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ENTROPY FOR C^*-ALGEBRAS WITH TRACIAL RANK ZERO

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Abstract. Voiculescu's topological approximation entropy is extended to automorphisms on unital simple C^*-algebras with tracial rank zero. Several expected properties are shown. We also consider the value of our entropy for a cat map on the non-commutative torus.

Introduction

In [14], D. Voiculescu defined a notion of topological entropy for an automorphism on a unital nuclear C^*-algebra, and then in [1], N. P. Brown extended it on an exact C^*-algebra. Another topological entropy for an automorphism on an AF-algebra was also defined in [14]. Roughly speaking, Brown-Voiculescu entropy is defined by measuring the growth of iterates of the automorphism by completely positive approximations of a faithful nuclear representation which factor through finite dimensional C^*-algebras. The other entropy is defined by measuring the growth of iterates of the automorphism by approximations of finite dimensional C^*-subalgebras. Hence the definition really makes sense for exact C^*-algebras in the former case and for AF-algebras in the latter case. However the class of AF-algebras is much smaller than the class of exact C^*-algebras.

In [9], H. Lin defined a notion of topological rank for C^*-algebras, which is called tracial rank. Roughly speaking, AF-algebras are C^*-algebras that can be approximated in norm by finite dimensional C^*-algebras. C^*-algebras with tracial rank zero are C^*-algebras that can be approximated in trace by finite dimensional C^*-subalgebras.

Inspired by this notion, we extended Voiculescu's topological approximation entropy. In this paper, the definition of our entropy is given for the case of unital simple C^*-algebras with tracial rank zero. The reason is the following. On the one hand, for non-simple C^*-algebras, the definition of the tracial rank is very complicated. Hence our entropy might be defined for arbitrary C^*-algebras with tracial rank zero, but this does not seem to be computable. On the other hand, the important and interesting examples are the cases of simple C^*-algebras. One example is mentioned below and is studied in this paper. We also show that several known properties of the topological approximation entropy function extend to our setting.

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One of our purposes is to consider a cat map on the non-commutative torus A_θ, which is a unital simple C^*-algebra with tracial rank zero by the result of G. A. Elliott and D. E. Evans in [5]. However we obtain only the estimate of our entropy under certain assumptions on the corresponding number θ. As a corollary, we obtain that if θ is an algebraic number of degree 2, then our assumptions hold.

1. Preliminary

The following conventions will be used. Let A be a C^*-algebra with tracial state space $T(A)$. If ω is a finite subset of A, then we write $\omega \Subset A$. For $a \in A$, $\varepsilon > 0$ and a subset $B \subset A$, we write $x \in_\varepsilon B$ if there is $b \in B$ such that $\|a - b\| < \varepsilon$.

Definition 1.1 (cf. [9], [10])**.** Let $a, b \in A_+$. We write $[a] \leq [b]$ if there exists $x \in A$ such that $a = x^*x$ and $xx^* \in \overline{bAb}$. Note that if p and q are two projections in A, then $[p] \leq [q]$ if and only if p is Murray-von Neumann equivalent to a subprojection of q. For $n \in \mathbb{N}$, we write $n[a] \leq [b]$ if there are mutually orthogonal positive elements $b_1, \ldots, b_n \in \overline{bAb}$ such that $[a] \leq [b_i]$ for $i = 1, \ldots, n$.

Definition 1.2 (cf. [9], [10])**.** A unital simple C^*-algebra A has *tracial rank zero* if for any $\omega \Subset A$, any $\delta > 0$ and any non-zero element $a \in A_+$, there exists a finite dimensional C^*-subalgebra B of A with $1_B = p$ such that

(1) $\|px - xp\| < \delta$ for all $x \in \omega$,
(2) $pxp \in_\delta B$ for all $x \in \omega$, and
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Remark 1.3 (cf. [9], [10]). Condition (3) roughly says that $1-p$ is arbitrarily small. Namely, if a unital simple C^*-algebra A has tracial rank zero, then the following holds: for any $\omega \Subset A$, any $\delta > 0$ and $\sigma > 0$, there exists a finite dimensional C^*-subalgebra B of A with $1_B = p$ such that

(1) $\|px - xp\| < \delta$ for all $x \in \omega$,
(2) $pxp \in_\delta B$ for all $x \in \omega$, and
(3′) $\tau(1 - p) < \sigma$

for any $\tau \in T(A)$.

