SYMMETRY OF BOUND AND ANTIBOUND STATES
IN THE SEMICLASSICAL LIMIT
FOR A GENERAL CLASS OF POTENTIALS

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Abstract. We consider the Schrödinger operator $-\hbar^2 \partial_x^2 + V(x)$ on a half-line, where $V$ is a compactly supported potential which is positive near the endpoint of its support. We prove that the eigenvalues and the purely imaginary resonances are symmetric up to an error $Ce^{-\delta/h}$.

1. Introduction

In this paper, we study spectral properties of the Schrödinger operator

$$P(h) = -\hbar^2 \partial_x^2 + V(x)$$

defined for $x$ in the half-line $(-\infty, B]$. Here $h > 0$ is the semiclassical parameter and $V(x)$ is a piecewise continuous real-valued potential supported in $[0, B]$.

The operator $P(h)$ with the Neumann boundary condition at $B$ is self-adjoint on $L^2(-\infty, B)$; therefore, its resolvent

$$R_V(\lambda) = (P(h) - \lambda^2)^{-1}, \quad \text{Im} \lambda > 0,$$

is a bounded operator from $L^2$ to $H^2$ for $\lambda^2$ not in the spectrum of $P(h)$. This resolvent can be extended meromorphically as an operator $L^2_{\text{comp}} \rightarrow H^2_{\text{loc}}$ to $\lambda \in \mathbb{C}$ with isolated poles of finite rank; these poles are called resonances. The reader is referred to [12] for details.) To each resonance $\lambda$ corresponds a resonant state; that is, a nonzero $u \in H^2_{\text{loc}}(-\infty, B)$ solving the equation $(P(h) - \lambda^2)u = 0$ with the Neumann boundary condition at the right endpoint and with the following outgoing condition at $-\infty$:

$$u(x) = Ae^{-i\lambda x/h} \text{ for all } x < 0 \text{ and some constant } A.$$

(Note that for $x < 0$, $u$ solves the free equation $(-\hbar^2 \partial_x^2 - \lambda^2)u = 0$, so it must be a linear combination of $e^{\pm i\lambda x/h}$.)

For $\text{Im} \lambda > 0$, the outgoing condition implies that $u$ is exponentially decreasing on the negative half-line and thus $u \in L^2$; therefore, $\lambda$ is a pole (of the resolvent) lying in the upper half-plane if and only if $\lambda^2$ is an eigenvalue of $P(h)$ on $L^2$. Since $P(h)$ is self-adjoint, all poles in the upper half-plane have to lie on the imaginary axis. There may be poles $\lambda$ with $\text{Im} \lambda < 0$ and $\text{Re} \lambda \neq 0$; however, we will restrict our attention to purely imaginary resonances:

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Definition 1. A positive number $k$ is called a bound state if $ik$ is a pole of the resolvent $R_V(\lambda)$, and an antibound state if $-ik$ is a pole.

We see from above that $k$ is an (anti)bound state if and only if there exists a nonzero solution $u$ of the problem
\begin{align*}
(P(h) + k^2)u &= 0, \quad (1) \\
ux|_{x=B} &= 0, \quad (2) \\
hux \pm kux|_{x=0} &= 0. \quad (3)
\end{align*}
The plus sign in (3) corresponds to an antibound state, and the minus sign corresponds to a bound state. We will also study Neumann eigenvalues of $P(h)$ on $[0, B]$, i.e., those $k$ for which there exists a nonzero solution to (1) with boundary conditions (2) and
\begin{equation}
ux|_{x=0} = 0. \quad (4)
\end{equation}

Since the space of solutions to (1) and (2) is always one dimensional, bound states, antibound states, and Neumann eigenvalues never coincide. However, Bindel and Zworski proved in [3] that bound and antibound states located away from zero coincide, modulo errors of order $e^{-\delta/h}$ for some $\delta > 0$, if the potential satisfies the following conditions:
\begin{align*}
\exists A > 0, V_0 > 0: V(x) &= V_0 \text{ for all } x \in [0, A], \\
\exists \epsilon > 0: V(x) &= 0 \text{ for all } x \in (A, A + \epsilon).
\end{align*}

