9(c). □

Proof of Main Theorem 1.1(b). By Main Theorem 1.1(a) it remains to prove that $b_N = 0$ for $n-k = 4s+1$.

Since N is k-connected, the composition $\pi_{k+1}(N) \to H_{k+1}(N) \xrightarrow{\rho_2} H_{k+1}(N;\mathbb{Z}_2)$ of the Hurewicz isomorphism and the reduction modulo 2 is an epimorphism. Hence by Theorem 2.5(a) for each $x \in H_{k+1}(N;\mathbb{Z}_2)$ there is an embedding $S^{k+1} \to N$ realizing x (because $n \geq 2k+3$). Since N is $(k+1)$-parallelizable, this embedding extends to an embedding $\overline{x} : S^{k+1} \times D^{n-k-1} \to N$.

Denote

$$T := S^{k+1} \times S^{n-k-1}, \quad t := t^{2n-k-1}_{k+1,n-k-1} \quad \text{and} \quad \gamma := b_T(1)t,$$

where $1 \in H_{k+1}(T;\mathbb{Z}_2)$ is the generator. Since $n-k \equiv 1 \mod 4$, by [Sk08, Theorem 3.9 and tables] γ is isotopic to t. Therefore

$$b_N = 0 \quad \text{because} \quad b_N(x)f = f\#_{\overline{x}}\gamma = f\#_{\overline{x}}t = f.$$

Here the parametric connected sums are well-defined because $2n-k-1 \geq n+k+1+3$. In order to prove the first equality we assume in the construction of γ that $S^n, S^{n-1} \subset \mathbb{R}^{2n-k-1}_+$. Then we may assume that γ is standardized. So $f\#_{\overline{x}}\gamma$ is obtained from f by linked connected summation along $\overline{x}$ with a composition $S^n \to S^{n-1} \times D^{n-k} \to S^{2n-k-1} - f(N)$ of two embeddings, the one representing $\Sigma^{n-3}\eta$ and the other representing $AD(x)$. Hence $b_N(x)f = f\#_{\overline{x}}\gamma$ by definition of b. □

Remark 2.10. Let N be a closed k-connected $(k+2)$-parallelizable n-manifold, $n \geq 2k+6$ and $k \geq 1$. Then every embedding $N \to \mathbb{R}^{2n-k-1}$ can be obtained from every other embedding by parametric connected summations with embeddings (γ is defined in the above proof; τ and $\varkappa$ are defined below):

- γ, $\varkappa$ and τ, provided $n-k$ is even;
- γ and $\varkappa$, provided $n-k$ is odd and $H_{k+1}(N)$ has no 2-torsion.[19]

Definition 2.11 of embeddings τ and $\varkappa$. Define the Hudson Torus $\varkappa : T^{k+2,n-k-2} \to \mathbb{R}^{2n-k-1}$ as in [Sk08, §2.2] or set $\varkappa := \psi_t(h^{-1}_{t,n}AD_t(1))$ for the generator $1 \in H_{k+2}(T^{k+2,n-k-2}) \cong \mathbb{Z}$ and the standard embedding $t = t^{2n-k-1}_{k+2,n-k-2}$; the map $h_{t,n}$ is an isomorphism.

Construct an embedding $\tau : T^{k+1,n-k-1} \to \mathbb{R}^{2n-k-1}$ for $n-k$ even as follows. Take a nonzero tangent vector field $v : S^{n-k-1} \to \mathbb{R}^{n-k}$ on S^{n-k-1}. We have $v(a) \perp a$. Define a map

$$\tau' : \mathbb{R}^2 \times \mathbb{R}^k \times S^{n-k-1} \to \mathbb{R}^{n-k} \times \mathbb{R}^k \times S^{n-k-1} \quad \text{by} \quad \tau'(x,y,s,a) = (xa + yv(a), s, a).$$

[19] It would be interesting to drop the latter condition; for this, one needs an explicit construction of embeddings whose Whitney invariants are in $\operatorname{im}\beta \subset H_{k+2}(N;\mathbb{Z}_2)$. For this, one needs an explicit construction of immersions $S^n \to \mathbb{R}^{2n}$.

Define an embedding τ to be the composition of the restriction of τ' and the standard inclusion:

$$T^{k+1,n-k-1} \to \mathbb{R}^{n-k} \times \mathbb{R}^k \times S^{n-k-1} \subset \mathbb{R}^{2n-k-1}.$$

Remark 2.10 follows from the proof of Main Theorem 1.1(a) because the Hurewicz homomorphism $\pi_{k+2}(N) \to H_{k+2}(N)$ is epimorphic and by the following easy result for $m = 2n-k, 2n-k-1$ (for $m = 2n-k-1$ this is essentially the same as the Summation Lemma 2.9(c)).

Let N be a closed orientable n-manifold and $m \le 2n-k$. The Whitney invariant $W_m : \mathrm{E}^m(N) \to H_{k+1}(N; \mathbb{Z}_{(n-k-1)})$ is defined as the composition of W_{2n-k} and the map $\mathrm{E}^m(N) \to \mathrm{E}^{2n-k}(N)$ induced by the inclusion $\mathbb{R}^m \to \mathbb{R}^{2n-k}$. If

$$f : N \to \mathbb{R}^{2n-k-1}, \quad g : T^{k+1,n-k-1} \to \mathbb{R}^{2n-k-1}, \quad s : S^{k+1} \times D^{n-k-1} \to N$$

are embeddings, then $W_m(f\#_s g) = W_m(f) + W_m(g)[s|_{S^{k+1}\times 0}]$, where $k+1 < n/2$ and $W_m(g)$ is considered as an element of $\mathbb{Z}_{(n-k-1)}$.

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