If a unital simple C^*-algebra A has tracial rank zero, then A has the following properties (the reader may be referred to Lin's book [10]):

- property (SP), i.e., every non-zero hereditary C^*-subalgebra of A contains a non-zero projection,
- real rank zero and stable rank one,
- the tracial state space $T(A)$ is non-empty,
- Blackadar's fundamental comparison property [2]; i.e., if two projections $p, q \in A$ satisfy $\tau(p) < \tau(q)$ for any $\tau \in T(A)$, then $[p] \leq [q]$.

2. Tracial topological approximation entropy

Throughout this paper, we always assume that A is a unital infinite dimensional simple C^*-algebra with tracial rank zero and α is an automorphism on A. We introduce two definitions of our entropy for α.

For given $\omega \Subset A$, $\delta > 0$ and $0 \neq a \in A_+$, we denote by $F(A; \omega, \delta, a)$ the set of all finite dimensional C^*-subalgebras B of A satisfying (1), (2) and (3) in Definition 1.2.

Definition 2.1. For $\omega \Subset A$, $\delta > 0$ and $0 \neq a \in A_+$, we define

$$\mathrm{r}(\omega; \delta, a) = \inf\{\operatorname{rank} B \mid B \in F(A; \omega, \delta, a)\},$$

where $\operatorname{rank} B$ is the dimension of a maximal abelian subalgebra of B. We define

$$\begin{aligned}
\operatorname{htr}(\alpha, \omega; \delta, a) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{r}\left(\bigcup_{i=0}^{n-1} \alpha^i(\omega); \delta, a\right), \\
\operatorname{htr}(\alpha, \omega; \delta) &= \sup_{0\neq a\in A_+} \operatorname{htr}(\alpha, \omega; \delta, a), \\
\operatorname{htr}(\alpha, \omega) &= \sup_{\delta>0} \operatorname{htr}(\alpha, \omega; \delta), \\
\operatorname{htr}(\alpha) &= \sup_{\omega\Subset A} \operatorname{htr}(\alpha, \omega).
\end{aligned}$$

Then $\operatorname{htr}(\alpha)$ will be called the *tracial topological approximation entropy of* α.

Let $\tau \in T(A)$. For $\omega \Subset A$, $\delta > 0$ and $\sigma > 0$, we denote by $F_\tau(A; \omega, \delta, \sigma)$ the set of all finite dimensional C^*-subalgebras B of A satisfying (1), (2) and (3′) in Remark 1.3.

Definition 2.2. Let $\tau \in T(A)$. For $\omega \Subset A$, $\delta > 0$ and $\sigma > 0$, we similarly define

$$\mathrm{r}_\tau(\omega; \delta, \sigma) = \inf\{\operatorname{rank} B \mid B \in F_\tau(A; \omega, \delta, \sigma)\}$$

and

$$\begin{aligned}
\operatorname{htr}_\tau(\alpha, \omega; \delta, \sigma) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{r}_\tau\left(\bigcup_{i=0}^{n-1} \alpha^i(\omega); \delta, \sigma\right), \\
\operatorname{htr}_\tau(\alpha, \omega; \delta) &= \sup_{\sigma>0} \operatorname{htr}_\tau(\alpha, \omega; \delta, \sigma), \\
\operatorname{htr}_\tau(\alpha, \omega) &= \sup_{\delta>0} \operatorname{htr}_\tau(\alpha, \omega; \delta), \\
\operatorname{htr}_\tau(\alpha) &= \sup_{\omega\Subset A} \operatorname{htr}_\tau(\alpha, \omega).
\end{aligned}$$

Then $\operatorname{htr}_\tau(\alpha)$ will be called the *tracial topological approximation entropy of* α *with respect to* τ.

Next we consider some relations between $\operatorname{htr}(\alpha)$ and $\operatorname{htr}_\tau(\alpha)$.

Proposition 2.3. *For any* $\tau \in T(A)$, *we have*

$$\operatorname{htr}_\tau(\alpha) \leq \operatorname{htr}(\alpha).$$

Proof. Let $\omega \Subset A$, $\delta > 0$ and $\sigma > 0$. Take $m \in \mathbb{N}$ with $m^{-1} < \sigma$. Since A is simple and has property (SP), there are mutually orthogonal and mutually equivalent non-zero projections $e_1, \ldots, e_m \in A$ (e.g., see [10, Lemma 3.5.7]). If

$$B \in F\left(A; \bigcup_{i=0}^{n-1} \alpha^i(\omega), \delta, e_1\right)$$

with $p = 1_B$, then we have

$$B \in F_\tau\left(A; \bigcup_{i=0}^{n-1} \alpha^i(\omega), \delta, \sigma\right)$$

because condition (3) implies that

$$\tau(1-p) \leq \tau(e_1) \leq m^{-1} < \sigma.$$

Hence we have

$$\mathrm{r}_\tau\left(\bigcup_{i=0}^{n-1}\alpha^i(\omega);\delta,\sigma\right)\leq \mathrm{r}\left(\bigcup_{i=0}^{n-1}\alpha^i(\omega);\delta,e_1\right).$$

