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SQUARE-MEAN ALMOST AUTOMORPHIC SOLUTIONS FOR SOME STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. The concept of square-mean almost automorphy for stochastic processes is introduced. The existence and uniqueness of square-mean almost automorphic solutions to some linear and non-linear stochastic differential equations are established provided the coefficients satisfy some conditions. The asymptotic stability of the unique square-mean almost automorphic solution in the square-mean sense is discussed.

1. Introduction

The concept of almost automorphy is a generalization of almost periodicity. It is introduced by Bochner [5] in relation to some aspects of differential geometry. Almost automorphic functions are characterized by the following property. Let f be a continuous function; given any sequence of real numbers $\{s_n'\}$, we can extract a subsequence $\{s_n\}$ such that for some function g

$$g(t) = \lim_{n\to\infty} f(t+s_n) \quad \text{and} \quad \lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$. Almost automorphy has been studied by many authors; see Veech [15, 16] for classical exposition; see Johnson [10], Shen and Yi [12], N'Guérékata [7] for recent developments, among others.

Recently, some authors have studied the almost periodic or pseudo almost periodic solutions to stochastic differential equations; see [2, 3, 4, 6, 9, 13, 14], among others. In this paper, we go one step further by introducing the concept of square-mean almost automorphic stochastic processes. Under some conditions of coefficients, we establish the existence and uniqueness of square-mean almost automorphic solutions for some stochastic differential equations.

The paper is organized as follows. In section 2, we introduce the notion of square-mean almost automorphic processes and study some of their basic properties. In sections 3 and 4, given some suitable conditions, we prove the existence and uniqueness of square-mean almost automorphic mild solutions to some linear and non-linear stochastic differential equations, respectively. In section 5, we discuss

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the asymptotic stability property of the unique square-mean almost automorphic solution in the square-mean sense.

2. Square-mean almost automorphic processes

Throughout this paper, we assume that $(\mathbb{H}, \|\cdot\|)$ is a real separable Hilbert space, $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, and $\mathcal{L}^2(\mathbf{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that

$$\mathbf{E}\|x\|^2 = \int_\Omega \|x\|^2 d\mathbf{P} < \infty.$$

For $x \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, let

$$\|x\|_2 := \left(\int_\Omega \|x\|^2 d\mathbf{P} \right)^{1/2}.$$

Then it is routine to check that $\mathcal{L}^2(\mathbf{P}, \mathbb{H})$ is a Hilbert space equipped with the norm $\|\cdot\|_2$.

Definition 2.1. A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ is said to be *stochastically continuous* if

$$\lim_{t \to s} \mathbf{E}\|X(t) - X(s)\|^2 = 0.$$

Definition 2.2. A stochastically continuous stochastic process $x : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ is said to be *square-mean almost automorphic* if every sequence of real numbers $\{s'_n\}$ has a subsequence $\{s_n\}$ such that for some stochastic process $y : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$

$$\lim_{n \to \infty} \mathbf{E}\|x(t + s_n) - y(t)\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbf{E}\|y(t - s_n) - x(t)\|^2 = 0$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ is denoted by $AA(\mathbb{R}; \mathcal{L}^2(\mathbf{P}, \mathbb{H}))$.

In the following lemma, we list some basic properties of square-mean almost automorphic stochastic processes.

Lemma 2.3. *If x, x_1 and x_2 are all square-mean almost automorphic stochastic processes, then*

(1) *$x_1 + x_2$ is square-mean almost automorphic.*
(2) *λx is square-mean almost automorphic for every scalar λ.*
(3) *There exists a constant $M > 0$ such that $\sup_{t \in \mathbb{R}} \|x(t)\|_2 \le M$. That is, x is bounded in $\mathcal{L}^2(\mathbf{P}, \mathbb{H})$.*

Proof. Since statements (1) and (2) are obvious, we only prove (3). If $\sup_{t \in \mathbb{R}} \|x(t)\|_2 = \infty$, then there exists a sequence of real numbers $\{s'_n\}$ such that

$$\lim_{n \to \infty} \|x(s'_n)\|_2^2 = \lim_{n \to \infty} \mathbf{E}\|x(s'_n)\|^2 = \infty.$$

Since $x \in AA(\mathbb{R}; \mathcal{L}^2(\mathbf{P}, \mathbb{H}))$, there exists a subsequence $\{s_n\} \subset \{s'_n\}$ and a stochastic process $y : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ such that

$$\lim_{n \to \infty} \mathbf{E}\|x(t + s_n) - y(t)\|^2 = 0 \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

In particular, when $t = 0$ in (2.1), it follows that

$$\lim_{n \to \infty} \mathbf{E}\|x(s_n)\|^2 < \infty,$$

a contradiction. ∎

Theorem 2.4. $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ *is a Banach space when it is equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}},$$

for $x \in AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$.

