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SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITIES IN THE L^1 MINKOWSKI PLANE

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Abstract. An isoperimetric inequality bounds from below the perimeter of a domain in terms of its area. A *quantitative* isoperimetric inequality is a stability result: it bounds from above the distance to an isoperimetric minimizer in terms of the isoperimetric deficit. In other words, it measures how close to a minimizer an almost optimal set must be.

The euclidean quantitative isoperimetric inequality has been thoroughly studied, in particular by Hall and by Fusco, Maggi and Pratelli, but the L^1 case has drawn much less attention.

In this note we prove two quantitative isoperimetric inequalities in the L^1 Minkowski plane with sharp constants and determine the extremal domains for one of them. It is usually (but not here) difficult to determine the extremal domains for a quantitative isoperimetric inequality: the only such known result is for the euclidean plane, due to Alvino, Ferone and Nitsch.

1. Statement of the results

We consider the plane $\mathbb{R}^2$ endowed with the L^1 metric

$$|(x_1, x_2) - (y_1, y_2)| = |x_1 - y_1| + |x_2 - y_2|.$$

We denote the boundary of a set by ∂. The notation $|\cdot|$ shall be used to denote the size of an object, whatever its nature. If A is a measurable plane set, then $|A|$ is its Lebesgue measure, also called its area; if v is a vector, $|v|$ is its L^1 norm; if γ is a curve, $|\gamma|$ is its L^1 length, namely

$$|\gamma| = \sup \sum_{i=1}^{n-1} |\gamma(t_i) - \gamma(t_{i+1})|$$

where the supremum is over all increasing sequences $(t_1, t_2, \ldots, t_n)$ taking values in the definition interval of γ. A curve is said to be (L^1-)rectifiable if its length is finite. For example, a curve defined on a segment whose coordinate functions are both monotonic is rectifiable, and its length is the distance between its endpoints. See e.g. [AT04] for more information on the length of curves in a metric space.

By a *domain* of the plane, we mean the closure of the bounded component of a Jordan curve. In particular, domains are compact and connected. All rectangles and squares considered are assumed to have their sides parallel to the coordinate

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axes. The square centered at 0 with side length 2λ is denoted by $B_\infty(\lambda)$: it is the λ ball of the L^∞ metric. Squares are known to minimize L^1 perimeter among plane domains of a given area; in particular, since the L^1 perimeter of a unit-area square is 4, the following isoperimetric inequality holds for all domains: $|\partial A| \geqslant 4|A|^{1/2}$.

Note that this kind of isoperimetric inequality can be recast as the problem of minimizing a boundary energy given by the integral over the boundary of the domain of a function defined on the set of possible tangent directions. The minimizing shape can be explicitly constructed and is known as the Wulff shape (or Wulff crystal, since it is often a polyhedron) of the integrand; see e.g. [Tay78, BM94].

The measure of the distance between compact plane sets A and B that we use in our main result is the L^∞ Hausdorff metric:

$$d_\infty(A,B) = \inf\{\lambda \geqslant 0 \mid A \subset B + B_\infty(\lambda) \text{ and } B \subset A + B_\infty(\lambda)\}.$$

Let us explain why this metric is natural here. One way to prove that almost isoperimetric domains are close to minimizers is to prove that they contain a minimizer of radius r and are included in another of radius R, with small difference of radii $R-r$ and the same center. In euclidean space, such inclusions imply that the domain under consideration is at Hausdorff distance at most $(R-r)/2$ from some ball; the Bonnesen inequality gives a bound on the possible values of $R-r$ in terms of the isoperimetric deficit, and implies results similar to ours in the euclidean case; see [Ber05, Bon29, Oss78]. However, balls and minimizers are different in the L^1 plane, so that if A is between concentric squares of radii R and r, one can only say that it is at L^1 Hausdorff distance $R-r$ from some square, while the bound on the L^∞ Hausdorff distance is the expected $(R-r)/2$.