Thus

$$\mathrm{htr}_\tau(\alpha,\omega;\delta,\sigma)\leq \mathrm{htr}(\alpha,\omega;\delta,e_1)\leq \mathrm{htr}(\alpha,\omega;\delta)$$

holds for any $\sigma>0$. Therefore we obtain $\mathrm{htr}_\tau(\alpha,\omega;\delta)\leq \mathrm{htr}(\alpha,\omega;\delta)$ and so $\mathrm{htr}_\tau(\alpha)\leq \mathrm{htr}(\alpha)$. □

The previous proposition naturally leads us to the following questions.

Question 2.4. Is there $\tau\in T(A)$ such that $\mathrm{htr}_\tau(\alpha)=\mathrm{htr}(\alpha)$?

Question 2.5. Does "*the variational principle*"

$$\mathrm{htr}(\alpha)=\sup_{\tau\in T(A)}\mathrm{htr}_\tau(\alpha)$$

hold?

We conclude this section with a partial answer to the above questions.

Proposition 2.6. *If A has a unique tracial state τ, then*

$$\mathrm{htr}_\tau(\alpha)=\mathrm{htr}(\alpha).$$

Proof. Let $\omega\Subset A$, $\delta>0$ and $0\neq a\in A_+$. By property (SP), there is a non-zero projection e in $\overline{aAa}$. Put $\sigma=\tau(e)>0$. If

$$B\in F_\tau\left(A;\bigcup_{i=0}^{n-1}\alpha^i(\omega),\delta,\sigma\right)$$

with $p=1_B$, then $\tau(1-p)<\sigma=\tau(e)$. Since A has the fundamental comparison property, the uniqueness of τ implies that $[1-p]\leq[e]$. Hence $1-p$ is equivalent to some projection of $\overline{aAa}$ and thus

$$B\in F\left(A;\bigcup_{i=0}^{n-1}\alpha^i(\omega),\delta,a\right).$$

Therefore we obtain

$$\mathrm{r}_\tau\left(\bigcup_{i=0}^{n-1}\alpha^i(\omega);\delta,\sigma\right)\geq \mathrm{r}\left(\bigcup_{i=0}^{n-1}\alpha^i(\omega);\delta,a\right).$$

Hence $\mathrm{htr}_\tau(\alpha,\omega;\delta)\geq \mathrm{htr}_\tau(\alpha,\omega;\delta,\sigma)\geq \mathrm{htr}(\alpha,\omega;\delta,a)$ for any non-zero $a\in A_+$. This gives the desired conclusion. □

3. Relations to other entropies

We first discuss the relation between our entropy $\mathrm{htr}(\alpha)$ and Voiculescu's topological approximation entropy $\mathrm{hat}(\alpha)$ in [14]. In this section, we will frequently use notation and results from [14].

Proposition 3.1. *If A is a unital simple AF-algebra, then*

$$\mathrm{htr}(\alpha)\leq \mathrm{hat}(\alpha).$$

Proof. Since $\mathrm{r}(\omega;\delta,a)\leq \mathrm{r}(\omega;\delta)$ for any $\omega\Subset A,\delta>0$ and $0\neq a\in A_+$, we have $\mathrm{htr}(\alpha,\omega;\delta,a)\leq \mathrm{ha}(\alpha,\omega;\delta)$, which implies $\mathrm{htr}(\alpha)\leq \mathrm{hat}(\alpha)$. □

We note that if $\tau \in T(A)$ is α-invariant, then the von Neumann algebra $\pi_\tau(A)''$ is hyperfinite, and we denote by $\widetilde{\tau}$ and $\widetilde{\alpha}$ the extended trace and automorphism on $\pi_\tau(A)''$, respectively. We similarly obtain the relation between $\mathrm{htr}_\tau(\alpha)$ and Voiculescu's approximation entropy $\mathrm{ha}_{\widetilde{\tau}}(\widetilde{\alpha})$ in [14].