Proof. By Lemma 2.3, $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ is a vector space, so it is easy to verify that $\|\cdot\|_\infty$ is a norm on $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$. We only need to show that $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ is complete with respect to the norm $\|\cdot\|_\infty$. To this end, assume that $\{x_n\} \subset AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$ and that x is the pointwise limit of x_n with respect to $\|\cdot\|_2$; i.e.

$$\lim_{n\to\infty} \|x_n(t) - x(t)\|_2 = 0 \quad \text{for each } t \in \mathbb{R}. \tag{2.2}$$

Note that this limit x always exists by the completeness of $\mathcal{L}^2(\mathbf{P},\mathbb{H})$ with respect to $\|\cdot\|_2$. Since $\{x_n\}$ is Cauchy with respect to $\|\cdot\|_\infty$, the convergence in (2.2) is actually uniform for $t \in \mathbb{R}$. We need to show that $x \in AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$.

First, we verify that x is stochastically continuous. In fact, by

$$x(t+\triangle t) - x(t) = x(t+\triangle t) - x_n(t+\triangle t) + x_n(t+\triangle t) - x_n(t) + x_n(t) - x(t),$$

it follows that

$$\begin{aligned}&\mathbf{E}\|x(t+\triangle t) - x(t)\|^2\\ \le{}& 3\mathbf{E}\|x(t+\triangle t) - x_n(t+\triangle t)\|^2 + 3\mathbf{E}\|x_n(t+\triangle t) - x_n(t)\|^2 + 3\mathbf{E}\|x(t) - x_n(t)\|^2.\end{aligned}$$

By the uniform convergence of x_n to x with respect to $\|\cdot\|_2$ and the stochastic continuity of x_n, the stochastic continuity of x follows.

Next, we prove that x is square-mean almost automorphic. Let $\{s'_n\}$ be an arbitrary sequence of real numbers; then by standard diagonal progress, we can extract a subsequence $\{s_n\} \subset \{s'_n\}$ such that for stochastic processes $y_i : \mathbb{R} \to \mathcal{L}^2(\mathbf{P},\mathbb{H})$

$$\lim_{n\to\infty} \mathbf{E}\|x_i(t+s_n) - y_i(t)\|^2 = 0 \tag{2.3}$$

for each $t \in \mathbb{R}$ and $i = 1, 2, \cdots$.

We observe that, for each $t \in \mathbb{R}$, the sequence of $\{y_i(t)\}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbf{P},\mathbb{H})$. Indeed, if we write

$$y_i(t) - y_j(t) = y_i(t) - x_i(t+s_n) + x_i(t+s_n) - x_j(t+s_n) + x_j(t+s_n) - y_j(t),$$

then we get

$$\begin{aligned}\mathbf{E}\|y_i(t) - y_j(t)\|^2 &\\ &\le 3\mathbf{E}\|y_i(t) - x_i(t+s_n)\|^2 + 3\mathbf{E}\|x_i(t+s_n) - x_j(t+s_n)\|^2\\ &\quad + 3\mathbf{E}\|x_j(t+s_n) - y_j(t)\|^2.\end{aligned}$$

By (2.2) and (2.3), the sequence of $\{y_i(t)\}$ is Cauchy.

Using the completeness of the space $\mathcal{L}^2(\mathbf{P},\mathbb{H})$, we denote by $y(t)$ the pointwise limit of $\{y_i(t)\}$. Let us prove now that

$$\lim_{n\to\infty} \mathbf{E}\|x(t+s_n) - y(t)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\|y(t-s_n) - x(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Indeed, for each $i = 1, 2, \cdots$, we have

$$\begin{aligned}&\mathbf{E}\|x(t+s_n) - y(t)\|^2 \\ &\quad \leq 3\mathbf{E}\|x(t+s_n) - x_i(t+s_n)\|^2 + 3\mathbf{E}\|x_i(t+s_n) - y_i(t)\|^2 + 3\mathbf{E}\|y_i(t) - y(t)\|^2.\end{aligned}$$

By (2.2) and (2.3),

$$\lim_{n\to\infty} \mathbf{E}\|x(t+s_n) - y(t)\|^2 = 0$$

for each $t \in \mathbb{R}$.

We can use the same step to prove that

$$\lim_{n\to\infty} \mathbf{E}\|y(t-s_n) - x(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. That is, $x(t)$ is square-mean almost automorphic. The proof is complete. $\square$

Definition 2.5. A function $f : \mathbb{R} \times \mathcal{L}^2(\mathbf{P}, \mathbb{H}) \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, $(t, x) \mapsto f(t, x)$, which is jointly continuous, is said to be *square-mean almost automorphic* in $t \in \mathbb{R}$ for each $x \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that for some function $\widetilde{f}$

$$\lim_{n\to\infty} \mathbf{E}\|f(t+s_n, x) - \widetilde{f}(t, x)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\|\widetilde{f}(t-s_n, x) - f(t, x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$.