It would certainly be possible to use the L^1 Hausdorff metric, and we expect that arguments of the same kind as those we use to prove Theorem 1, but more involved, would give a constant better than the 1/16 obtained using the inequality $d_1 \leqslant 2d_\infty$ and Theorem 1.

Theorem 1. *Let A be a domain of the L^1 Minkowski plane whose boundary is a rectifiable curve, and assume that*

$$|\partial A|^2 \leqslant (16+\varepsilon)|A|. \tag{1}$$

Then there is a square S such that

$$d_\infty(A,S)^2 \leqslant \frac{\varepsilon|A|}{64}. \tag{2}$$

We shall also see that Theorem 1 is sharp and show that up to L^1 isometry and homothety the domains that achieve the bound are rectangles and squares with one square deleted at a corner.

A second possible measure of the distance between domains of the same area, which presents the advantage of being suitable for higher dimensions as well, is simply the gap between their area and the area of their intersection. In this respect we prove the following.

Proposition 1. *Let A be a domain of the L^1 Minkowski plane whose boundary is a rectifiable curve, and assume that* (1) *holds with ε sufficiently small. Then there is a square S such that $|S| = |A|$ and*

$$|S \cap A| \geqslant \left(1 - \frac{\sqrt{\varepsilon}}{4} + O(\varepsilon)\right)|A|. \tag{3}$$

In terms of Fraenkel asymmetry, this reads:

$$\frac{|S\Delta A|}{|A|} \leqslant \frac{\sqrt{\varepsilon}}{2} + O(\varepsilon). \tag{4}$$

We shall see that the constant 1/2 in (4) is sharp.

Surprisingly enough, it seems that these results are new, although similar ones can be deduced from the much more general [FMP09] (but with a non-optimal constant) and the Bonnesen-like inequalities of [PWZ93] (but only when A is convex).

2. Proof of the inequalities

Assume that A satisfies (1) for some ε and let R be the smallest rectangle containing A. This rectangle plays the role of a convex hull.

Lemma 1. *We have* $|\partial A| \geqslant |\partial R|$.

Proof. Since R is minimal, each of its sides contains a point of the boundary of A. Denote such points by r_1, r_2, r_3, r_4 so that r_i and r_{i+1} lie on two adjacent sides of R for all i (modulo 4). It is possible that $r_i = r_{i+1}$ for some i, but this does not affect what follows.

There are four curves γ_i in ∂A that connect r_i to r_{i+1} and meet only at their endpoints (see Figure 1). Similarly, the boundary of R is made up of four curves η_i connecting r_i to r_{i+1}. Since R is a rectangle, the η_i are L^1 geodesics. The length of γ_i is at least $|r_i - r_{i+1}| = |\eta_i|$, so that

$$|\partial A| = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4| \geqslant |\eta_1| + |\eta_2| + |\eta_3| + |\eta_4| = |\partial R|.$$

□

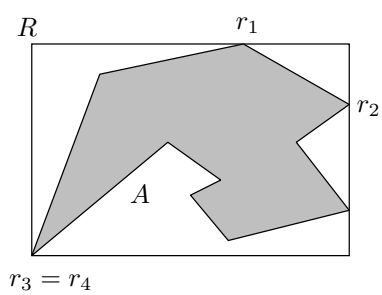

Figure 1. The L^1 perimeter of A is at least that of R.

Let ℓ and α be such that $\ell - 2\alpha$ and $\ell + 2\alpha$ are the side lengths of R. In other words, if ℓ_1 and ℓ_2 are respectively the smaller and greater side lengths of R, then $\ell = (\ell_1 + \ell_2)/2$ and $\alpha = (\ell_2 - \ell_1)/4$.

Lemma 2. *We have*

$$|A| \leqslant \ell^2 \leqslant \frac{16+\varepsilon}{16}|A| \tag{5}$$

and

$$\alpha^2 \leqslant \frac{\varepsilon|A|}{64}. \tag{6}$$

Proof. From the previous lemma we have $|\partial A| \geqslant 4\ell$, so that using (1) we get $16\ell^2 \leqslant (16+\varepsilon)|A|$. Since $A \subset R$, we have $|A| \leqslant |R| = \ell^2 - 4\alpha^2$ and (5) follows.

