ARITHMETICAL RANK OF TORIC IDEALS ASSOCIATED TO GRAPHS

ANARGYROS KATSABEKIS

(Communicated by Bernd Ulrich)

Abstract. Let \( I_G \subset K[x_1,\ldots,x_m] \) be the toric ideal associated to a finite graph \( G \). In this paper we study the binomial arithmetical rank and the \( G \)-homogeneous arithmetical rank of \( I_G \) in 2 cases:

(1) \( G \) is bipartite,

(2) \( I_G \) is generated by quadratic binomials.

In both cases we prove that the binomial arithmetical rank and the \( G \)-homogeneous arithmetical rank coincide with the minimal number of generators of \( I_G \).

1. Introduction

Let \( G \) be a finite, connected and undirected graph having no loops and no multiple edges on the vertex set \( V(G) = \{v_1,\ldots,v_n\} \), and let \( E(G) = \{e_1,\ldots,e_m\} \) be the set of edges of \( G \). The incidence matrix of \( G \) is the \( n \times m \) matrix \( M_G = (a_{i,j}) \) defined by

\[
a_{i,j} = \begin{cases} 
1, & \text{if } v_i \text{ is one of the vertices in } e_j, \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( K \) be an algebraically closed field, and let \( A_G = \{a_1,\ldots,a_m\} \) be the set of vectors in \( \mathbb{Z}^n \), where \( a_i = (a_{i,1},\ldots,a_{i,n}) \) for \( 1 \leq i \leq m \). The toric ideal \( I_G \) associated to \( G \) is the kernel of the \( K \)-algebra homomorphism

\[
\phi : K[x_1,\ldots,x_m] \to K[t_1,\ldots,t_n]
\]
given by

\[
\phi(x_i) = t_1^{a_{i,1}} \cdots t_n^{a_{i,n}} \quad \text{for all } i = 1,\ldots,m.
\]

The ideal \( I_G \) is prime, and therefore \( \text{rad}(I_G) = I_G \). The toric variety \( \mathcal{V}(I_G) \) associated to \( G \) is the set

\[
\{(u_1,\ldots,u_m) \in K^m | F(u_1,\ldots,u_m) = 0, \forall F \in I_G\}
\]
of zeroes of \( I_G \). For every graph \( G \) the variety \( \mathcal{V}(I_G) \) is an extremal toric variety, i.e. the vector configuration \( A_G \) is extremal; see Remark 2.1.

The polynomial ring \( K[x_1,\ldots,x_m] \) has a natural \( G \)-graded structure given by setting \( \deg_G(x_i) = a_i \) for \( i = 1,\ldots,m \). For \( u = (u_1,\ldots,u_m) \in \mathbb{N}^m \), we define the
G-degree of the monomial $x^u := x_1^{u_1} \cdots x_m^{u_m}$ to be
\[
deg_G(x^u) := u_1 \mathbf{a}_1 + \cdots + u_m \mathbf{a}_m \in \mathbb{N}A_G,
\]
where $\mathbb{N}A_G$ is the semigroup generated by $A_G$. Remark that $\mathbb{N}A_G$ is pointed, i.e. zero is the only invertible element. A polynomial $F \in K[x_1, \ldots, x_m]$ is called $G$-homogeneous if the monomials in nonzero terms of $F$ have the same $G$-degree. An ideal $I$ is $G$-homogeneous if it is generated by $G$-homogeneous polynomials.

The toric ideal $I_G$ is generated by all the binomials $x^{u_+} - x^{u_-}$ such that
\[
de_{G}(x^{u_+}) = \deg_G(x^{u_-}),
\]
where $u_+ \in \mathbb{N}^m$ and $u_- \in \mathbb{N}^m$ denote the positive and negative part of $u = u_+ - u_-$, respectively (see [11]).

A basic problem in commutative algebra asks one to compute the smallest integer $s$ for which there exist polynomials $F_1, \ldots, F_s$ in $I_G$ such that $I_G = \text{rad}(F_1, \ldots, F_s)$. This integer is called the \textit{arithmetical rank} of $I_G$ and will be denoted by $\text{ara}(I_G)$. An usual approach to this problem is to restrict to a certain class of polynomials and ask how many polynomials from this class can generate the toric ideal up to radical. Restricting the polynomials to the class of binomials we arrive at the notion of the \textit{binomial arithmetical rank} of $I_G$, denoted by $\text{bar}(I_G)$. Also, if all of the polynomials $F_1, \ldots, F_s$ satisfying $I_G = \text{rad}(F_1, \ldots, F_s)$ are $G$-homogeneous, the smallest integer $s$ is called \textit{the $G$-homogeneous arithmetical rank} of $I_G$ and will be denoted by $\text{ara}_G(I_G)$. From the definitions, the generalized Krull’s principal ideal theorem and the graded version of Nakayama’s Lemma, we deduce the following inequalities for a toric ideal $I_G$:
\[
\text{ht}(I_G) \leq \text{ara}(I_G) \leq \text{ara}_G(I_G) \leq \text{bar}(I_G) \leq \mu(I_G).
\]
Here $\text{ht}(I_G)$ denotes the height and $\mu(I_G)$ denotes the minimal number of generators of $I_G$. When $\text{ht}(I_G) = \mu(I_G)$ the ideal $I_G$ is called a \textit{complete intersection}.

A case of particular interest is when $\text{bar}(I_G) = \text{ht}(I_G)$. When $K$ is a field of characteristic zero, this is equivalent to saying that $I_G$ is complete intersection; see [11]. Complete intersection bipartite graphs have been characterized in [4], [8]. In most cases, when $G$ is bipartite, the equality $\text{bar}(I_G) = \text{ht}(I_G)$ does not hold. In section 3 we prove that $\text{bar}(I_G) = \text{ara}_G(I_G) = \mu(I_G)$ for any bipartite graph $G$. In addition we show that the equality $\text{bar}(I_G) = \text{ara}_G(I_G) = \mu(I_G)$ also holds for any graph $G$ such that the toric ideal $I_G$ is generated by quadratic binomials.

2. Basics on toric ideals associated to graphs

Let $G$ be a graph. A \textit{walk} of length $q$ of $G$ is a finite sequence of the form
\[
\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{q-1}, v_q\}, \{v_q, v_{q+1}\});
\]
this walk is \textit{closed} if $v_{q+1} = v_1$. An \textit{even} closed walk is a closed walk of even length. A \textit{cycle} of $G$ is a closed walk
\[
\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})
\]
with $v_i \neq v_j$ for all $1 \leq i < j \leq q$. Notice that if $e_i = \{v_{i_k}, v_{i_l}\}$ is an edge of $G$, then $\phi(x_{i_k}) = t_i t_{i_l}$. Given an even closed walk $\Gamma = (e_{i_1}, \ldots, e_{i_{2q}})$ of $G$ with each $e_k \in E(G)$, we have that
\[
\phi(\prod_{k=1}^{q} x_{i_{2k-1}}) = \phi(\prod_{k=1}^{q} x_{i_{2k}}),
\]
and therefore the binomial

\[ f_\Gamma := \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}} \]

belongs to \( I_G \). Remark that if \( \Gamma \) is an even cycle of \( G \), then the monomials \( M = \prod_{k=1}^{q} x_{i_{2k-1}} \) and \( N = \prod_{k=1}^{q} x_{i_{2k}} \) are squarefree. From Proposition 3.1 in [12] we have that

\[ I_G = \langle \{ f_\Gamma | \Gamma \text{ is an even closed walk of } G \} \rangle. \]