Theorem 2.6. *Let $f : \mathbb{R} \times \mathcal{L}^2(\mathbf{P}, \mathbb{H}) \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, $(t, x) \mapsto f(t, x)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, and assume that f satisfies the Lipschitz condition in the following sense:*

$$\mathbf{E}\|f(t, x) - f(t, y)\|^2 \leq L\mathbf{E}\|x - y\|^2$$

for all $x, y \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ and for each $t \in \mathbb{R}$, where $L > 0$ is independent of t. Then for any square-mean almost automorphic process $x : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, the stochastic process $F : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ given by $F(t) := f(t, x(t))$ is square-mean almost automorphic.

Proof. Let $\{s'_n\}$ be a sequence of real numbers. By the almost automorphy of f and x, we can extract a subsequence $\{s_n\}$ of $\{s'_n\}$ such that for some function $\widetilde{f}$ and for each $t \in \mathbb{R}$ and $x \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$,

$$\lim_{n\to\infty} \mathbf{E}\|f(t+s_n, x) - \widetilde{f}(t, x)\|^2 = 0, \tag{2.4}$$

and for some function y and for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{E}\|x(t+s_n) - y(t)\|^2 = 0. \tag{2.5}$$

Let us consider the function $\widetilde{F} : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ defined by $\widetilde{F}(t) := \widetilde{f}(t, y(t))$, $t \in \mathbb{R}$. Note that

$$\begin{aligned}F(t+s_n) - \widetilde{F}(t) &= f(t+s_n, x(t+s_n)) - f(t+s_n, y(t)) \\ &\quad + f(t+s_n, y(t)) - \widetilde{f}(t, y(t)),\end{aligned}$$

so we have

$$\begin{aligned}
&\mathbf{E}\|F(t+s_n)-\widetilde{F}(t)\|^2\\
&\quad\le 2\mathbf{E}\|f(t+s_n,x(t+s_n))-f(t+s_n,y(t))\|^2+2\mathbf{E}\|f(t+s_n,y(t))-\widetilde{f}(t,y(t))\|^2\\
&\quad\le 2L\mathbf{E}\|x(t+s_n)-y(t)\|^2+2\mathbf{E}\|f(t+s_n,y(t))-\widetilde{f}(t,y(t))\|^2.
\end{aligned}$$

We can deduce from (2.4) and (2.5) that

$$\lim_{n\to\infty}\mathbf{E}\|F(t+s_n)-\widetilde{F}(t)\|^2=0,\ \text{for each } t\in\mathbb{R}.$$

Similarly we can prove that $\lim_{n\to\infty}\mathbf{E}\|\widetilde{F}(t-s_n)-F(t)\|^2=0$ for each $t\in\mathbb{R}$, which proves the square-mean almost automorphy of $F(t)$. □

3. The linear stochastic differential equations

Consider the following linear stochastic differential equation

$$\mathrm{d}x(t)=Ax(t)\mathrm{d}t+f(t)\mathrm{d}t+g(t)\mathrm{d}W(t),\ t\in\mathbb{R}, \tag{3.1}$$

where A is an infinitesimal generator which generates a $\mathcal{C}_0$-semigroup $(T(t)_{t\ge0})$ such that

$$\|T(t)\|\le Ke^{-\omega t},\ \text{for all } t\ge 0 \tag{3.2}$$

with $K>0$, $\omega>0$. In addition, $f:\mathbb{R}\to\mathcal{L}^2(\mathbf{P},\mathbb{H})$, $g:\mathbb{R}\to\mathcal{L}^2(\mathbf{P},\mathbb{H})$ are stochastic processes, and $W(t)$ is a two-sided standard one-dimensional Brown motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbf{P},\mathcal{F}_t)$, where $\mathcal{F}_t=\sigma\{W(u)-W(v);u,v\le t\}$.

Definition 3.1. An $\mathcal{F}_t$-progressively measurable process $\{x(t)\}_{t\in\mathbb{R}}$ is called a *mild solution* of (3.1) if it satisfies the corresponding stochastic integral equation

$$x(t)=T(t-a)x(a)+\int_a^t T(t-s)f(s)\mathrm{d}s+\int_a^t T(t-s)g(s)\mathrm{d}W(s),$$

for all $t\ge a$ and each $a\in\mathbb{R}$.

Theorem 3.2. *Given $f,g\in AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$, (3.1) has a unique square-mean almost automorphic mild solution.*

Proof. It is well known (see [1]) that for given $a\in\mathbb{R}$ and given initial value x_a at 'time' a, the process

$$x(t)=T(t-a)x_a+\int_a^t T(t-s)f(s)\mathrm{d}s+\int_a^t T(t-s)g(s)\mathrm{d}W(s),\quad t\ge a, \tag{3.3}$$

is the unique mild solution to (3.1) with the initial value condition $x(a)=x_a$. So to prove the existence of a square-mean almost automorphic mild solution, we need to find an initial value x_a such that the stochastic process given by (3.3) is square-mean almost automorphic.

