LAMPLIGHTER GRAPHS DO NOT ADMIT
HARMONIC FUNCTIONS OF FINITE ENERGY

AGELOS GEORGAKOPOULOS

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ABSTRACT. We prove that a lamplighter graph of a locally finite graph over a
finite graph does not admit a non-constant harmonic function of finite Dirichlet
energy.

1. INTRODUCTION

The wreath product \( G \wr H \) of two groups \( G, H \) is a well-known concept. Cayley
graphs of \( G \wr H \) can be obtained in an intuitive way by starting with a Cayley
graph of \( G \) and associating with each of its vertices a lamp whose possible states
are indexed by the elements of \( H \); see below. Graphs obtained this way are called
lamplighter graphs. A well-known special case are the Diestel-Leader \([7]\) graphs
\( \text{DL}(n,n) \).

Kaimanovich and Vershik \([11, \text{Sections 6.1, 6.2}]\) proved that lamplighter graphs
of infinite grids \( \mathbb{Z}^d, d \geq 3 \) admit non-constant, bounded, harmonic functions. Their
construction had an intuitive probabilistic interpretation related to random walks
on these graphs, which triggered a lot of further research on lamplighter graphs.
For example, spectral properties of such groups are studied in \([5,10,13]\), and other
properties related to random walks are studied in \([8,9,17]\). Harmonic functions
on lamplighter graphs and the related Poisson boundary are further studied e.g.
in \([3,12,18]\). Finally, Lyons, Pemantle and Peres \([14]\) proved that the lamplighter
graph of \( \mathbb{Z} \) over \( \mathbb{Z}_2 \) has the surprising property that random walk with a drift
towards a fixed vertex can move outwards faster than simple random walk.

It is known that the existence of a non-constant harmonic function of finite
Dirichlet energy implies the existence of a non-constant bounded harmonic function
\([19, \text{Theorem 3.73}]\). Given the aforementioned impact that bounded harmonic
functions on lamplighter graphs have had, we ask whether these graphs have non-
constant harmonic functions of finite Dirichlet energy. For lamplighter graphs on a
grid it is known that no such harmonic functions can exist, since such graphs are
ameanable and thus admit no non-constant harmonic functions of finite Dirichlet
energy \([16]\). A. Karlsson (oral communication) asked whether this is also the case
for graphs of the form \( T \setminus \mathbb{Z}_2 \), where \( T \) is any regular tree. In this paper we give an
affirmative answer to this question. In fact, the actual result is much more general:
Theorem 1.1. Let $G$ be a connected locally finite graph and let $H$ be a connected finite graph with at least one edge. Then $G \wr H$ does not admit any non-constant harmonic function of finite Dirichlet energy.

Indeed, we do not need to assume that any of the involved graphs is a Cayley graph. Lamplighter graphs on general graphs can be defined as in the usual case when all graphs are Cayley graphs; see the next section.

It is easy to prove, and well-known, that the non-existence of non-constant harmonic functions in a graph is equivalent to the uniqueness of electrical currents. Thus, in a lamplighter graph $G \wr H$ as in Theorem 1.1 electrical currents of finite energy are unique.

Classes of graphs that do admit non-constant harmonic functions of finite Dirichlet energy are known; see [1, 2, 4].

As an intermediate step to the proof of Theorem 1.1 we prove a result (Lemma 3.1 below) that strengthens a theorem of Markvorsen, McGuinness and Thomassen [15] and might be applicable in order to prove that other classes of graphs do not admit non-constant Dirichlet-finite harmonic functions.

2. Definitions

We will be using the terminology of Diestel [6]. For a finite path $P$ we let $|P|$ denote the number of edges in $P$. For a graph $G$ and a set $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of $G$ induced by the vertices in $U$. If $G$ is finite, then its diameter $\text{diam}(G)$ is the maximum distance, in the usual graph metric, of two vertices of $G$.

Let $G$, $H$ be connected graphs, and suppose that every vertex of $G$ has a distinct lamp associated with it, the set of possible states of each lamp being the set of vertices $V(H)$ of $H$. At the beginning, all lamps have the same state $s_0 \in V(H)$, and a “lamplighter” is standing at some vertex of $G$. In each unit of time the lamplighter is allowed to choose one of two possible moves: either walk to a vertex of $G$ adjacent to the vertex $x \in V(G)$ he is currently at or switch the current state $s \in V(H)$ of $x$ into one of the states $s' \in V(H)$ adjacent with $s$. The lamplighter graph $G \wr H$ is, then, a graph whose vertices correspond to the possible configurations of this game and whose edges correspond to the possible moves of the lamplighter. More formally, the vertex set of $G \wr H$ is the set of pairs $(C,x)$ where $C : V(G) \to V(H)$ is an assignment of states such that $C(v) \neq s_0$ holds for only finitely many vertices $v \in V(G)$, and $x$ is a vertex of $G$ (the current position of the lamplighter). Two vertices $(C,x)$ and $(C',x')$ of $G \wr H$ are joined by an edge if (precisely) one of the following conditions holds:

- $C = C'$ and $xx' \in E(G)$, or
- $x = x'$, all vertices except $x$ are mapped to the same state by $C$ and $C'$, and $C(x)C'(x) \in E(H)$.