Let \( u = (u_1, \ldots, u_m) \in \mathbb{Z}^m \) be a vector. Then the support of \( u \), denoted by \( \text{supp}(u) \), is the set \( \{ i \in \{1, \ldots, m\} | u_i \neq 0 \} \). For a monomial \( x^u \) we define \( \text{supp}(x^u) := \text{supp}(u) \). A non-zero vector \( u = (u_1, \ldots, u_m) \in \ker\mathbb{Z}(M_G) \) is called a circuit of \( A_G \) if its support is minimal with respect to inclusion and all the coordinates of \( u \) are relatively prime, where \( \ker\mathbb{Z}(M_G) = \{ v \in \mathbb{Z}^m | M_G v^t = 0^t \} \). The binomial \( x^u - x^v \in I_G \) associated to a vector \( u \in \ker\mathbb{Z}(M_G) \) is also called a circuit. A binomial \( B = x^u - x^v \in I_G \) is called primitive if there exists no other binomial \( x^w - x^y \in I_G \) such that \( x^w \) divides \( x^u \) and \( x^y \) divides \( x^v \). For a circuit \( B = x^u - x^v \in I_G \) we have, from Corollary 8.1.4 in [13], that \( B = f_\Gamma \) for an even closed walk \( \Gamma \) of \( G \), since every circuit is also primitive.

If \( G \) is a bipartite graph, then \( G \) has no odd cycles, so, from Proposition 4.2 in [12], a binomial \( f_\Gamma \), where \( \Gamma \) is an even closed walk of \( G \), is a circuit if and only if \( \Gamma \) is an even cycle.

For the rest of this section we recall some fundamental material from [7]. We shall denote by \( C_G \) the set of circuits of \( A_G \). Let

\[ \mathcal{C} := \{ E \subset \{1, \ldots, m\} | \text{supp}(u_+) = E \text{ or } \text{supp}(u_-) = E \text{ where } u \in C_G \}, \]

and let \( \mathcal{C}_{\text{min}} \) be the set of minimal elements of \( \mathcal{C} \).

To every toric ideal \( I_G \) we associate the rational polyhedral cone

\[ \sigma = \text{pos}_\mathbb{Q}(A_G) := \{ \lambda_1 a_1 + \cdots + \lambda_m a_m | \lambda_i \in \mathbb{Q}_{\geq 0} \}. \]

A face \( \mathcal{F} \) of \( \sigma \) is any set of the form

\[ \mathcal{F} = \sigma \cap \{ x \in \mathbb{Q}^n : cx = 0 \}, \]

where \( c \in \mathbb{Q}^n \) and \( cx \geq 0 \) for all \( x \in \sigma \). Given an edge \( e_i = \{ v_{i_1}, v_{i_2} \} \) of \( G \), we have that \( \text{pos}_\mathbb{Q}(a_i) \) is a face of \( \sigma \) with defining vector \( c = (c_1, \ldots, c_n) \in \mathbb{Z}^n \) having coordinates

\[ c_j = \begin{cases} 0, & \text{if } j = i_1, i_2, \\ 1, & \text{otherwise.} \end{cases} \]

Remark 2.1. For every graph \( G \) the vector configuration \( A_G \) is extremal, i.e. for any \( B \not\subseteq A_G \) we have \( \text{pos}_\mathbb{Q}(B) \not\subseteq \text{pos}_\mathbb{Q}(A_G) \).

To see this consider a set \( B = \{ a_{i_1}, \ldots, a_{i_k} \} \not\subseteq A_G \) and assume that the vector \( a_j \) is not in \( B \). Let \( c \) be the defining vector of the face \( \text{pos}_\mathbb{Q}(a_j) \). If \( a_j \) belongs to \( \text{pos}_\mathbb{Q}(B) \), then \( a_j = \lambda_1 a_{i_1} + \cdots + \lambda_k a_{i_k} \), where \( \lambda_1, \ldots, \lambda_k \) are nonnegative rationals and there is at least one \( \lambda_i \) different from zero. Thus \( c a_j = \lambda_1 (ca_{i_1}) + \cdots + \lambda_k (ca_{i_k}) \), and therefore \( 0 = \lambda_1 (ca_{i_1}) + \cdots + \lambda_k (ca_{i_k}) \). But

\[ \lambda_1 (ca_{i_1}) + \cdots + \lambda_k (ca_{i_k}) > 0, \]

a contradiction. Consequently \( a_j \) does not belong to \( \text{pos}_\mathbb{Q}(B) \), so \( \text{pos}_\mathbb{Q}(B) \) is a proper subset of \( \text{pos}_\mathbb{Q}(A_G) \).
For a subset $E$ of $\{1, \ldots, m\}$ we denote by $\sigma_E$ the subcone $\text{pos}_Q(a_i|i \in E)$ of $\sigma$. We adopt the convention that $\sigma_0 = \{\emptyset\}$. The relative interior of $\sigma_E$, denoted by $\text{relint}_Q(\sigma_E)$, is the set of all strictly positive rational linear combinations of $a_i$, $i \in E$.

**Definition 2.2** ([7]). We associate to $G$ the simplicial complex $\Delta_G$ with vertices the elements of $C_{\text{min}}$. Let $T \subset C_{\text{min}}$; then $T \in \Delta_G$ if

$$\bigcap_{E \in T} \text{relint}_Q(\sigma_E) \neq \emptyset.$$ 

In particular, $\{E, E'\} \in \Delta_G$ if and only if there exists a circuit $u \in C_G$ such that $\text{supp}(u_+) = E$ and $\text{supp}(u_-) = E'$.

Let $J$ be a subset of $\Omega := \{0, 1, \ldots, \dim(\Delta_G)\}$. A set $M = \{T_1, \ldots, T_s\}$ of simplices of $\Delta_G$ is called a $J$-matching in $\Delta_G$ if $T_k \cap T_l = \emptyset$ for every $1 \leq k, l \leq s$ and $\dim(T_k) \in J$ for every $1 \leq k \leq s$; see also Definition 2.1 in [7]. Let $\text{supp}(M) = \bigcup_{i=1}^s T_i$, which is a subset of the vertices $C_{\text{min}}$. A $J$-matching $M$ in $\Delta_G$ is called a maximal $J$-matching if $\text{supp}(M)$ has the maximum possible cardinality among all $J$-matchings.

Given a maximal $J$-matching $M = \{T_1, \ldots, T_s\}$ in $\Delta_G$, we shall denote by $\text{card}(M)$ the cardinality $s$ of the set $M$. In addition, by $\delta(\Delta_G) J$ we denote the minimum of the set

$$\{\text{card}(M)|M \text{ is a maximal } J\text{-matching in } \Delta_G\}.$$ 

It follows from the definitions that if $\Delta_G = \bigcup_{i=1}^k \Delta_G^i$, then

$$\delta(\Delta_G) J = \sum_{i=1}^k \delta(\Delta_G^i) J,$$

where $\Delta_G^i$ are the connected components of $\Delta_G$.