Let $\{s'_n\}$ be an arbitrary sequence of real numbers. Since f and g are square-mean almost automorphic, there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ such that for certain stochastic processes $\widetilde{f}$ and $\widetilde{g}$

$$\lim_{n\to\infty}\mathbf{E}\|f(t+s_n)-\widetilde{f}(t)\|^2=0,\qquad \lim_{n\to\infty}\mathbf{E}\|\widetilde{f}(t-s_n)-f(t)\|^2=0$$

and

$$\lim_{n\to\infty}\mathbf{E}\|g(t+s_n)-\widetilde{g}(t)\|^2=0,\qquad \lim_{n\to\infty}\mathbf{E}\|\widetilde{g}(t-s_n)-g(t)\|^2=0$$

hold for each $t\in\mathbb{R}$.

Now we consider $u(t):=\int_{-\infty}^t T(t-s)f(s)\mathrm{d}s$, defined as

$$\lim_{r\to-\infty}\int_r^t T(t-s)f(s)\mathrm{d}s.$$

From [8, Theorem 3.1], we know that $\int_r^t T(t-s)f(s)\mathrm{d}s$ exists for each $r<t$. Moreover, if we let $\widetilde{u}(t):=\int_{-\infty}^t T(t-s)\widetilde{f}(s)\mathrm{d}s$, we have

$$u(t+s_n)\to\widetilde{u}(t)\quad\text{and}\quad \widetilde{u}(t-s_n)\to u(t)\quad\text{in } \mathcal{L}^2(\mathbf{P},\mathbb{H}),\quad \text{as } n\to\infty$$

for each $t\in\mathbb{R}$. This indicates that u is square-mean almost automorphic. Note that $u(a)=\int_{-\infty}^a T(a-s)f(s)\mathrm{d}s$. If $t\geq a$, then

$$\begin{aligned}\int_a^t T(t-s)f(s)\mathrm{d}s &= \int_{-\infty}^t T(t-s)f(s)\mathrm{d}s-\int_{-\infty}^a T(t-s)f(s)\mathrm{d}s\\ &= u(t)-T(t-a)u(a);\end{aligned}$$

i.e.,

$$u(t)=T(t-a)x(a)+\int_a^t T(t-s)f(s)\mathrm{d}s.$$

If we choose initial value $x_a=u(a)$, then the process $x(t)$ given by (3.3) is square-mean almost automorphic. In fact, denote

$$\hat{u}(t):=T(t-a)x(a)+\int_a^t T(t-s)\widetilde{f}(s)\mathrm{d}s$$

and

$$\tilde{x}(t):=T(t-a)x(a)+\int_a^t T(t-s)\widetilde{f}(s)\mathrm{d}s+\int_a^t T(t-s)\widetilde{g}(s)\mathrm{d}W(s).$$

Then, for each $t\in\mathbb{R}$, we have

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{E}\|x(t+s_n)-\tilde{x}(t)\|^2\\
=\;&\lim_{n\to\infty}\mathbf{E}\|T(t+s_n-a)x(a)+\int_a^{t+s_n}T(t+s_n-s)f(s)\mathrm{d}s\\
&+\int_a^{t+s_n}T(t+s_n-s)g(s)\mathrm{d}W(s)\\
&-T(t-a)x(a)-\int_a^t T(t-s)\widetilde{f}(s)\mathrm{d}s-\int_a^t T(t-s)\widetilde{g}(s)\mathrm{d}W(s)\|^2\\
=\;&\lim_{n\to\infty}\mathbf{E}\|u(t+s_n)-\hat{u}(t)+\left(\int_{-\infty}^{t+s_n}-\int_{-\infty}^a\right)T(t+s_n-s)g(s)\mathrm{d}W(s)\\
&\quad-\left(\int_{-\infty}^t-\int_{-\infty}^a\right)T(t-s)\widetilde{g}(s)\mathrm{d}W(s)\|^2.\end{aligned}$$