In this paper, we prove a similar result with more general assumptions on the potential:

Theorem 1. Suppose that $V$ is a piecewise continuous real-valued potential supported in $[0, B]$ and satisfying the following bump condition:
\begin{equation}
\exists A > 0: V(x) > 0 \text{ for all } x \in (0, A]. \quad (5)
\end{equation}

Fix two constants $0 < c_k < C_k < \infty$. Then there exist constants $C, \delta > 0, h_0 > 0$ such that for $h < h_0$ and any $k \in [c_k, C_k]$:
\begin{enumerate}
\item If $k$ is a Neumann eigenvalue, then there exist a bound state $k_+$ and an antibound state $k_-$ such that $|k - k_\pm| \leq Ce^{-\delta/h}$.
\item If $k$ is a bound or an antibound state, then there exists a Neumann eigenvalue $k_0$ such that $|k - k_0| \leq Ce^{-\delta/h}$.
\end{enumerate}

The bump condition (5) cannot be disposed of completely, as illustrated by the numerical experiments performed using [2]. Figure 1 shows two potentials on the whole line, each supported in $[-2, 2]$, and the corresponding bound states (denoted by squares) and antibound states (denoted by circles). The vertical coordinate of each (anti)bound state on the picture corresponds to its value $k$; the horizontal coordinate corresponds to the value of $h^{-1}$ used. We see that the conclusion of the theorem does not appear to hold for the potential on the left, which does not satisfy the bump condition; at the same time, it is true for the potential on the right. Theorem 1, formulated for the half-line case, applies to these numerical experiments on the whole line since for even potentials, the set of their (anti)bound states is composed of these states for the positive half-line with Dirichlet condition and the same states for the Neumann condition; the theorem above can be applied with
The study of resonances in one dimension has a long tradition going back to the origins of quantum mechanics; see for instance [8]. One of the first studies of their distribution was conducted by Regge [10]; since then, there have been many mathematical results on the topic, including [1], [4], [5], [6], [7], [9], [11], and [13]. Concerning antibound states, Hitrik has shown in [6] that for a positive compactly supported potential, there are no antibound states in the semiclassical limit. This agrees with our result since there are no bound states in this case. Simon proved in [11] that between any two bound states, there must be an odd Dirichlet condition in place of (2). (However, condition (1) cannot be replaced by the Dirichlet condition in the theorem.)

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Figure 1. Bound and antibound states for two spline potentials (splinepot([0, -0.4, -1, -0.2, -1, -0.4, 0], [-2, -1.5, -1, 0, 1, 1.5, 2]) and splinepot([0, 0.2, -1, -0.2, -1, 0.2, 0], [-2, 1.5, -1, 0, 1, 1.5, 2]))
number of antibound states; the following corollary of this result follows almost immediately using the methods we develop to prove Theorem 1:

**Theorem 2.** Consider the half-line problem with a bounded compactly supported potential $V$ (which does not need to satisfy any positivity condition). Then for each two bound states $0 < k_1 < k_2$, the interval $(k_1, k_2)$ contains at least one antibound state. In particular, if there are $n$ bound states in some subinterval of $(0, \infty)$, then there are at least $n - 1$ antibound states in the same subinterval.