**Example 2.3.** Consider the complete graph $K_4$ on the vertex set $\{v_1, \ldots, v_4\}$. We consider one variable $x_{ij}$, $1 \leq i < j \leq 4$, for each edge $\{v_i, v_j\}$ of $K_4$ and form the polynomial ring $K[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. From Proposition 4.2 in [12] we have that the toric ideal $I_{K_4}$ has 3 circuits, namely $f_{12} = x_{12}x_{34} - x_{14}x_{23}$, $f_{13} = x_{12}x_{34} - x_{13}x_{24}$ and $f_{14} = x_{13}x_{24} - x_{14}x_{23}$, corresponding to the 3 even cycles $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, respectively, of $K_4$. In fact $I_{K_4}$ is minimally generated by two of the above binomials, so it is a complete intersection of height 2. The simplicial complex $\Delta_{K_4}$ has three vertices, namely $E_1 = \{12, 34\}$, $E_2 = \{14, 23\}$ and $E_3 = \{13, 24\}$. It consists of all subsets of the set $\{E_1, E_2, E_3\}$. There are four maximal $\{0, 1\}$-matchings in $\Delta_{K_4}$, namely $\{\{E_1, E_2\}, \{E_3\}\}$, $\{\{E_1, E_3\}, \{E_2\}\}$, $\{\{E_2, E_3\}, \{E_1\}\}$ and $\{\{E_1\}, \{E_2\}, \{E_3\}\}$. We have that $\delta(\Delta_{K_4})_{\{0,1\}} = 2$, which is attained by $\text{maximal } \{0, 1\}$-matching $\{\{E_1, E_2\}, \{E_3\}\}$. In addition, $\delta(\Delta_{K_4})_{\{0,1,2\}} = 1$, which is attained by the maximal $\{0, 1, 2\}$-matching $\{\{E_1, E_2, E_3\}\}$.

Using the fact that $\Delta_G$ is an extremal vector configuration and also using two results from [7], namely Theorem 4.6 and Theorem 3.5, we get the following theorem:

**Theorem 2.4.** For a toric ideal $I_G$ we have $\delta(\Delta_G)_{\{0,1\}} \leq \text{bar}(I_G)$ and $\delta(\Delta_G) \Omega \leq \text{ara}_G(I_G)$. 

3. Arithmetical rank

Let $I_G \subset K[x_1, \ldots, x_m]$ be the toric ideal associated to a graph $G$. A binomial $B \in I_G$ is called indispensable if every system of binomial generators of $I_G$ contains $B$ or $-B$, while a monomial $M$ is called indispensable if every system of binomial generators of $I_G$ contains a binomial $B$ such that $M$ is a monomial of $B$. Let $N_G$ be the monomial ideal generated by all $x_u$ for which there exists a nonzero $x_u - x_w \in I_G$. From Proposition 3.1 in [2] we have that the set of indispensable monomials is the unique minimal generating set of $N_G$. The following lemma will be useful in the proof of Theorem 3.2 and Proposition 3.4.

**Lemma 3.1.** Assume that either $G$ is a bipartite graph or $I_G$ is generated by quadratic binomials. Let $T_G = \{M_1, \ldots, M_r\}$ be the set of indispensable monomials; then $C_{min} = \{\text{supp}(M_1), \ldots, \text{supp}(M_r)\}$.

**Proof.** First consider the case where $G$ is a bipartite graph. From Theorem 3.2 in [5] we have that $I_G$ is minimally generated by all binomials of the form $f_T$, where $\Gamma$ is an even cycle of $G$ with no chord. Combining the above theorem and Theorem 2.3 in [10], we obtain that a binomial $x_u^+ - x_u^- \in I_G$ is indispensable if and only if it is of the form $f_T$, for an even cycle $\Gamma$ of $G$ with no chord. Notice that in some cases there are circuits of $I_G$ of the form $f_T$ for an even cycle $\Gamma$ of $G$ with a chord. If $B_1, \ldots, B_s$ are the indispensable binomials of $I_G$, then the toric ideal $I_G$ is generated by the indispensable binomials. In addition, the monomials of $B_i$, $1 \leq i \leq s$, are all indispensable and also they form $T_G$. We will prove that

$$C_{min} \subset \{\text{supp}(M_1), \ldots, \text{supp}(M_r)\}.$$ 

Let $E \in C_{min}$ and let $\sigma = \text{pos}_G(A_G)$. From Theorem 4.6 in [2] the simplicial complexes $\Delta_G$ and $D_\sigma$ are identical; see [11] for the definition of the last complex. Using the fact that $I_G$ is generated by the binomials $B_1, \ldots, B_s$ and Corollary 5.7 in [5], we have that there is a monomial $M_i$ such that $\text{cone}(M_i) = \sigma_E$. For the definition and results about the cone of a monomial, see [3]. But $\text{cone}(M_i) = \sigma_{\supp(M_i)}$, so in this case all vectors belong to an extreme ray of $\sigma$, so $\sigma_E = \sigma_{\supp(M_i)}$ and therefore $E = \text{supp}(M_i)$. Thus

$$C_{min} \subset \{\text{supp}(M_1), \ldots, \text{supp}(M_r)\}.$$ 

Now consider a set $E = \text{supp}(M_i)$, $1 \leq i \leq r$, and we will prove that it also belongs to $C_{min}$. Suppose not; then there is an $E' \nsubseteq E$ such that $E' = \text{supp}(x_u^+)$ or $E' = \text{supp}(x_u^-)$, where $x_u^+ - x_u^- \in I_G$ is a circuit. Without loss of generality we can assume that $E' = \text{supp}(x_u^+)$. The monomials $x_u^+, x_u^-$ are squarefree and also $x_u^+$ divides $M_i$, since $E' \nsubseteq E$, a contradiction to the fact that $M_i$ is indispensable.

Assume now that $I_G$ is generated by quadratic binomials. Let $\{B_1, \ldots, B_s\}$ be a quadratic set of generators of $I_G$, and let $S_G$ be the set of monomials appearing in the binomials $B_1, \ldots, B_s$. We will prove that $S_G$ coincides with $T_G$; i.e. $S_G$ is the minimal generating set of the ideal $N_G$. Every monomial $N$ of $S_G$ belongs to the ideal $N_G$. On the other hand, for a monomial $x_u \in N_G$ there exists a monomial $x_w$ such that $x_u - x_w \in I_G$. But $I_G = (B_1, \ldots, B_s)$, so there is a monomial $N' \in S_G$ which divides $x_u$, and therefore $S_G$ is a set of generators for the ideal $N_G$. In addition $S_G$ is a minimal generating set, since every monomial $N$ of $S_G$ is quadratic. Thus $S_G$ is the set of indispensable monomials.
Using the fact that $I_G$ is generated by the binomials $B_1, \ldots, B_s$ and Corollary 5.7 in [5], we can easily prove that

$$C_{\min} \subset \{\text{supp}(M_1), \ldots, \text{supp}(M_r)\}.$$ 

It remains to prove that

$$\{\text{supp}(M_1), \ldots, \text{supp}(M_r)\} \subset C_{\min}.$$ 

Let $E = \text{supp}(M_i)$, $1 \leq i \leq r$, and assume that $E$ does not belong to $C_{\min}$. Then there is an $E' \subseteq E$, i.e., $E' = \{i\}$ is a singleton, and a circuit $x_i^{t_0} - N_i \in I_G$ such that $E' = \text{supp}(x_i^{t_0})$ and $E' \cap \text{supp}(N_i) = \emptyset$. Let $R = A_G - \{a_i\} \subseteq A_G$. Then $\text{deg}_{G}(N_i)$ belongs to $\text{pos}_Q(R)$, so $g_i a_i$ also belongs to $\text{pos}_Q(R)$, since $g_i a_i = \text{deg}_{G}(N_i)$, and therefore $a_i \in \text{pos}_Q(R)$. Thus $\text{pos}_Q(A_G) = \text{pos}_Q(R)$, a contradiction to the fact that $A_G$ is an extremal vector configuration.