Let $\tilde{W}(\sigma) := W(\sigma + s_n) - W(s_n)$ for each $\sigma \in \mathbb{R}$. Note that $\tilde{W}$ is also a Wiener process and has the same distribution as W. Hence by changing $\sigma = s - s_n$ and by Ito's isometry property of stochastic integrals (see [11, p. 29], for example), we have

$$\begin{aligned}
& \mathbf{E}\|x(t+s_n) - \tilde{x}(t)\|^2 \\
\leq\ & 3\mathbf{E}\|u(t+s_n) - \hat{u}(t)\|^2 + 3\mathbf{E}\left\|\int_{-\infty}^{t} T(t-\sigma)[g(\sigma+s_n) - \widetilde{g}(\sigma)]\mathrm{d}\tilde{W}(\sigma)\right\|^2 \\
& + 3\mathbf{E}\left\|\int_{-\infty}^{a} T(t-\sigma)[g(\sigma+s_n) - \widetilde{g}(\sigma)]\mathrm{d}\tilde{W}(\sigma)\right\|^2 \\
\leq\ & 3\mathbf{E}\|u(t+s_n) - \hat{u}(t)\|^2 + 3\left[\int_{-\infty}^{t} \|T(t-\sigma)\|^2\mathbf{E}\|g(\sigma+s_n) - \widetilde{g}(\sigma)\|^2\mathrm{d}\sigma\right] \\
& + 3\left[\int_{-\infty}^{a} \|T(t-\sigma)\|^2\mathbf{E}\|g(\sigma+s_n) - \widetilde{g}(\sigma)\|^2\mathrm{d}\sigma\right].
\end{aligned}$$

Noting that g is square-mean almost automorphic and by the exponential dissipation property (3.2) of $T(t)$, we immediately obtain that

$$\lim_{n\to\infty} \mathbf{E}\|x(t+s_n) - \tilde{x}(t)\|^2 = 0.$$

And we can show in a similar way that

$$\lim_{n\to\infty} \mathbf{E}\|\tilde{x}(t-s_n) - x(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. So far, the existence has been proved.

We finally prove the uniqueness of the square-mean almost automorphic solution of (3.1). Assume that $x(t)$ and $y(t)$ are both square-mean almost automorphic solutions of (3.1) with different initial values $x(a)$ and $y(a)$ at 'time' a. That is, for $t \geq a$,

$$\begin{aligned}
x(t) &= T(t-a)x(a) + \int_a^t T(t-s)f(s)\mathrm{d}s + \int_a^t T(t-s)g(s)\mathrm{d}W(s), \\
y(t) &= T(t-a)y(a) + \int_a^t T(t-s)f(s)\mathrm{d}s + \int_a^t T(t-s)g(s)\mathrm{d}W(s),
\end{aligned}$$

and $x(a) \neq y(a)$. Let $z(t) = x(t) - y(t)$. Then $z(t)$ satisfies the equation

$$\mathrm{d}z(t) = Az(t)\mathrm{d}t, \quad t \geq a,$$

with initial condition $z(a) = x(a) - y(a)$. Hence

$$z(t) = T(t-a)z(a)$$

and

$$\|z(t)\| \leq Ke^{-\omega(t-a)}\|z(a)\|, \quad \text{for all } t \geq a.$$

So $z(t) \to 0$, as $t \to +\infty$. Since $z(t) \in AA(\mathbb{R}; \mathcal{L}^2(\mathbf{P}, \mathbb{H}))$, for any sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ such that for some progress $\tilde{z}(t)$,

$$\lim_{n\to\infty} z(t+s_n) = \tilde{z}(t) \quad \text{and} \quad \lim_{n\to\infty} \tilde{z}(t-s_n) = z(t) \tag{3.4}$$

for each $t \in \mathbb{R}$. In particular, if $\lim_{n\to\infty} s'_n = \infty$, then $\tilde{z}(t) \equiv 0$ by the first equality of (3.4). Hence $z(t) \equiv 0$ by the second equality of (3.4), so we must have $x(a) = y(a)$, a contradiction. The proof is complete. $\square$

4. The non-linear stochastic differential equations

Consider the following non-linear stochastic differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}W(t),\ t \in \mathbb{R}, \tag{4.1}$$

where $f : \mathbb{R} \times \mathcal{L}^2(\mathbf{P}, \mathbb{H}) \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, $g : \mathbb{R} \times \mathcal{L}^2(\mathbf{P}, \mathbb{H}) \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$, and $W(t)$ is a two-sided standard one-dimensional Brown motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \le t\}$.

As in the previous section, we also assume that A generates a $\mathcal{C}_0$-semigroup $(T(t)_{t\ge 0})$ such that

$$\|T(t)\| \le Ke^{-\omega t},\ \text{for all } t \ge 0 \tag{4.2}$$

with $K > 0$, $\omega > 0$.

Definition 4.1. An $\mathcal{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbb{R}}$ is called a *mild solution* of (4.1) if it satisfies the corresponding stochastic integral equation

$$x(t) = T(t-r)x(r) + \int_r^t T(t-s)f(s, x(s))\mathrm{d}s + \int_r^t T(t-s)g(s, x(s))\mathrm{d}W(s),$$

for all $t \ge r$ and each $r \in \mathbb{R}$.