The proof of Theorem 1 works as follows: we study the evolution (in $x$) of the vectors $(u, hu_x)$ for the three solutions of (1) with initial data at $x = 0$ satisfying the conditions (3) and (4). The idea is to look at these three vectors at $x = A$. Since $V(x) + k^2 \geq 0$ on the interval $(0, A)$, the transition matrix for the considered vectors from $x = 0$ to $x = A$ will have an expanding and a contracting direction. (In fact, if we introduce rescaling $\tilde{x} = x/h$, then the behavior of the original system for small $h$ is similar to the behavior of the rescaled system for large $\tilde{x}$, and the latter will be similar to the behavior of the geodesic flow on a two-dimensional manifold of negative curvature.) It turns out that our three vectors lie in a certain angle between the expanding and the contracting directions, from which it follows that they will stay in this angle for later times (Lemma 2); what is more, their polar angles will get exponentially close to each other (Lemma 7). Finally, we can study how the polar angles of the considered vectors change with $k$ (Lemma 4); it follows (Lemma 8) that the polar angle for the solution with Neumann initial data at $x = 0$ will strictly increase in $k$ and the polar angle for the solution with the same data at $x = B$ will decrease in $k$. The proof is then completed by a pertrubation argument (Lemma 5).

The detailed proofs of Theorems 1 and 2 are given in Section 3. Both are elementary and use certain properties of ordinary differential equations presented in Section 2.

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## 2. Preliminaries

Throughout this section, $I$ is an interval in $\mathbb{R}$, $V(x) \in L^\infty(I; \mathbb{R})$, $u(x), v(x) \in H^2(I; \mathbb{R})$, $h > 0$, and $P(h) = -h^2 \partial_x^2 + V(x)$. Any solution to the equation $P(h)u = 0$ is determined by the vector $(u, hu_x)$ at any $x$; we will sometimes view this vector in polar coordinates:

**Definition 2.** Define the length $L(u)$ and the **polar angle** $\theta(u)$ by the equations

\[
\begin{align*}
    u &= L(u) \cos \theta(u), \\
    hu_x &= L(u) \sin \theta(u).
\end{align*}
\]

Here $\theta(u)$ lies in the circle $S^1 = \mathbb{R}/(2\pi \mathbb{Z})$.

**Lemma 1.** Define the **Wronskian** $W(u, v)$ by

\[ W(u, v) = h(uv_x - vu_x). \]

Then

\[
\begin{align*}
    W(u, v) &= L(u)L(v) \sin(\theta(v) - \theta(u)), \\
    h\partial_x W(u, v) &= v \cdot P(h)u - u \cdot P(h)v.
\end{align*}
\]
Note that $W(u, v)$ is just the oriented area of the parallelogram spanned by the vectors $(u, hu_x)$ and $(v, hu_x)$. The next lemma tells us that if the vector $(u, hu_x)$ falls inside a certain angle in the plane at the initial time, then it will stay inside that angle for all later times:

**Lemma 2.** Suppose that $a^2 \leq V(x) \leq b^2$ for all $x \in I$ and some constants $a, b > 0$. Let $u$ be a solution to $P(h)u = 0$ and define

$$W_+(u) = W(u, e^{bz/h}), \quad W_-(u) = W(e^{-ax/h}, u).$$

Let $x_0$ be a point in $I$ and assume that $W_+(u), W_-(u) \geq 0$ at $x_0$. Then for $x \geq x_0$, the functions $W_\pm(u)$ are increasing in $x$ and

$$u \geq \frac{L(u)}{\sqrt{1 + b^2}}.$$

**Proof.** We have

$$e^{-bx/h}W_+(u) = bu - hu_x, \quad e^{ax/h}W_-(u) = au + hu_x.$$ 

Therefore, $W_+(u), W_-(u) \geq 0$ yields $|hu_x| \leq bu$ and thus (5). Next,

$$P(h)e^{bx/h} = e^{bx/h}(V(x) - b^2) \leq 0,$$

$$P(h)e^{-ax/h} = e^{-ax/h}(V(x) - a^2) \geq 0.$$

Using (7), we see that $\partial_x W_\pm \geq 0$ as long as $u \geq 0$. It remains to prove that $u(x) \geq 0$ for $x \geq x_0$. Suppose this is false and let $x_1 = \inf\{x \geq x_0 \mid u(x) < 0\}$. Then $u$ is not identically zero; since it solves a second order linear ODE, $L(u) > 0$ everywhere. But $u \geq 0$ on $[x_0, x_1]$, so $W_\pm$ are increasing on this interval. In particular, $W_\pm \geq 0$ at $x_1$ and thus (8) holds at this point. However, by the choice of $x_1$ we have $u(x_1) = 0$, which contradicts $L(u) > 0$. \hfill $\Box$