Let $F \subset I_G$ be a set of binomials. We shall denote by $S(b)_F$ the graph with vertices the elements of $\text{deg}_{G}^0(b) = \{x^u \mid \text{deg}_{G}(x^u) = b\}$ and edges the sets \{\text{support}(\text{deg}_{G}(x^u)) = b\} whenever $x^u - x^v$ is a monomial multiple of a binomial in $F$. The next theorem computes the binomial arithmetical rank and the $G$-homogeneous arithmetical rank of $I_G$ for a bipartite graph $G$.

**Theorem 3.2.** Let $G$ be a bipartite graph; then $\text{bar}(I_G) = \mu(I_G)$ and $\text{ara}_G(I_G) = \mu(I_G)$.

**Proof.** First we will prove that $\{E, E'\}$ is an edge of $\Delta_G$ if and only if there is an indispensable binomial $x^{u_+} - x^{u_-} \in I_G$ with $\text{supp}(u_+) = E$ and $\text{supp}(u_-) = E'$. The one implication is easy. Let $x^{u_+} - x^{u_-} \in I_G$ be an indispensable binomial with $\text{supp}(u_+) = E$ and $\text{supp}(u_-) = E'$. Then $x^{u_+} - x^{u_-} = f_t$ for an even cycle $\Gamma$ of $G$ with no chord. But $f_t$ is a circuit, and therefore $\{E, E'\}$ is an edge. Conversely, consider an edge $\{E, E'\}$ of $\Delta_G$; then there is a circuit $x^{u_+} - x^{u_-} \in I_G$ such that $\text{supp}(u_+) = E$ and $\text{supp}(u_-) = E'$. Let $T_G = \{M_1, \ldots, M_r\}$ be the set of indispensable monomials. Then, from Lemma 3.1, there are indispensable monomials $M_i$, $M_j$ such that $E = \text{supp}(M_i)$ and $E' = \text{supp}(M_j)$. But $x^{u_+}$, $x^{u_-}$ are squarefree and also $M_i$, $M_j$ are minimal generators of $N_G$, so $x^{u_+} = M_i$, $x^{u_-} = M_j$ and therefore both monomials $x^{u_+}$, $x^{u_-}$ are indispensable. Let $b = \text{deg}_{G}(x^{u_+}) = \text{deg}_{G}(x^{u_-})$.

If $B_1, \ldots, B_s$ are the indispensable binomials of $I_G$, then $F := \{B_1, \ldots, B_s\}$ is a generating set of $I_G$, and therefore the graph $S(b)_F$ is connected; see Theorem 3.2 in [3]. Suppose that $x^{u_+} - x^{u_-}$ is not indispensable. Then there exists a path

\[
(\{x^{u_+} = x^{u_0}, x^{u_1}\}, \{x^{u_1}, x^{u_2}\}, \ldots, \{x^{u_{t-1}}, x^{u_t} = x^{u_-}\}),
\]

in $S(b)_F$ connecting the vertices $x^{u_0}$ and $x^{u_t}$. Now consider the binomial $x^{u_0} - x^{u_1}$. There is a binomial $B_t$ and a monomial $P$ such that $x^{u_0} - x^{u_1} = PB_t$, since \{\text{support}(\text{deg}(x^{u_0})), \text{support}(\text{deg}(x^{u_1}))\} is an edge of $S(b)_F$. If $B_t = x^{w_0} - x^{w_1}$, then $x^{u_0} = P x^{w_0}$ and therefore $x^{w_0}$ divides $x^{u_0}$. But $x^{u_0}$ is indispensable, so $P = 1$, and therefore the binomial $x^{w_0} - x^{w_1}$ is indispensable. Thus the monomial $x^{w_0}$ is indispensable. Moreover $x^{w_1} - x^{w_2}$ is indispensable, since \{\text{support}(\text{deg}(x^{w_0})), \text{support}(\text{deg}(x^{w_1}))\} is an edge of $S(b)_F$ and $x^{w_1}$ is indispensable, as well as all the binomials $x^{u_{i-1}} - x^{u_i}$, $3 \leq i \leq t$. Consequently there are at least two indispensable binomials with the same $G$-degree, contradicting Theorem 3.4 in [2]. Recall that the $G$-degree of a binomial $x^u - x^v \in I_G$ is defined to be $\text{deg}_{G}(x^u - x^v) := \text{deg}_{G}(x^u)$. 

\[\]
Example 3.3. Let $C_m$ be a connected component of $\Delta_B$, 1 \leq i \leq s$, be a connected component of $\Delta_G$. Thus every connected component of $\Delta_G$ is studied. Remark that if $\text{rank}(I_G)$ arises in the above minimal generating set of $\Delta_G$, then $\text{ara}(I_G) = \mu(I_G)$. Therefore, from Theorem 2.4, we have that $\text{ara}(I_G) = \mu(I_G)$. Thus $\text{ara}(I_G) = \mu(I_G)$.

Consequently,

$$\delta(\Delta_G)_{[0,1]} = \sum_{i=1}^{s} \delta(\Delta_G)_{[0,1]} = s,$$

i.e. $\delta(\Delta_G)_{[0,1]} = \mu(I_G)$. From Theorem 2.4 we have that $\text{bar}(I_G) = \mu(I_G)$ and therefore $\text{bar}(I_G) = \mu(I_G)$.

In addition, $\delta(\Delta_G)_{\Omega} = \delta(\Delta_G)_{[0,1]}$, since $\dim(\Delta_G) = 1$. So $\delta(\Delta_G)_{\Omega} = \mu(I_G)$, and therefore, from Theorem 2.4, we have that $\text{ara}(I_G) = \mu(I_G)$. Thus $\text{ara}(I_G) = \mu(I_G)$.

Example 3.3. Let $G = K_{3,3}$ be the complete bipartite graph on the vertex set $\{v_1, \ldots, v_6\}$ with 9 edges:

$\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}$.