Theorem 4.2. *Assume f and g are square-mean almost automorphic processes in $t \in \mathbb{R}$ for each $x \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$. Moreover f and g satisfy Lipschitz conditions in x uniformly for t; that is, for all $x, y \in \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ and $t \in \mathbb{R}$,*

$$\mathbf{E}\|f(t, x) - f(t, y)\|^2 \le L\mathbf{E}\|x - y\|^2,$$
$$\mathbf{E}\|g(t, x) - g(t, y)\|^2 \le L'\mathbf{E}\|x - y\|^2,$$

for constants $L, L' > 0$. Then (4.1) *has a unique square-mean almost automorphic mild solution, provided $\frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega} < 1$.*

Proof. By Definition 4.1, the stochastic process $x : \mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{H})$ is a solution to (4.1) if and only if it satisfies the stochastic integral equation

$$x(t) = T(t-r)x(r) + \int_r^t T(t-s)f(s, x(s))\mathrm{d}s + \int_r^t T(t-s)g(s, x(s))\mathrm{d}W(s).$$

If we let $r \to -\infty$ in the above integral equation, by the exponential dissipation condition of T, (4.2), then we obtain that the stochastic process x is a solution to (4.1) if and only if x satisfies the stochastic integral equation

$$x(t) = \int_{-\infty}^t T(t-s)f(s, x(s))\mathrm{d}s + \int_{-\infty}^t T(t-s)g(s, x(s))\mathrm{d}W(s).$$

To seek the square-mean almost automorphic mild solution, let us consider the non-linear operator $\mathcal{S}$ acting on the Banach space $AA(\mathbb{R}; \mathcal{L}^2(\mathbf{P}, \mathbb{H}))$ given by

$$(\mathcal{S}x)(t) := \int_{-\infty}^t T(t-s)f(s, x(s))\mathrm{d}s + \int_{-\infty}^t T(t-s)g(s, x(s))\mathrm{d}W(s).$$

If we can show that the operator $\mathcal{S}$ maps $AA(\mathbb{R}; \mathcal{L}^2(\mathbf{P}, \mathbb{H}))$ into itself and it is a contraction mapping, then by the Banach fixed point theorem, we can conclude that there is a unique square-mean almost automorphic mild solution to the equation (4.1).

Let us consider the non-linear operators $\mathcal{S}_1 x$ and $\mathcal{S}_2 x$ acting on the Banach space $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ given by

$$(\mathcal{S}_1 x)(t) := \int_{-\infty}^{t} T(t-s) f(s,x(s))\mathrm{d}s$$

and

$$(\mathcal{S}_2 x)(t) := \int_{-\infty}^{t} T(t-s) g(s,x(s))\mathrm{d}W(s),$$

respectively. By Theorem 2.6, $F_1(t) := f(t,x(t))$ and $F_2(t) := g(t,x(t))$ are square-mean almost automorphic if x is; then by the proof of Theorem 3.2, we know that $\mathcal{S}_1 x$ and $\mathcal{S}_2 x$ are square-mean almost automorphic if F_1 and F_2 are. That is, the operator $\mathcal{S}$ maps $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ into itself.

Next we show that $\mathcal{S}$ is a contraction mapping on $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$. For $x_1, x_2 \in AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ and each $t \in \mathbb{R}$ we have

$$\begin{aligned}
\mathbf{E}\|(\mathcal{S}x_1)(t)-(\mathcal{S}x_2)(t)\|^2 &= \mathbf{E}\| \int_{-\infty}^{t} T(t-s)[f(s,x_1(s)) - f(s,x_2(s))]\mathrm{d}s \\
&\quad + \int_{-\infty}^{t} T(t-s)[g(s,x_1(s)) - g(s,x_2(s))]\mathrm{d}W(s)\|^2 \\
&\le 2K^2 \mathbf{E}\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,x_1(s)) - f(s,x_2(s))\|\mathrm{d}s\right)^2 \\
&\quad + 2\mathbf{E}\left\|\int_{-\infty}^{t} T(t-s)[g(s,x_1(s)) - g(s,x_2(s))]\mathrm{d}W(s)\right\|^2.
\end{aligned}$$

We first evaluate the first term of the right-hand side by the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}
& \mathbf{E}\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,x_1(s)) - f(s,x_2(s))\|\mathrm{d}s\right)^2 \\
= \;& \mathbf{E}\left(\int_{-\infty}^{t} (e^{\frac{-\omega(t-s)}{2}})(e^{\frac{-\omega(t-s)}{2}})\|f(s,x_1(s)) - f(s,x_2(s))\|\mathrm{d}s\right)^2 \\
\le \;& \mathbf{E}\left[\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,x_1(s)) - f(s,x_2(s))\|^2\mathrm{d}s\right)\right] \\
\le \;& \left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbf{E}\|f(s,x_1(s)) - f(s,x_2(s))\|^2\mathrm{d}s\right) \\
\le \;& L\cdot\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbf{E}\|x_1(s) - x_2(s)\|^2\mathrm{d}s\right) \\
\le \;& L\cdot\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)^2 \sup_{s\in\mathbb{R}}\mathbf{E}\|x_1(s) - x_2(s)\|^2 \\
\le \;& \frac{L}{\omega^2}\cdot\sup_{s\in\mathbb{R}}\mathbf{E}\|x_1(s) - x_2(s)\|^2.
\end{aligned}$$