In the next section, we will use the following crude estimate on how fast the solutions of an ODE can grow:

**Lemma 3.** Assume that $|V(x)| \leq M$ for $x \in I$ and that $u$ is a solution to $P(h)u = 0$. Take $x_0, x_1 \in I$; then

$$L(u)|_{x = x_1} \leq (1 + M)|x_0 - x_1|/(2h) \cdot L(u)|_{x = x_0}.$$

**Proof.** Without loss of generality we may assume that $x_1 > x_0$. We have $L(u)^2 = u^2 + (hu_x)^2$; thus

$$h\partial_x(L(u)^2) = 2hu_x(1 + V(x)) \leq (1 + M)L(u)^2,$$

and the lemma is proven by Gronwall’s inequality. \hfill $\Box$

**Lemma 4.** Assume that $u(x, k)$ is a family of solutions to $(P(h) + k^2)u = 0$, $x_0, x_1 \in I$, and $u(x_0, k)$ and $u_x(x_0, k)$ are independent of $k$. Let $\Theta_1(k) = \Theta(u(x, k))|_{x = x_1}, \quad L_1(k) = L(u(x, k))|_{x = x_1}$. Then

$$\Theta_1'(k) = \frac{2k}{hL_1(k)^2} \int_{x_0}^{x_1} u(x, k)^2 \, dx.$$

**Proof.** We have $W(u, u_k)|_{x = x_1} = L_1(k)^2\Theta_1'$. (To see that, differentiate the formulas in Definition 2 in $k$ and use the definition of the Wronskian.) Now, we differentiate the equation $(P(h) + k^2)u = 0$ in $k$ to get $(P(h) + k^2)u_k = -2ku$. It remains to apply (7) together with $W(u, u_k)|_{x = x_0} = 0$. \hfill $\Box$
Lemma 5. Assume that $\Phi$ is a $C^1$ map from the interval $I = [K_0, K_1]$ to the circle $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ and $\Phi'(k) \geq \delta > 0$ for all $k \in I$. Suppose that $\Psi : I \to S^1$ is a continuous map such that $|\Psi(k)| \leq \varepsilon < \pi$ for all $k$. Put $\nu = \varepsilon/\delta$ and $I_\nu = [K_0 + \nu, K_1 - \nu]$. Then:

1. If $k_0 \in I_\nu$ has $\Phi(k_0) = 0$, then there exists $k_1 \in I$ with $\Phi(k_1) = \Psi(k_1)$ and $|k_0 - k_1| \leq \nu$.
2. If $k_1 \in I_\nu$ has $\Phi(k_1) = \Psi(k_1)$, then there exists $k_0 \in I$ with $\Phi(k_0) = 0$ and $|k_0 - k_1| \leq \nu$.

Proof. We can lift $\Phi$ and $\Psi$ to continuous maps $\bar{\Phi}, \bar{\Psi} : I \to \mathbb{R}$; then $|\bar{\Psi}| \leq \varepsilon$ and $\bar{\Phi}(k') - \bar{\Phi}(k) \geq \delta(k' - k)$ for $k' \geq k$.

1. We have $\bar{\Phi}(k_0) = 2\pi m$ for some $m \in \mathbb{Z}$. Then $\bar{\Phi}(k_0 + \nu) \geq 2\pi m + \delta \nu \geq 2\pi m + \bar{\Psi}(k_0 + \nu)$ and $\bar{\Phi}(k_0 - \nu) \leq 2\pi m + \bar{\Psi}(k_0 - \nu)$; it remains to apply the intermediate value theorem.