We consider one variable $x_{ij}, 1 \leq i < j \leq 6$, for each edge $\{v_i, v_j\}$ of $G$ and form the polynomial ring $K[x_{ij} | 1 \leq i < j \leq 6]$. The toric ideal $I_G$ is minimally generated by 9 binomials:

$$x_{14}x_{26} - x_{16}x_{24}, \ x_{15}x_{36} - x_{16}x_{35}, \ x_{25}x_{36} - x_{26}x_{35}, \ x_{24}x_{36} - x_{26}x_{43}, \ x_{14}x_{25} - x_{15}x_{24}, \ x_{15}x_{26} - x_{16}x_{25},$$

$$x_{24}x_{35} - x_{25}x_{34}, \ x_{14}x_{26} - x_{16}x_{24}, \ x_{15}x_{36} - x_{16}x_{35}, \ x_{25}x_{36} - x_{26}x_{35} + x_{14}x_{25} - x_{15}x_{24}.$$ 

The simplicial complex $\Delta_G$ has 18 vertices corresponding to the 18 monomials arising in the above minimal generating set of $I_G$ and 9 edges corresponding to the 9 minimal generators of $I_G$. From Theorem 3.2 we have that $\text{ara}(I_G) = \text{bar}(I_G) = 9$. The height of $I_G$ equals $9 - 6 + 1 = 4$; see Proposition 3.2 in [12]. For the arithmetical rank of $I_G$, we have that $4 \leq \text{ara}(I_G) \leq 7$, since $I_G$ equals the radical of the ideal generated by

$$x_{14}x_{26} - x_{16}x_{24} + x_{15}x_{36} - x_{16}x_{35}, \ x_{25}x_{36} - x_{26}x_{35} + x_{14}x_{25} - x_{15}x_{24},$$

$$x_{24}x_{36} - x_{26}x_{43}, \ x_{14}x_{26} - x_{16}x_{24}, \ x_{15}x_{36} - x_{16}x_{35}, \ x_{25}x_{36} - x_{26}x_{35} + x_{14}x_{25} - x_{15}x_{24},$$

An interesting case occurs when $I_G$ is generated by quadratic binomials. In [9] a combinatorial criterion for the toric ideal $I_G$ to be generated by quadratic binomials is studied. Remark that if $B = x_{ij}x_{kl} - x_{ki}x_{lj}$ is a quadratic binomial in $I_G$, then $B = f_{\Gamma}$ for an even cycle $\Gamma$ of $G$ of length 4. We are going to compute the binomial arithmetical rank and the $G$-homogeneous arithmetical rank of such an ideal.

We shall denote by $\Delta_{\text{ind}}(A_G)$ the indispensable complex of $A_G$. For the definition and results about the indispensable complex, see [2].

Proposition 3.4. Let $I_G$ be a toric ideal generated by quadratic binomials. Then
(1) \( \{E, E'\} \) is a connected component of \( \Delta_G \) if and only if there is an indispensable quadratic binomial \( x^u_+ - x^u_- \in I_G \) with \( \text{supp}(x^u_+) = E \) and \( \text{supp}(x^u_-) = E' \).

(2) Every connected component of \( \Delta_G \) is either an edge or a 2-simplex.

**Proof.** (1) The first goal is to prove that \( \{E, E'\} \) is an edge of \( \Delta_G \) if and only if there is a quadratic binomial \( x^u_+ - x^u_- \in I_G \) with \( \text{supp}(x^u_+) = E \) and \( \text{supp}(x^u_-) = E' \). The one implication follows from the fact that if \( x^u_+ - x^u_- \in I_G \) is a quadratic binomial, then \( \text{supp}(u) \) is minimal with respect to inclusion, since \( A_G \) is extremal, and therefore the binomial \( x^u_+ - x^u_- \) is a circuit. Conversely let \( \{E, E'\} \) be an edge of \( \Delta_G \) and let \( x^u_+ - x^u_- \in I_G \) be a circuit with \( \text{supp}(x^u_+) = E \) and \( \text{supp}(x^u_-) = E' \), where \( u = (u_1, \ldots, u_m) \in \mathbb{Z}^m \). From Corollary 8.4.16 in \cite{13} we have that \( |u_i| \leq 2 \).

Let \( \{B_1, \ldots, B_s\} \) be a quadratic set of generators of \( I_G \), and let \( T_G = \{M_1, \ldots, M_r\} \) be the indispensable monomials. We have that both \( E, E' \) belong to \( C_{\text{min}} \), and therefore, from Lemma 3.1, \( E = \text{supp}(M_i) \) and \( E' = \text{supp}(M_j) \). But \( M_i \) and \( M_j \) are quadratic monomials, so \( E \) and \( E' \) consist of exactly 2 elements. Let \( E = \{k, l\} \) and \( E' = \{p, q\} \). We will consider three cases:

(i) If the monomials \( x^u_+ \) and \( x^u_- \) are squarefree, then \( x^u_+ - x^u_- \) is a quadratic binomial with \( \text{supp}(x^u_+) = E \) and \( \text{supp}(x^u_-) = E' \).

(ii) If \( x^u_+ = x_1^2x_l^2 \) and \( x^u_- = x_1^2x_q^2 \), then \( \text{deg}_G(x^u_+) = \text{deg}_G(x^u_-) \), and therefore \( 2a_k + 2a_l = 2a_p + 2a_q \). Thus \( a_k + a_l = a_p + a_q \), so the binomial \( x_kx_l - x_px_q \) belongs to \( I_G \) and also \( \text{supp}(x_kx_l) = E \) and \( \text{supp}(x_px_q) = E' \).

(iii) If \( x^u_+ = x_1^2x_l^2 \) and \( x^u_- = x_1^2x_q^2 \), then \( \text{deg}_G(x^u_+) = \text{deg}_G(x^u_-) \), and therefore \( 2a_k + a_l = 2a_p + a_q \). Assume that \( a_k - a_p \neq 0 \). Notice that every nonzero coordinate of the above vector equals either 1 or \(-1\). We have that \( a_q - a_l = 2(a_k - a_p) \), so every nonzero coordinate of the vector \( a_q - a_l \) equals either 2 or \(-2\), a contradiction. Thus \( a_k - a_p = 0 \), so \( a_k = a_p \), and therefore \( k = p \), a contradiction to the fact that \( \text{supp}(u_+) \cap \text{supp}(u_-) = \emptyset \). Similarly the assumption \( x^u_+ = x_kx_l^2 \) or \( x^u_- = x_px_q^2 \) again leads to a contradiction.

The second goal is to prove that \( \{E, E'\} \) is a connected component of \( \Delta_G \) if and only if there is an indispensable quadratic binomial \( M_i - M_j \in I_G \) with \( \text{supp}(M_i) = E \) and \( \text{supp}(M_j) = E' \). Suppose that the binomial \( M_i - M_j \in I_G \) is indispensable with \( \text{supp}(M_i) = E \) and \( \text{supp}(M_j) = E' \). From Theorem 3.4 in \cite{2} we have that \( \{M_i, M_j\} \) is a facet of the indispensable complex \( \Delta_{\text{ind}(A_G)} \). Assume that \( \{E, E'\} \) is not a connected component of \( \Delta_G \). Let us suppose that \( \{E, E''\} \) is an edge of \( \Delta_G \). Then there exists a quadratic binomial \( x^u_+ - x^u_- \in I_G \) with \( \text{supp}(x^u_+) = E \) and \( \text{supp}(x^u_-) = E'' \). Moreover, \( x^u_+ = M_i \) and \( x^u_- = M_k \), since the monomials \( x^u_+ \), \( x^u_- \) are quadratic and therefore indispensable. Notice that \( \text{deg}_G(M_i) = \text{deg}_G(M_k) \).