Proposition 3.2. *For any α-invariant tracial state τ on A, we have*

$$\mathrm{ha}_{\widetilde{\tau}}(\widetilde{\alpha}) \leq \mathrm{htr}_\tau(\alpha).$$

Proof. For $\omega \Subset A$ and $\delta > 0$, we consider $\mathrm{r}_{\widetilde{\tau}}(\pi_\tau(\omega); \delta)$. We take $\sigma > 0$ such that

$$\sigma = \frac{\delta^2}{6 \max\{\|x\|^2 : x \in \omega\}}.$$

If $B \in F_\tau(A; \omega, \delta/\sqrt{2}, \sigma)$ with $p = 1_B$, then $\pi_\tau(\omega) \subset_\delta \pi_\tau(B \oplus \mathbb{C}p^\perp)$ in the 2-norm with respect to $\widetilde{\tau}$. Indeed, for each $x \in \omega$, there is $b \in B$ such that $\|pxp - b\| < \delta/\sqrt{2}$. Then

$$\begin{aligned}\|x - b\|_2^2 &= \tau((px^*p - b^*)(pxp - b)) + \tau(px^*p^\perp xp) + \tau(p^\perp x^* pxp^\perp) + \tau(p^\perp x^* p^\perp xp^\perp) \\ &\leq \|pxp - b\|^2 + 3\|x\|^2\sigma \\ &\leq \delta^2/2 + \delta^2/2 = \delta^2.\end{aligned}$$

It follows that

$$\mathrm{r}_{\widetilde{\tau}}\left(\bigcup_{i=0}^{n-1} \widetilde{\alpha}^i(\pi_\tau(\omega)); \delta\right) \leq \mathrm{r}_\tau\left(\bigcup_{i-1}^{n-1} \alpha^i(\omega); \delta/\sqrt{2}, \sigma\right) + 1.$$

By using a Kolmogorov-Sinai type theorem [14, Proposition 1.4], we infer that $\mathrm{ha}_{\widetilde{\tau}}(\widetilde{\alpha}) \leq \mathrm{htr}_\tau(\alpha)$. □

Remark 3.3. By Proposition 3.2 and [14, Proposition 1.5], we obtain

$$H(\widetilde{\alpha}) \leq \mathrm{ha}_{\widetilde{\tau}}(\widetilde{\alpha}) \leq \mathrm{htr}_\tau(\alpha),$$

where $H(\widetilde{\alpha})$ is the Connes-Størmer entropy in [4]. By [14, Proposition 3.6, 3.7], we also have

$$\mathrm{h}_\tau(\alpha) = \mathrm{h}_{\widetilde{\tau}}(\widetilde{\alpha}) \leq \mathrm{hcpa}_{\widetilde{\tau}}(\widetilde{\alpha}) \leq \mathrm{ha}_{\widetilde{\tau}}(\widetilde{\alpha}) \leq \mathrm{htr}_\tau(\alpha),$$

where $\mathrm{h}_\tau(\alpha)$ is the Connes-Narnhofer-Thirring entropy in [3] and $\mathrm{hcpa}_{\widetilde{\tau}}(\widetilde{\alpha})$ is Voiculescu's completely positive approximation entropy in [14].

4. Basic properties and examples

The proofs of [14, Proposition 1.3] and [14, Proposition 1.4] immediately adapt to our context to yield the next two propositions. The reader may refer to [1, Proposition 2.5] and [1, Proposition 2.6].

Proposition 4.1. *If $k \in \mathbb{Z}$, then*

$$\mathrm{htr}(\alpha^k) = |k|\mathrm{htr}(\alpha).$$

In addition, if we have an α-invariant tracial state τ on A, then

$$\mathrm{htr}_\tau(\alpha^k) = |k|\mathrm{htr}_\tau(\alpha).$$

Proposition 4.2. *If $\{\omega_\lambda\}_{\lambda\in\Lambda}$ is a net (partially ordered by inclusion) of finite sets of A such that*

$$\bigcup_{\substack{\lambda\in\Lambda\\ n\in\mathbb{Z}}} \alpha^n(\omega_\lambda)$$

generates A as a C^*-algebra, then

$$\mathrm{htr}(\alpha) = \sup_{\lambda\in\Lambda} \mathrm{htr}(\alpha, \omega_\lambda).$$

In addition, if we have an α-invariant tracial state τ on A, then

$$\mathrm{htr}_\tau(\alpha) = \sup_{\lambda\in\Lambda} \mathrm{htr}_\tau(\alpha, \omega_\lambda).$$

The tensor products in the following proposition are minimal tensor products, which ensures having tracial topological rank zero of the resulting tensor product by [8, Theorem 3.8].