As to the second term, by Ito's isometry property of stochastic integrals, we have

$$\begin{aligned}
&\mathbf{E}\left\|\int_{-\infty}^{t} T(t-s)[g(s,x_1(s))-g(s,x_2(s))]\mathrm{d}W(s)\right\|^2\\
&= \mathbf{E}\left[\int_{-\infty}^{t}\|T(t-s)[g(s,x_1(s))-g(s,x_2(s))]\|^2\mathrm{d}s\right]\\
&\le \mathbf{E}\left[\int_{-\infty}^{t}\|T(t-s)\|^2\|g(s,x_1(s))-g(s,x_2(s))\|^2\mathrm{d}s\right]\\
&\le K^2\int_{-\infty}^{t} e^{-2\omega(t-s)}\mathbf{E}\|g(s,x_1(s))-g(s,x_2(s))\|^2\mathrm{d}s\\
&\le K^2L'\cdot\left(\int_{-\infty}^{t} e^{-2\omega(t-s)}\mathrm{d}s\right)\sup_{s\in\mathbb{R}}\mathbf{E}\|x_1(s)-x_2(s)\|^2\\
&\le \frac{K^2L'}{2\omega}\cdot\sup_{s\in\mathbb{R}}\mathbf{E}\|x_1(s)-x_2(s)\|^2.
\end{aligned}$$

Thus, it follows that, for each $t\in\mathbb{R}$,

$$\mathbf{E}\|(\mathcal{S}x_1)(t)-(\mathcal{S}x_2)(t)\|^2 \le \left(\frac{2K^2L}{\omega^2}+\frac{K^2L'}{\omega}\right)\cdot\sup_{s\in\mathbb{R}}\mathbf{E}\|x_1(s)-x_2(s)\|^2;$$

that is,

$$\|(\mathcal{S}x_1)(t)-(\mathcal{S}x_2)(t)\|_2^2 \le \eta\cdot\sup_{s\in\mathbb{R}}\|x_1(s)-x_2(s)\|_2^2 \tag{4.3}$$

with $\eta := \frac{2K^2L}{\omega^2}+\frac{K^2L'}{\omega}$. Note that

$$\sup_{s\in\mathbb{R}}\|x_1(s)-x_2(s)\|_2^2 \le (\sup_{s\in\mathbb{R}}\|x_1(s)-x_2(s)\|_2)^2 \tag{4.4}$$

and (4.3) together with (4.4) give, for each $t\in\mathbb{R}$,

$$\|\mathcal{S}(x_1)(t)-\mathcal{S}(x_2)(t)\|_2 \le \sqrt{\eta}\ \|x_1-x_2\|_\infty.$$

Hence

$$\|\mathcal{S}x_1-\mathcal{S}x_2\|_\infty = \sup_{t\in\mathbb{R}}\|\mathcal{S}(x_1)(t)-\mathcal{S}(x_2)(t)\|_2 \le \sqrt{\eta}\ \|x_1-x_2\|_\infty.$$

Since $\eta<1$, it follows that $\mathcal{S}$ is a contraction mapping on $AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$. Therefore, there exists a unique $v\in AA(\mathbb{R};\mathcal{L}^2(\mathbf{P},\mathbb{H}))$ such that $\mathcal{S}v=v$, which is the unique solution to (4.1). The proof is now complete. □

Remark 4.3. If f and g in (3.1) and (4.1) are almost periodic in t, then the unique square-mean almost automorphic solution obtained in Theorems 3.2 and 4.2 is actually almost periodic; see [2, 3, 4, 6, 9, 13].

5. Stability of the unique square-mean almost automorphic solution

In the previous section, for the non-linear stochastic differential equation (4.1), we obtain that it has a unique square-mean almost automorphic solution. In this section, we will show that the unique square-mean almost automorphic solution is asymptotically stable in the square-mean sense and that any other solutions converge to it exponentially fast. First, let us state the definition of asymptotic stability.