Thus \( \text{deg}_G(M_i) = \text{deg}_G(M_j) = \text{deg}_G(M_k) \), since the binomial \( M_i - M_j \) belongs to \( I_G \), and therefore \( \text{deg}_G(M_i) = \text{deg}_G(M_j) \). So \( \{M_i, M_j, M_k\} \) is a facet of \( \Delta_{\text{ind}(A_G)} \), a contradiction to the fact that \( \{M_i, M_j\} \) is a facet of \( \Delta_{\text{ind}(A_G)} \). Consequently \( \{E, E'\} \) is a connected component of \( \Delta_G \). Conversely assume that \( \{E, E'\} \) is a connected component of \( \Delta_G \). Then there is a quadratic binomial \( x^u_+ - x^u_- \in I_G \) with \( \text{supp}(x^u_+) = E \) and \( \text{supp}(x^u_-) = E' \). In fact \( x^u_+ = M_i \) and \( x^u_- = M_j \) for some indispensable monomials \( M_i \), \( M_j \). Suppose that the above binomial is not indispensable. Then, since \( M_i \) is indispensable, there
is an \( t \in \{1, \ldots, s\} \) such that \( E_t = M_t - M_k \). Set \( E'' = \text{supp}(M_k) \in C_{\min} \). We have that \( \{E, E''\} \) and \( \{E', E''\} \) are edges of \( \Delta_G \), since also \( M_j - M_k \) \( \in I_G \), a contradiction to the fact that \( \{E, E'\} \) is a connected component.

(2) Notice that \( \Delta_G \) has no connected components which are singletons. To see this consider \( E = \text{supp}(M_i) \in C_{\min} \). Then there is an \( l \in \{1, \ldots, s\} \) such that \( E_l = M_l - M_j \). Consequently \( \{\text{supp}(M_i), \text{supp}(M_j)\} \) is an edge of \( \Delta_G \).

Next we will show that \( \{E, E', E''\} \) is a 2-simplex of \( \Delta_G \) if and only if there are quadratic binomials \( M_i - M_j, M_j - M_k, M_l - M_k \) in \( I_G \) with \( \text{supp}(M_i) = E \), \( \text{supp}(M_j) = E' \) and \( \text{supp}(M_k) = E'' \). The one implication is easily derived from the fact that if \( \{E, E', E''\} \) is a 2-simplex of \( \Delta_G \), then every 2-element subset of it is an edge. Conversely we have that

\[
\deg_G(M_i) = \deg_G(M_j) = \deg_G(M_k)
\]

belongs to the intersection

\[
\text{relint}_Q(\sigma_E) \cap \text{relint}_Q(\sigma_{E'}) \cap \text{relint}_Q(\sigma_{E''}).
\]

Thus \( \{E, E', E''\} \) is a 2-simplex of \( \Delta_G \).

Finally, we prove that if \( \{E, E', E''\} \) is a 2-simplex of \( \Delta_G \), then it is a connected component. There are quadratic binomials \( M_i - M_j, M_j - M_k \) and \( M_i - M_k \) in \( I_G \), where \( \text{supp}(M_i) = E \), \( \text{supp}(M_j) = E' \) and \( \text{supp}(M_k) = E'' \). Let us suppose that \( M_i = x_{i_1}x_{i_2}, M_j = x_{j_1}x_{j_2} \) and \( M_k = x_{k_1}x_{k_2} \). In addition there are even cycles \( \Gamma_1, \Gamma_2, \Gamma_3 \) of \( G \) of length 4 such that \( M_i - M_j = f_{\Gamma_1}, M_i - M_k = f_{\Gamma_2} \) and \( M_j - M_k = f_{\Gamma_3} \). The cycle \( \Gamma_1 \) has 4 edges, namely \( e_{i_1}, e_{i_2}, e_{j_1} \) and \( e_{j_2} \), the cycle \( \Gamma_2 \) has 4 edges, namely \( e_{i_1}, e_{i_2}, e_{k_1} \) and \( e_{k_2} \), and \( \Gamma_3 \) has 4 edges, namely \( e_{j_1}, e_{j_2}, e_{k_1} \) and \( e_{k_2} \). Notice that the edges \( e_{i_1} \) and \( e_{i_2} \) have no common vertex. The above three cycles have the same vertex set \( V \) consisting of 4 vertices. Moreover these are the only even cycles of length 4 with vertex set \( V \). Let \( K_4 \) be the induced subgraph of \( G \) on the above vertex set. It is a complete graph with 4 vertices and edges

\[
E(K_4) = \{e_{i_1}, e_{i_2}, e_{j_1}, e_{j_2}, e_{k_1}, e_{k_2}\}.
\]

If, for example, \( \{E, E''\} \) is an edge of \( \Delta_G \), then there is a quadratic binomial \( M_i - M_t \) \( \in I_G \) with \( \text{supp}(M_t) = E'' \in C_{\min} \). Furthermore \( M_i - M_t = f_{\Gamma_4} \) for an even cycle \( \Gamma_4 \) of \( G \) of length 4. The vertex set of \( \Gamma_4 \) is \( V \), since \( e_{i_1} \) and \( e_{i_2} \) have no common vertex, and therefore \( \Gamma_4 \) coincides with either \( \Gamma_1 \) or \( \Gamma_2 \). Thus \( M_t \) equals either \( M_j \) or \( M_k \), so \( E'' = E' \) or \( E'' = E'' \). Consequently \( \{E, E', E''\} \) is a connected component of \( \Delta_G \).

\[\square\]

Remark 3.5. (1) To every connected component \( \{E, E'\} \) of \( \Delta_G \) we can associate an indispensable binomial \( f_{\Gamma} = x_{i}x_{j} - x_{k}x_{l} \) \( \in I_G \) for an even cycle \( \Gamma \) of \( G \) of length 4, where \( E = \{i, j\} \) and \( E' = \{k, l\} \), and also the induced subgraph \( H \) of \( G \) on the vertex set of \( \Gamma \). The subgraph \( H \) is not a complete graph. Moreover the toric ideal \( I_H \) is a complete intersection of height 1 and is generated by \( f_{\Gamma} \).

(2) \( \{E, E', E''\} \) is a 2-simplex of \( \Delta_G \) if and only if there are quadratic binomials \( M_i - M_j, M_i - M_k, M_j - M_k \) in \( I_G \) with \( \text{supp}(M_i) = E \), \( \text{supp}(M_j) = E' \) and \( \text{supp}(M_k) = E'' \).

(3) To every connected component of \( \Delta_G \) which is a 2-simplex we can associate a complete subgraph \( K_4 \) of \( G \) of order 4. The toric ideal \( I_{K_4} \) is minimally generated by two binomials \( f_{\Gamma_1} \) and \( f_{\Gamma_2} \), where \( \Gamma_1 \) and \( \Gamma_2 \) are even cycles of length 4 on the vertex set of \( K_4 \).
Proposition 3.6. Let $\Gamma = (e_1, e_p, e_j, e_q)$ be an even cycle of a graph $G$ such that the induced subgraph $\mathcal{H}$ of $G$ on the vertex set of $\Gamma$ is not a complete graph. If $H$ is a nonzero polynomial in $I_{\mathcal{H}}$, then there exist monomials $M$, $N$ of $H$ such that $x_i x_j$ divides $M$ and $x_p x_q$ divides $N$.