Next we have

$$\begin{aligned} 16\ell^2 &\leqslant (16+\varepsilon)(\ell^2 - 4\alpha^2), \\ 0 &\leqslant \varepsilon\ell^2 - 4(16+\varepsilon)\alpha^2, \\ \alpha^2 &\leqslant \frac{\varepsilon\ell^2}{4(16+\varepsilon)}, \\ \alpha^2 &\leqslant \frac{\varepsilon}{64}|A|, \end{aligned}$$

and we are done. □

Note that this lemma is sufficient for deducing the L^1 isoperimetric inequality and its equality case: if $\varepsilon = 0$, then $\alpha = 0$ and $|A| = \ell^2$.

2.1. **Proof of Theorem 1.** We have $\min_S d_\infty(A,S) \geqslant \alpha$ (where S runs over all squares; see Figure 2), and if there is equality, Lemma 2 is sufficient to conclude the result. We therefore assume $\delta := \min_S d_\infty(A,S) > \alpha$.

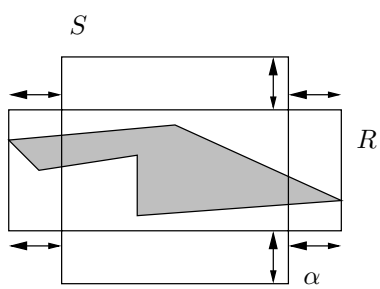

FIGURE 2. The closest square to R.

The following is the main step of the proof; it improves the inequality $|A| \leqslant \ell^2 - 4\alpha^2$ obtained in Lemma 2 by a term that is a function of δ.

Lemma 3. *We have either*

$$|A| \leqslant \ell^2 - 4\alpha^2 - 8\delta(\delta - \alpha)$$

or

$$|A| \leqslant \ell^2 - 4\alpha^2 - 4\delta^2.$$

Proof. Choose the origin so that R has its bottom side at height 0. Let S_η be the square that is at distance δ from each short side of R (in particular the centers of R and S_η have the same horizontal projection) and whose bottom side is at height η.

We consider negative values of η, more precisely $\eta \in [-\delta, 3\delta - 4\alpha]$. In this way, R (and thus A) is contained in an L^∞ neighborhood of size δ around S_η; thus there is some point $p_\eta \in S_\eta$ that is at L^∞ distance at least δ from A.

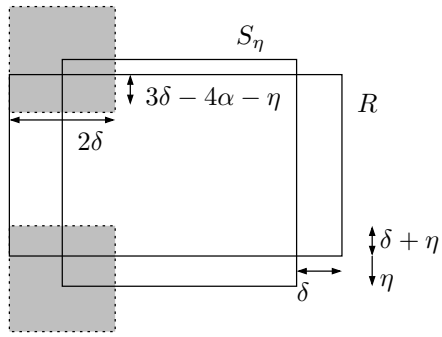

FIGURE 3. The domain A avoids one of the grey squares.

This precludes A from intersecting a square centered at p_η. Having in mind to bound A from above, we can directly assume the worst case that for each η, p_η is a corner of S_η, either always on its left side or always on its right side. Up to a horizontal reflection we can directly assume that p_η is either the upper or the lower left corner of S_η as in Figure 3.

If p_η is the lower left corner of S_η, then there is a $2\delta \times (\delta + \eta)$ sub-rectangle of R excluded. Otherwise, there is a $2\delta \times (3\delta - 4\alpha - \eta)$ sub-rectangle of R excluded. In computing these values we have assumed that $\eta < \delta$ and $3\delta - 4\alpha - \eta < \delta$ respectively, otherwise there is simply an excluded square of area $4\delta^2$.

Let x be the supremum of the η such that p_η is a lower corner. There is an excluded sub-rectangle of area $2\delta \times (\delta + x)$, and for all $\eta > x$ the point p_η must be a higher corner so that there is another excluded sub-rectangle of area $2\delta \times (3\delta - 4\alpha - x)$.