2. Similar to the previous statement, we have $\bar{\Phi}(k_1) = 2\pi m + \bar{\Psi}(k_1)$ for some $m \in \mathbb{Z}$ and $\bar{\Phi}(k_1 + \nu) \geq 2\pi m \geq \bar{\Phi}(k_1 - \nu)$. 

Lemma 6. Assume that $\Phi$ is a $C^1$ map from some interval $I$ to the circle $S^1 = \mathbb{R}/(2\pi \mathbb{Z})$ with $\Phi'(k) > 0$ for all $k \in I$. Let $\Psi : I \to S^1$ be a continuous map such that $\Psi(k) \neq 0$ for all $k \in I$. If $k_1 < k_2$ are two roots of the equation $\Phi = 0$, then the interval $(k_1, k_2)$ contains at least one root of the equation $\Phi = \Psi$.

Proof. As in the previous lemma, lift $\Phi$ and $\Psi$ to maps $\bar{\Phi}, \bar{\Psi} : I \to \mathbb{R}$; we can make $0 < \Psi(k) < 2\pi$ for all $k \in I$. Since $\Phi' > 0$ everywhere, we have $\bar{\Phi}(k_j) = 2\pi m_j$, where $m_1 < m_2$ are some integers. Therefore, $\bar{\Phi}(k_1) < 2\pi m_1 + \bar{\Psi}(k_1)$ and $\bar{\Phi}(k_2) > 2\pi m_1 + \bar{\Psi}(k_2)$; it remains to apply the intermediate value theorem.

3. Proofs of the theorems

We assume in this section that $0 < c_k' < k \leq C_k'$ for some constants $c_k' < c_k$ and $C_k' > C_k$; the constants in our estimates will depend on $c_k'$ and $C_k'$. (We need to expand the interval $[c_k, C_k]$ a little bit to be able to apply Lemma 5.)

Consider the solutions $u_\pm, u_0, u_1(x, k)$ to the equation (1) in $[0, B]$ with the initial data

$$u_{\pm 0}(0, k) = 1, \quad \partial_x u_0(0, k) = 0, \quad h \partial_x u_{\pm}(0, k) = \pm k,$$

$$u_1(B, k) = 1, \quad \partial_x u_1(B, k) = 0.$$

Define $\Theta_0(k), \Theta_\pm(k)$, and $\Theta_1(k)$ to be the polar angles of vectors $(u, hu_x)$ at $x = A$ for $u = u_0, u_\pm, u_1$. Then $k > 0$ is

- a Neumann eigenvalue if $u_0$ and $u_1$ are linearly dependent; that is (recalling that they solve the same second order ODE), if $2(\Theta_0(k) - \Theta_1(k)) = 0$;
- a bound state if $2(\Theta_+(k) - \Theta_1(k)) = 0$;
- an antibound state if $2(\Theta_-(k) - \Theta_1(k)) = 0$.

Here we count angles modulo $2\pi$.

To prove Theorem 1 it suffices to use Lemma 5 (for $\Phi = 2(\Theta_0 - \Theta_1)$ and $\Psi = 2(\Theta_0 - \Theta_\pm)$) together with the following two facts:

Lemma 7. For some constants $C_1$ and $\delta_1 > 0$ independent of $h$ and $k$,

$$|2(\Theta_0(k) - \Theta_\pm(k))| \leq C_1 e^{-\delta_1/h} \text{ for all } k \in [c_k', C_k'].$$

Lemma 8. We have $\Theta_0'(k) - \Theta_1'(k) \geq 1/C_2 > 0$ for all $k \in [c_k', C_k']$ and some constant $C_2$ independent of $h$ and $k$. 

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We first prove Lemma 1. Put \( b = \max_{[0,A]} V(x) \), \( k_b = \sqrt{k^2 + b} \), \( \psi_0(x) = e^{-kx/h} \), \( \psi_+(x) = e^{kx/h} \), and consider the Wronskians
\[
W_+(u) = W(u, \psi_+), \quad W_0(u) = W(\psi_0, u).
\]
These are nonnegative for \( u = u_0, u_\pm \) at \( x = 0 \). Then by Lemma 2 all these six functions are nonnegative and increasing in \( x \) for \( 0 \leq x \leq A \).