Proof. For the toric ideal $I_{\mathcal{H}}$ we have, from Proposition 4.13 in [11], that $I_{\mathcal{H}} = I_G \cap K[x_i | e_i \in E(\mathcal{H})]$. Since $I_{\mathcal{H}} = (f_f)$, there is a nonzero polynomial $C \in K[x_i | e_i \in E(\mathcal{H})]$ such that $H = Cf_f$. The polynomial $C$ has a unique representation as a sum of terms $C = C_1 + \cdots + C_s$. Notice that the monomials of two distinguished terms $C_k$ and $C_l$ are different. We have that

$$H = C_1 x_i x_j + \cdots + C_s x_i x_j - C_1 x_p x_q - \cdots - C_s x_p x_q. \tag{3.1}$$

Assume that $H$ has no term whose monomial is $x_p x_q$. This implies that in the above expression of $H$ all the terms of the form $C_k x_p x_q$ should be cancelled. But these terms cannot cancel by themselves, so terms of the form $C_k x_i x_j$ are used to cancel them. We claim that a term $C_k x_i x_j$ can be used to cancel at most one term of the form $-C_l x_p x_q$. Assume that $C_k x_i x_j$ is used to cancel the terms $-C_l x_p x_q$ and $-C_{k'} x_p x_q$. Let $M_1$, $M_2$ and $M_3$ be the monomials of the terms $C_k x_i x_j$, $-C_l x_p x_q$ and $-C_{k'} x_p x_q$, respectively. Then $M_1 = M_2$ and $M_1 = M_3$, so $M_2 = M_3$, a contradiction. But $x_p x_q$ divides no monomials of $H$, so there are two cases:

1. In expression (3.1) every term cancels. Therefore $H$ is equal to zero, a contradiction.
2. In (3.1) every term of the form $-C_l x_p x_q$ cancels, but there exist terms of the form $C_{k'} x_i x_j$, where $C_{k'}$ is different from $C_k$. Notice that the monomials of such a term coincides with the monomial of a suitable term $-C_l x_p x_q$. Thus every term $C_{k'} x_i x_j$ is divided by $x_p x_q$, a contradiction.

The next lemma will be useful in the proof of Theorem 3.8.

Lemma 3.7. Let $G$ be a graph with edges $E(G) = \{e_1, \ldots, e_m\}$, $\Gamma$ an even cycle of length 4 and $\mathcal{H}$ the induced subgraph of $G$ on the vertex set of $\Gamma$. If $F \subset I_G$ is a set of $G$-homogeneous polynomials which generates $I_G$ up to radical, then $F \cap K[x_i | e_i \in E(\mathcal{H})]$ that generates $I_{\mathcal{H}}$ up to radical.

Proof. Let $\{v_{i_1}, \ldots, v_{i_4}\}$ be the vertices of $\Gamma$. The rational polyhedral cone $pos_{\mathbb{Q}}(A_\mathcal{H})$ is a face of $pos_{\mathbb{Q}}(A_G)$ with defining vector $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ having coordinates

$$c_j = \begin{cases} 0, & \text{if } j = i_1, i_2, i_3, i_4, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, from Proposition 3.2 in [11], we have that $F \cap K[x_i | e_i \in E(\mathcal{H})]$ generates $I_{\mathcal{H}}$ up to radical. \qed

The following theorem determines the binomial arithmetical rank and the $G$-homogeneous arithmetical rank of a toric ideal $I_G$ generated by quadratic binomials.

Theorem 3.8. Let $G$ be a graph. If $I_G$ is generated by quadratic binomials, then

1. $\text{bar}(I_G) = \mu(I_G)$ and
2. $\text{ara}_G(I_G) = \mu(I_G)$.
Proof: (1) Let $g \geq 0$ be the number of indispensable binomials of $I_G$. Then, from Proposition 3.4 (1), the simplicial complex $\Delta_G$ has exactly $g$ connected components which are edges.

We will show that $\Delta_G$ has $\frac{s-g}{2}$ connected components which are 2-simplices, where $s = \mu(I_G)$. Let $B = \{B_1, \ldots, B_s\}$ be a minimal set of quadratic generators of $I_G$, and let $\{M_1, \ldots, M_s\}$ be the set of indispensable monomials. Notice that $B$ has $s-g$ binomials which are not indispensable. Given a 2-simplex $\{E, E', E''\}$ of $\Delta_G$, there are quadratic binomials $f_{E_1} = M_i - M_j$, $f_{E_2} = M_k - M_l$, $f_{E_3} = M_j - M_k$ in $I_G$ with $\text{supp}(M_i) = E$, $\text{supp}(M_j) = E'$ and $\text{supp}(M_k) = E''$. Remark that the binomials $f_{E_1}$, $f_{E_2}$ and $f_{E_3}$, as well as $-f_{E_1}$, $-f_{E_2}$ and $-f_{E_3}$, are not indispensable. In fact the minimal generating set $B$ contains exactly two of the binomials $f_{E_1}$, $-f_{E_1}$, $f_{E_2}$, $-f_{E_2}$, $f_{E_3}$ and $-f_{E_3}$, since the monomials $M_i$, $M_j$, $M_k$ are indispensable and $\{E, E', E''\}$ is a connected component of $\Delta_G$. Let $t$ be the number of connected components which are 2-simplices. Then $B$ contains at least 2t binomials which are not indispensable. So $2t \leq s - g$. On the other hand, if $B_i = M_i - M_j$ is not indispensable, then $\{\text{supp}(M_i)\}$ is an edge which is not a connected component of $\Delta_G$. Therefore there is a monomial $M_t$ such that $\{\text{supp}(M_t)\}$ is a connected component of $\Delta_G$. Moreover, there exists a $p \in \{1, \ldots, s\}$ such that $B_p$ or $-B_p$ equals either $M_i - M_j$ or $M_j - M_k$. But $\text{supp}(M_t)$ is a minimal generating set, so there exist exactly two binomials in $B$ whose monomials are $M_i$, $M_j$ and $M_k$. Thus $\Delta_G$ has at least $\frac{s-g}{2}$ connected components, so $\frac{s-g}{2} \leq t$. Consequently $t = \frac{s-g}{2}$.

For every connected component $\Delta_G^i$ of $\Delta_G$ which is an edge, we have $\delta(\Delta_G^i)_{\{0,1\}} = 1$, while for every connected component $\Delta_G^i$ of $\Delta_G$ which is a 2-simplex, we have $\delta(\Delta_G^i)_{\{0,1\}} = 2$. Consequently,

$$\delta(\Delta_G)_{\{0,1\}} = g + 2 \frac{s-g}{2} = s,$$

i.e. $\delta(\Delta_G)_{\{0,1\}} = \mu(I_G)$, and therefore, from Theorem 2.4, $\text{bar}(I_G) = \mu(I_G)$.