Summing up, either there are excluded sub-rectangles of total area at least $2\delta \times 4(\delta - \alpha)$ or there is an excluded sub-rectangle of area at least $4\delta^2$, and we get the desired bounds on $|A|$. $\square$

We can now conclude the proof of Theorem 1. First, if $|A| \leqslant (\ell^2 - 4\alpha^2) - 8\delta(\delta - \alpha)$, then we have

$$\begin{aligned}
|\partial A|^2 &\leqslant (16+\varepsilon)|A|, \\
(4\ell)^2 &\leqslant (16+\varepsilon)\big(\ell^2 - 4\alpha^2 - 8\delta(\delta-\alpha)\big), \\
0 &\leqslant \varepsilon\ell^2 - 4(16+\varepsilon)(\alpha^2 + 2\delta(\delta-\alpha)), \\
\alpha^2 + 2\delta(\delta-\alpha) &\leqslant \frac{\varepsilon\ell^2}{4(16+\varepsilon)}, \\
2\delta^2 - 2\alpha\delta + \alpha^2 &\leqslant \frac{\varepsilon}{64}|A|,
\end{aligned}$$

with the last inequality coming from (5).

Since the function $x \mapsto 2\delta^2 - 2x\delta + x^2$ is minimal when $x = \delta$, we have $2\delta^2 - 2\alpha\delta + \alpha^2 \geqslant 2\delta^2 - 2\delta^2 + \delta^2 = \delta^2$, so that $\delta^2 \leqslant \frac{\varepsilon}{64}|A|$.

In the case where $|A| \leqslant \ell^2 - 4\alpha^2 - 4\delta^2$, we get

$$\begin{aligned} |\partial A|^2 &\leqslant (16+\varepsilon)(\ell^2 - 4\alpha^2 - 4\delta^2), \\ 16\ell^2 &\leqslant 16\ell^2 + \varepsilon\ell^2 - 4(16+\varepsilon)(\alpha^2+\delta^2), \\ \alpha^2+\delta^2 &\leqslant \frac{\varepsilon\ell^2}{4(16+\varepsilon)}, \\ \delta^2 &\leqslant \frac{\varepsilon}{64}|A|. \end{aligned}$$

2.2. **Proof of Proposition 1.** Let $\mu = \max_S |S \cap A|/|A|$ where S runs over all squares having the same area as A.

Lemma 4. *One of the following holds:*

$$\begin{aligned} \mu &\geqslant 2 - \frac{\ell^2 - 4\alpha^2}{|A|}, \\ \mu &\geqslant 2 - \frac{\ell + 2\alpha}{\sqrt{|A|}}. \end{aligned}$$

Proof. Define S_0 to be a square that shares a corner of R and intersects its interior, and which has the same area as A (see Figure 4). The definition of μ implies that $|A \cap S_0| \leqslant \mu|A|$.

If $\sqrt{|A|} \leqslant \ell - 2\alpha$, we have

$$\begin{aligned} |A| &\leqslant |A \cap S_0| + |R \setminus S_0| \\ &\leqslant \mu|A| + (\ell+2\alpha)(\ell - 2\alpha - \sqrt{|A|}) + \sqrt{|A|}(\ell + 2\alpha - \sqrt{|A|}) \\ &\leqslant \mu|A| + \ell^2 - 4\alpha^2 - \sqrt{|A|}(\ell+2\alpha) + \sqrt{|A|}(\ell+2\alpha) - |A| \\ &\leqslant \ell^2 - 4\alpha^2 + (\mu - 1)|A| \end{aligned}$$

and then $\mu|A| \geqslant 2|A| - (\ell^2 - 4\alpha^2)$.

Otherwise, we get

$$\begin{aligned} |A| &\leqslant \mu|A| + (\ell - 2\alpha)(\ell + 2\alpha - \sqrt{|A|}) \\ &\leqslant \mu|A| + \sqrt{|A|}(\ell + 2\alpha - \sqrt{|A|}) \end{aligned}$$

and thus $\mu|A| \geqslant 2|A| - (\ell + 2\alpha)\sqrt{|A|}$.