This definition of $G \wr H$ coincides with that of Erschler [9].

The blow-up of a vertex $v \in V(G)$ in $L = G \wr H$ is the set of vertices of $L$ of the form $(C,v)$. Similarly, the blow-up of a subgraph $T$ of $G$ is the subgraph of $L$ spanned by the blow-ups of the vertices of $T$. Given a vertex $x \in V(L)$ we let $[x]$ denote the vertex of $G$, the blow-up of which contains $x$.

An edge of $L$ is a switching edge if it corresponds to a move of the lamplighter that switches a lamp, more formally, if it is of the form $(C,v)(C',v)$. For a switching
We start with a lemma that might be applicable in order to prove that other classes of graphs do not admit non-constant Dirichlet-finite harmonic functions. This strengthens a result of [15, Theorem 7.1].

**Lemma 3.1.** Let $G$ be a connected locally finite graph such that for every two disjoint rays $S,Q$ in $G$ there is a constant $c$ and a sequence $(P_i)_{i \in \mathbb{N}}$ of pairwise edge-disjoint $S$–$Q$ paths such that $|P_i| \leq ci$. Then $G$ does not admit a non-constant harmonic function of finite energy.

**Proof.** Let $G$ be a locally finite graph that admits a non-constant harmonic function $\phi$ of finite energy; it suffices to find two rays $S,Q$ in $G$ that do not satisfy the condition in the assertion.

Since $\phi$ is non-constant, we can find an edge $x_0x_1$ satisfying $\phi(x_1) > \phi(x_0)$. By the definition of a harmonic function, it is easy to see that $x_0x_1$ must lie in a double ray $D = \ldots x_{-1}x_0x_1 \ldots$ such that $\phi(x_i) \geq \phi(x_{i-1})$ for every $i \in \mathbb{Z}$; indeed, every vertex $x \in V(G)$ must have a neighbour $y$ such that $\phi(y) \geq \phi(x)$.

Define the subrays $S = x_0x_1x_2\ldots$ and $Q = x_0x_1x_2\ldots$ of $D$. Now suppose there is a sequence $(P_i)_{i \in \mathbb{N}}$ of pairwise edge-disjoint $S$–$Q$ paths such that $|P_i| \leq ci$ for some constant $c$.

Note that by the choice of $D$ there is a bound $u > 0$ such that $u_i := |\phi(s_i) - \phi(q_i)| \geq u$ for every $i$, where $s_i \in V(S)$ and $q_i \in V(Q)$ are the endvertices of $P_i$.

For every edge $e = xy$ let $f(e) := |\phi(y) - \phi(x)|$. Let $X_i$ be the set of edges $e$ in $P_i$ such that $f(e) > 0.99u_i$, and let $Y_i$ be the set of all other edges in $P_i$. As $|P_i| \leq ci$ by assumption, the edges in $Y_i$ contribute less than $0.9u_i$ to $u_i$; thus $\sum_{e \in X_i} f(e) > 0.1u_i$ must hold. But since $f(e) > 0.99u_i$ for every $e \in X_j$, we have $\sum_{e \in X_j} w_\phi(e) > 0.1 \times 0.99u_i^2$. As the sets $X_j$ are pairwise edge-disjoint, and as the series $\sum_1 1/i$ is not convergent, this contradicts the fact that $\sum_{e \in E(G)} w_\phi(e)$ is finite.

We now apply Lemma 3.1 to prove our main result.

**Proof of Theorem 1.1.** We will show that $L := G \setminus H$ satisfies the condition of Lemma 3.1 from which then the assertion follows. So let $S,Q$ be any two disjoint rays of $L$.

Since $L$ is connected we can find a double ray $D$ in $L$ that contains a tail $S'$ of $S$ and a tail $Q'$ of $Q$. Let $s_0$ (respectively, $q_0$) be the first vertex of $S'$ (resp. $Q'$). Let $V_0$ be the set of vertices of $G$, the blow-up of which meets the path $s_0Dq_0$. Note that $V_0$ induces a connected subgraph of $G$, because the lamplighter only moves along the edges of $G$. Thus we can choose a spanning tree $T_0$ of $G[V_0]$.