(2) Let $F \subset I_G$ be a set of $G$-homogeneous polynomials which generate $I_G$ up to radical. Let $\Delta_G^i = \{E, E', \Delta_G^i\}$ be two connected components which are edges, and let $\mathcal{H}_i$ and $\mathcal{H}_j$, respectively, be the corresponding induced subgraphs. Let $E = \{k, l\}$ and $E' = \{p, q\}$; then $I_{\mathcal{H}_i} = (f_E)$, where $f_E = x_kx_l - x_px_q$. The cycle $\Gamma$ has 4 edges, namely $e_k$, $e_l$, $e_p$ and $e_q$. From Proposition 3.6 every nonzero $H \in I_{\mathcal{H}_i}$ is a polynomial in at least 4 variables, namely $x_k$, $x_l$, $x_p$ and $x_q$. We will prove that every nonzero polynomial $H$, which belongs to $I_{\mathcal{H}_i}$, does not belong to $I_{\mathcal{H}_j}$. Assume that there is a nonzero polynomial $H \in I_{\mathcal{H}_i}$ which belongs to $I_{\mathcal{H}_j}$. From Proposition 4.13 in [11], we have that $I_{\mathcal{H}_j} = I_G \cap K[x_i^e_i \in E(\mathcal{H}_j)]$. But $H$ belongs to $I_{\mathcal{H}_i}$, so $H$ is a polynomial in the ring $K[x_i^e_i \in E(\mathcal{H}_i)]$, and therefore every edge of $\Gamma$ is also an edge of $\mathcal{H}_j$. Thus the indispensable binomial $f_E$ belongs to $I_{\mathcal{H}_j}$, and therefore, from Proposition 3.4 (1), we have that $\{E, E'\}$ is a connected component of $\Delta_G^i$, a contradiction. Given a connected component of $\Delta_G$, which is an edge, and the corresponding induced subgraph $\mathcal{H}$ of $G$, there exists, from Lemma 3.7, at least one $G$-homogeneous polynomial $F \in I_{\mathcal{H}}$ in $F$. The simplicial complex $\Delta_G$ has $g$ connected components which are edges, so $F$ has at least $g$ $G$-homogeneous polynomials, say $F_1, \ldots, F_g$, belonging to the corresponding toric ideals $I_{\mathcal{H}_i}$, $1 \leq i \leq g$. 

**ARITHMETICAL RANK OF TORIC IDEALS ASSOCIATED TO GRAPHS 11**
Remark that if $G$ has a complete subgraph $K_4$, then every polynomial $F_i$, $1 \leq i \leq g$, does not belong to the toric ideal $I_{K_4}$, since $I_{K_4} = I_G \cap K[x_r | e_r \in E(H_i)]$, and every $H_i$ is not a complete graph.

Let $\Delta^p_G$, $\Delta^q_G$ be two connected components which are 2-simplices, and let $K_{4,p}$ and $K_{4,q}$, respectively, the corresponding induced subgraphs. We will prove that every nonzero polynomial $H$, which is in the ideal $I_{K_{4,p}}$, does not belong to $I_{K_{4,q}}$. Let $I_{K_{4,p}} = (f_1, f_2)$, where $f_1 = x_i x_{i_2} - x_i x_i$ and $f_2 = x_i x_{i_2} - x_i x_i$. Assume that $H \in I_{K_{4,p}}$ is a nonzero polynomial which belongs to $I_{K_{4,q}}$. Since $H$ belongs to $I_{K_{4,p}}$, every monomial of $H$ is of the form $C x_i x_{i_2}$ or $N x_i x_i$ or $Q x_i x_i$, for appropriate monomials $C, N$ and $Q$. But $I_{K_{4,q}} = I_G \cap K[x_j | e_j \in E(K_{4,q})]$, so $K_{4,q}$ has at least 2 edges coming from $I_{K_{4,p}}$. These edges are $e_i$, $e_i$, $e_i$, and $e_i$. Notice that the edges $e_i$, $e_i$, $e_i$, and $e_i$ have no common vertex. The same holds for $e_i$, $e_i$, and $e_i$. But $K_{4,q}$ is a complete graph, so $K_{4,p} = K_{4,q}$, a contradiction.

Given a connected component of $\Delta_G$, which is a 2-simplex, and the corresponding induced subgraph $K_4$ of $G$, there exist, from Lemma 3.7, at least two $G$-homogeneous polynomial $H_1, H_2 \in I_{K_4}$ in $F$. The simplicial complex $\Delta_G$ has $\frac{s - g}{2}$ connected components which are 2-simplices, so $F$ has also at least $2 \frac{s - g}{2} = s - g$ polynomials, which are $G$-homogeneous, say $H_1, \ldots, H_{s-g}$, belonging to the corresponding toric ideals $I_{K_{4,i}}$, $1 \leq i \leq s - g$. Thus

$$\text{ara}_G(I_G) \geq g + (s - g) = s;$$

i.e. $\text{ara}_G(I_G) \geq \mu(I_G)$, and therefore $\text{ara}_G(I_G) = \mu(I_G)$. \hfill \Box

Example 3.9. Consider the complete graph $K_n$, $n \geq 4$, on the vertex set $\{v_1, \ldots, v_n\}$. We consider one variable $x_{ij}$, $1 \leq i < j \leq n$, for each edge $\{v_i, v_j\}$ of $K_n$ and form the polynomial ring $K[x_{ij} | 1 \leq i < j \leq n]$. The toric ideal $I_{K_n}$ is the kernel of the $K$-algebra homomorphism

$$\phi : K[x_{ij} | 1 \leq i < j \leq n] \rightarrow K[t_1, \ldots, t_n]$$

given by

$$\phi(x_{ij}) = t_it_j.$$

From Proposition 3.2 in [12] the height of $I_{K_n}$ equals $\binom{n}{2} - n = \frac{n(n-3)}{2}$, i.e. the number of edges minus the number of vertices. It is well known (see for example Proposition 9.2.1 in [13]) that

$$B = \{x_{ij}x_{kl} - x_{il}x_{jk}, x_{ik}x_{jl} - x_{il}x_{jk} | 1 \leq i < j < k < l \leq n\}$$

is a minimal generating set for $I_{K_n}$. The toric ideal $I_{K_n}$ has no indispensable binomials, and therefore every connected component of $\Delta_{K_n}$ is a 2-simplex. Thus $\Delta_{K_n}$ has $3\binom{n}{4}$ vertices and $\binom{n}{4}$ connected components, which are 2-simplices, corresponding to all complete subgraphs of $K_n$ of order 4. For the minimal number of generators we have that

$$\mu(I_{K_n}) = 2 \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{12}.$$

Consequently

$$\text{bar}(I_{K_n}) = \text{ara}_{K_n}(I_{K_n}) = \frac{n(n-1)(n-2)(n-3)}{12}.$$
Using the result of Eisenbud-Evans and Storch that \( \text{ara}(I_{K_n}) \) is bounded above by the number of variables of \( K[x_{ij} | 1 \leq i < j \leq n] \), we get that
\[
\frac{n(n-3)}{2} \leq \text{ara}(I_{K_n}) \leq \frac{n(n-1)}{2}.
\]

For the polynomials which minimally generate \( I_{K_n} \) up to radical, we know, from Theorem 5.8 in [5], that there must be at least \( 3\binom{n}{4} \) monomials in at least \( 2\binom{n}{4} \) \( K_n \)-homogeneous components.

Acknowledgment

The author would like to thank the referee for helpful comments.

References


Department of Mathematics, University of the Aegean, 83200 Karlovassi, Samos, Greece

E-mail address: katsabek@aegean.gr