Our first goal is to get an exponential lower bound on the length \( L(u) \) for \( u = u_0, u_\pm \) at \( x = A \). For \( u_0 \), note that by (4)
\[
L(u_0) \geq \frac{W(\psi_0, u_0)}{L(\psi_0)} \geq \frac{W_0(u_0)|_{x=0}}{L(\psi_0)} \geq \frac{1}{C} e^{kx/h}.
\]
The same applies to \( u_+ \). However, \( u_- \) needs more careful analysis since \( W_0(u_-) = 0 \) at \( x = 0 \). For that, take \( 0 < t < 1 \) and put \( a = \min_{[t,A]} V(x) > 0 \), \( k_a = \sqrt{k^2 + a} \), \( \psi_-(x) = e^{-kax/h} \), and \( W_-(u) = W(\psi_-, u) \). First, we have by Lemma 3
\[
L(u_-) \geq e^{-(1+k^2+b)x/(2h)} . \ L(u_-)|_{x=0}.
\]
Next, \( W_0(u_-) \geq 0 \) and \( W_+(u_-) \geq 0 \), so by (3)
\[
W_-(u_-) \geq (k_a - k)u_- \psi_- \geq \frac{1}{C} L(u_-) \psi_-
\]
Finally, we apply Lemma 2 on the interval \([A,A]\) to get
\[
L(u_-)|_{x=A} \geq \frac{W_-(u_-)|_{x=tA}}{L(\psi_-)|_{x=A}} \geq \frac{1}{C} e^{(k(1-t)-(1+k^2+b)t)A/h}.
\]
For \( t \) small enough and all \( k \), \( k(1-t)-(1+k^2+b)t \geq 0 \), so we have
\[
L(u_-)|_{x=A} \geq \frac{1}{C} > 0.
\]
The next step is to use that \( u_0 \) and \( u_\pm \) solve the same equation (1) and thus \( W(u_0, u_\pm) \) is constant in \( x \). Therefore, at \( x = A \) we have by (0)
\[
|\sin(\theta(u_\pm) - \theta(u_0))| = \frac{|W(u_0, u_\pm)|}{L(u_0)L(u_\pm)} \leq Ce^{-kA/h}.
\]
That finishes the proof of Lemma 7.

To prove Lemma 8 first note that by Lemma 3 \( \Theta'(k) \leq 0 \) and
\[
\Theta'_0(k) \geq \frac{1}{ChL(u_0)^2}|_{x=A} \int_0^A |u_0(x,k)|^2 dx.
\]
By (8), \( u_0 \geq L(u_0)/C \). Also, by Lemma 3 \( L(u_0) \geq e^{C(x-A)/h}L(u_0)|_{x=A} \) for \( 0 \leq x \leq A \); thus
\[
\int_0^A |u_0(x,k)|^2 dx \geq \frac{1}{C} \int_0^A e^{C(x-A)/h}(L(u_0)^2|_{x=A}) dx \geq \frac{h}{C} L(u_0)^2|_{x=A}
\]
and Lemma 3 is proven, which finishes the proof of Theorem 1.

To prove Theorem 2 let \( \Phi_\pm(k) = \theta(u_\pm)|_{x=B} \); a bound state corresponds to \( 2\Phi_+ = 0 \) and an antibound state corresponds to \( 2\Phi_- = 0 \). Since \( \theta(u_\pm)|_{x=0} \) is increasing with \( k \), by an argument similar to the proof of Lemma 3 we get \( \Phi'_+(k) > 0 \) for all \( k \). Moreover, \( 2(\Phi_+(k) - \Phi_-(k)) \) is never zero, as this would correspond to \( u_+ \) and \( u_- \) being linearly dependent. We may now apply Lemma 5 with \( \Phi = 2\Phi_+ \) and \( \Psi = 2(\Phi_+ - \Phi_-) \).
References


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