For $i = 1, 2, \ldots$ we construct an $S'$–$Q'$ path $P_i$ as follows. Let $s_i$ be the first vertex of $S'$ not in the blow-up of $V_{i-1}$, and let $q_i$ be the first vertex of $Q'$ not in
the blow-up of $V_{i-1}$. Let $V_i := V_{i-1} \cup \{s_i, q_i\}$, and extend $T_{i-1}$ into a spanning tree $T_i$ of $G[V_i]$ by adding two edges incident with $s_i$ and $q_i$, respectively; such edges do exist: their blow-up contains the edges of $S', Q'$ leading into $s_i, q_i$, respectively.

We now construct an $s_i-q_i$ path $P_i$. Pick a switching edge $e = s_is'_i$ incident with $s_i$. Then let $X_i$ be the unique path in $L$ from $s'_i$ to a vertex $q_i^+$ with $[q_i^+] = [q_i]$ such that $X_i$ is contained in the blow-up of $T_i$. Pick a switching edge $f = q_i^+q_i^-$ incident with $q_i^+$. Then follow the unique path $Y_i$ in $L$ from $q_i^-$ to a vertex $s_i^+$ with $[s_i^+] = [s_i]$ such that $Y_i$ is contained in the blow-up of $T_i$. Let $e' = s_i^+s_i^-$ be the switching edge incident with $s_i^+$ such that $[e'] = [e]$. Finally, let $Z_i$ be a path from $s_i^-$ to the unique vertex $q_i'$ with $[q_i q_i'] = [f]$, such that the interior of $Z_i$ is contained in the blow-up of $V_{i-1}$ and $Z_i$ has minimum length under all paths with these properties. Such a path exists because every lamp at a vertex in $G - V_{i-1}$ has the same state in $s_i^-$ and $q_i'$; indeed, the lamps in $G - V_i$ were never switched in the above construction, the lamp at $[s_i]$ was switched twice on the way from $s_i$ to $s_i^-$ using the same switching edge $[e]$, which means that its state in both endpoints of $Z_i$ coincides with that in $s_i$ and $q_i$, and finally the lamp at $[q_i']$ has the same state in both endpoints of $Z_i$, namely the state $[f]$ leads to. Now set $P_i := s_is'_iX_is'_iq_i^+Y_is_i+s_iZ_iq_iq_i'$.

It is not hard to check that the paths $P_i$ are pairwise disjoint. Indeed, let $i < j \in \mathbb{N}$. Then, by the choice of the vertices $s_j, q_j$ and the construction of $P_j$, it follows that for every inner vertex $x$ of $P_j$, the configuration of $x$ differs from the configuration of any vertex in $P_i$ in at least one of the two lamps at $[s_j]$ and $[q_j]$.

It remains to show that there is a constant $c$ such that $|P_i| \leq ci$ for every $i$. To prove this, note that $|P_i| = |X_i| + |Y_i| + |Z_i| + 4$; we will show that the latter three subpaths grow at most linearly with $i$, which then implies that this is also true for $P_i$.

Firstly, note that $diam(T_i) - diam(T_{i-1}) \leq 2$ since $V(T_i) := V(T_{i-1}) \cup \{s_i, q_i\}$. By the choice of $X_i$, we have $|X_i| \leq diam(T_i)$, from which it follows that there is a constant $c_1$ such that $|X_i| \leq c_1i$. By the same argument, we have $|Y_i| \leq c_1i$.

It remains to bound the length of $Z_i$. For this, note that if $T$ is a finite tree and $v, w \in V(T)$, then there is a $v$-$w$ walk $W$ in $T$ containing all edges of $T$ and satisfying $|W| \leq \frac{3}{2}E(T)$; indeed, starting at $v$, one can first walk around the “perimeter” of $T$ traversing every edge precisely once in each direction (2 $E(T)$ edges), and then move “straight” from $v$ to $w$ (at most $E(T)$ edges). Thus, in order to choose $Z_i$, we could put a lamplighter at the vertex and configuration indicated by $s_i^-$ and let him move in $T_i \subset G$ along such a walk $W$ from $[s_i^-]$ to $[q_i']$, and every time he visits a new vertex $x$ let him change the state of $x$ to the state indicated by $q_i'$. This bounds the length of $Z_i$ from above by $3|E(T_i)|diam(H)$, and since $|E(T_i)| - |E(T_{i-1})| = 2$ and $H$ is fixed, we can find a constant $c_2$ such that $|Z_i| \leq c_2i$ for every $i$. This completes the proof that $|P_i|$ grows at most linearly with $i$.

Thus we can now apply Lemma 3.1 to prove that $G \wr H$ does not admit a non-constant harmonic function of finite energy.

\begin{problem}
Does the assertion of Theorem 1.1 still hold if $H$ is an infinite locally finite graph?
\end{problem}

Lemma 3.1 might be applicable in order to prove that other classes of graphs also do not admit non-constant Dirichlet-finite harmonic functions. For example, it yields an easy proof of the (well-known) fact that infinite grids have this property.
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References


Technische Universität Graz, Steyrergasse 30, 8010, Graz, Austria