□

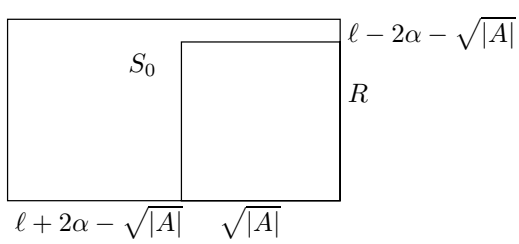

FIGURE 4. The domain A is included in R and cannot meet a too-large proportion of S_0.

We now prove Proposition 1. If the first conclusion in Lemma 4 holds, using Lemma 2 it follows that

$$\begin{aligned}\mu &\geqslant 2-\frac{(\ell^2-4\alpha^2)(16+\varepsilon)}{16\ell^2}\\ &\geqslant 2-\frac{16+\varepsilon}{16}=1-\frac{\varepsilon}{16}.\end{aligned}$$

If the second conclusion holds, using Lemma 2 we get

$$\begin{aligned}\mu &\geqslant 2-\frac{\ell}{\sqrt{|A|}}-\frac{2\alpha}{\sqrt{|A|}}\\ &\geqslant 2-\sqrt{1+\frac{\varepsilon}{16}}-\frac{\sqrt{\varepsilon}}{4}\\ &\geqslant 1-\frac{\sqrt{\varepsilon}}{4}+O(\varepsilon).\end{aligned}$$

But for all sufficiently small ε, this second expression is smaller than $1-\varepsilon/16$, and Proposition 1 is proved.

3. Sharpness

Two examples showing the sharpness of Theorem 1 stem from its proof.

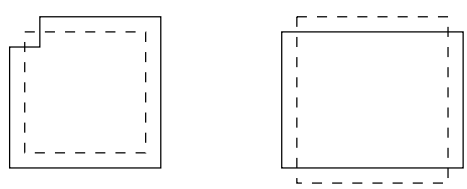

Figure 5. Two domains that are almost isoperimetric and as far as possible from squares: a square with a small corner deleted and a rectangle with sides of almost the same length.

The first one is the domain S'_δ obtained from the unit square by deleting a $2\delta\times 2\delta$ square from one corner ($\delta<1/2$). We have $|S'_\delta|=1-4\delta^2$ and $|\partial S'_\delta|=4$ so that (1) holds with

$$\varepsilon=\frac{64\delta^2}{1-4\delta^2},$$

and we have $\inf_S d_\infty(S,S'_\delta)=\delta$ so that equality holds in (2).

The second example is the rectangle R_α whose side lengths are $1-2\alpha$ and $1+2\alpha$ (where $\alpha<1/2$). We have $|R_\alpha|=1-4\alpha^2$, $|\partial R_\alpha|=4$ and $\inf_S d_\infty(S,R_\alpha)=\alpha$ so that (2) is an equality once again.

Let us show that S'_δ and R_α are the only possible (up to homothety and L^1 isometry) examples realizing equality in both (1) and (2) for the same ε. In the first case of Lemma 3, for $2\delta^2-2\alpha\delta+\alpha^2\leqslant\delta^2$ to be an equality it is necessary that $\alpha=\delta$, so A must be equal to R (otherwise R would have smaller isoperimetric deficit and the same distance to squares). In the second case of the lemma, one is led to $\alpha=0$ in the last lines of the proof of Theorem 1, so R is a square and, according to the proof of Lemma 3, A is contained in a S'_δ having the same isoperimetric deficit and the same minimal rectangle. They must therefore be equal.

Finally, R_α shows asymptotic sharpness of Proposition 1:

$$\sup_{|S|=|R_\alpha|} |S \cap R_\alpha| = (1-2\alpha)\sqrt{1-4\alpha^2} = 1 - 2\alpha + o(\alpha)$$

and

$$1 - \frac{1}{4}\sqrt{\varepsilon} = 1 - 2\alpha + o(\alpha)$$

when ε takes the extremal value $64\alpha^2/(1-4\alpha^2)$.

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