Definition 5.1. The unique square-mean almost automorphic solution $x_{aa}(t)$ of (4.1) is said to be *stable in the square-mean sense* if for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbf{E}\|x_c(t) - x_{aa}(t)\|^2 < \epsilon, \quad t \geq 0,$$

whenever $\|c - x_{aa}(0)\|^2 < \delta$, where $x_c(t)$ stands for the solution of (4.1) with initial condition $x_c(0) = c$. The solution $x_{aa}(t)$ is said to be *asymptotically stable in the square-mean sense* if it is stable in the square-mean sense and

$$\lim_{t\to\infty} \mathbf{E}\|x_c(t) - x_{aa}(t)\|^2 = 0.$$

Theorem 5.2. *Assume that the assumptions of Theorem* 4.2 *hold, then the unique square-mean almost automorphic solution* $x_{aa}(t)$ *of* (4.1) *is asymptotically stable in the square-mean sense.*

Proof. We actually will prove more general results. Assume that $x(t)$ and $y(t)$ are two solutions of (4.1) with initial values $x(0)$ and $y(0)$, respectively. Note that, by the exponential dissipation of $T(t)$, we have

$$\begin{aligned}
\|y(t) - x(t)\|^2 &= \|T(t)[y(0) - x(0)] + \int_0^t T(t-s)[f(s, y(s)) - f(s, x(s))]\mathrm{d}s \\
&\quad + \int_0^t T(t-s)[g(s, y(s)) - g(s, x(s))]\mathrm{d}W(s)\|^2 \\
&\leq 3K^2 e^{-2\omega t}\|y(0) - x(0)\|^2 + 3\left[\int_0^t K e^{-\omega(t-s)}\|f(s, y(s)) - f(s, x(s))\|\mathrm{d}s\right]^2 \\
&\quad + 3\left[\int_0^t K e^{-\omega(t-s)}[g(s, y(s)) - g(s, x(s))]\mathrm{d}W(s)\right]^2. \qquad (5.1)
\end{aligned}$$

Since f satisfies the Lipschitz condition, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& 3\left[\int_0^t K e^{-\omega(t-s)}\|f(s, y(s)) - f(s, x(s))\|\mathrm{d}s\right]^2 \\
\leq\ & 3K^2L^2\left(\int_0^t e^{-\omega(t-s)}\|y(s) - x(s)\|\mathrm{d}s\right)^2 \\
=\ & 3K^2L^2\left(\int_0^t (e^{\frac{-\omega(t-s)}{2}})(e^{\frac{-\omega(t-s)}{2}})\|y(s) - x(s)\|\mathrm{d}s\right)^2 \\
\leq\ & 3K^2L^2(\int_0^t e^{-\omega(t-s)}ds)(\int_0^t e^{-\omega(t-s)}\|y(s) - x(s)\|^2 ds) \\
\leq\ & \frac{3K^2L^2}{\omega}\int_0^t e^{-\omega(t-s)}\|y(s) - x(s)\|^2\mathrm{d}s, \qquad (5.2)
\end{aligned}$$

for all $t \geq 0$. By Ito's isometry property of stochastic integrals, it follows that

$$\begin{aligned}
& 3\left[\int_0^t K e^{-\omega(t-s)}[g(s, y(s)) - g(s, x(s))]\mathrm{d}W(s)\right]^2 \\
\leq\ & 3K^2L'^2\int_0^t e^{-2\omega(t-s)}\|y(s) - x(s)\|^2\mathrm{d}s \qquad (5.3)
\end{aligned}$$

for all $t \geq 0$.

Let $Y(t) := \mathbf{E}\|y(t) - x(t)\|^2$ and $k := 3K^2\hat{L}^2(1+\frac{1}{\omega})$ with $\hat{L} := \max\{L, L'\}$. Note that $e^{-2\omega t} \leq e^{-\omega t}$ for $t \geq 0$ and by (5.1), (5.2), (5.3), we have

$$Y(t) \leq 3K^2 e^{-\omega t} Y(0) + k \int_0^t e^{-\omega(t-s)} Y(s) \mathrm{d}s. \tag{5.4}$$

The $Y(t)$ in inequality (5.4) can be controlled by $\widetilde{Y}(t)$, which satisfies

$$\begin{aligned} \dot{\widetilde{Y}}(t) &= -\omega \widetilde{Y}(t) + k\widetilde{Y}(t), \\ \widetilde{Y}(0) &= 3K^2 Y(0). \end{aligned}$$

Hence $\widetilde{Y}(t) \to 0$ exponentially fast if $-\omega + k < 0$, that is, if

$$\omega > 3K^2\hat{L}^2(1 + \frac{1}{\omega}). \tag{5.5}$$

Note that (5.5) holds if and only if

$$\omega^2 > 3K^2\hat{L}^2(\omega + 1),$$

which always holds. Therefore, $Y(t)$ converges to 0 exponentially fast.

In particular, if we set $x(t) = x_{aa}(t)$ in the above arguments, we obtain that the unique square-mean almost automorphic solution $x_{aa}(t)$ of (4.1) is asymptotically stable in the square-mean sense. The proof is complete. □

Remark 5.3. By the proof of the stability theorem, we actually obtain that any solution of (4.1) (and hence (3.1)) is asymptotically stable in the square-mean sense.

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