Proposition 4.3. *If A_i are unital simple C^*-algebras with tracial topological rank zero, $\alpha_i \in \mathrm{Aut}(A_i)$ and $\tau_i \in T(A_i)$ for $i = 1, 2$, then*

$$\mathrm{htr}(\alpha_1 \otimes \alpha_2) \leq \mathrm{htr}(\alpha_1) + \mathrm{htr}(\alpha_2),$$
$$\mathrm{htr}_{\tau_1\otimes\tau_2}(\alpha_1 \otimes \alpha_2) \leq \mathrm{htr}_{\tau_1}(\alpha_1) + \mathrm{htr}_{\tau_2}(\alpha_2).$$

Proof. The proposition is obtained by combining the proof of [14, Proposition 1.9] with [8, Theorem 3.8]. For $i = 1, 2$, take $\omega_i \Subset A_i$, $\delta > 0$ and $0 \neq a \in A_1 \otimes A_2$ such that $\|x\| \leq 1$ for any $x \in \omega_i$. By [8, Lemma 3.6], there are non-zero positive elements $a_1 \in A_1$ and $a_2 \in A_2$ such that $[a_1 \otimes a_2] \leq [a]$. Then it follows from the proof of [8, Theorem 3.8] that if

$$B_i \in F\left(A; \bigcup_{j=0}^{n-1} \alpha_i^j(\omega_i), \delta/2, a_i\right)$$

for $i = 1, 2$, then

$$B_1 \otimes B_2 \in F\left(A; \bigcup_{j=0}^{n-1} \alpha_1^j(\omega_1) \otimes \alpha_2^j(\omega_2), \delta, a\right).$$

Hence we have

$$\log \mathrm{r}\left(\bigcup_{j=0}^{n-1} \alpha_1^j(\omega_1) \otimes \alpha_2^j(\omega_2), \delta, a\right) \leq \log \mathrm{r}\left(\bigcup_{j=0}^{n-1} \alpha_1^j(\omega_1), \delta/2, a_1\right) + \log \mathrm{r}\left(\bigcup_{j=0}^{n-1} \alpha_2^j(\omega_2), \delta/2, a_2\right).$$

Therefore the desired conclusion is obtained by using Proposition 4.2 . The other inequality is also checked similarly. □

Example 4.4. Let $A_n = M_n(\mathbb{C})$ be the C^*-algebra of $n \times n$ matrices with a unique tracial state τ_n. Let $A = A_n^{\otimes\mathbb{Z}}, \tau = \tau_n^{\otimes\mathbb{Z}}$ and α be the non-commutative Bernoulli shift on A. Then it immediately follows that

$$\mathrm{htr}(\alpha) = \mathrm{htr}_\tau(\alpha) = \log n.$$

Example 4.5. In [11], H. Lin and N. C. Phillips proved that if X is an infinite compact metric space with finite covering dimension and T is a minimal homeomorphism on X, then the associated crossed product C^*-algebra $A = C^*(\mathbb{Z}, X, T)$ has tracial rank zero if and only if the image of $K_0(A)$ in $\mathrm{Aff}(T(A))$ is dense. So in this case we wonder whether the equality

$$\mathrm{htr}(\mathrm{Ad}u_T) = \mathrm{h}_{\mathrm{top}}(T)$$

holds, where u_T is the implementing unitary of T in A.

The inequality $\mathrm{htr}(\mathrm{Ad}u_T) \geq \mathrm{h}_{\mathrm{top}}(T)$ is easily checked. Indeed, let ν be a T-invariant probability measure and τ_ν be the induced tracial state on A. Then by Remark 3.3 and [12, Corollary 8.1.2], we have

$$\mathrm{htr}(\mathrm{Ad}u_T) \geq \mathrm{h}_{\tau_\nu}(\mathrm{Ad}u_T) = \mathrm{h}_\nu(T).$$

Hence it follows from the classical variational principle that

$$\mathrm{htr}(\mathrm{Ad}u_T) \geq \sup_\nu \mathrm{h}_\nu(T) = \mathrm{h}_{\mathrm{top}}(T).$$

5. Cat maps on non-commutative tori

In this section, we consider automorphisms on non-commutative tori. The main result is that, under certain diophantine constraints, the tracial topological approximation entropy is bounded above by the classical entropy of the corresponding automorphism of the commutative torus.

For an irrational number θ, let A_θ be the irrational rotation algebra generated by two unitaries U and V satisfying

$$VU = e^{2\pi\theta\sqrt{-1}}UV.$$

For $\mu = (k, \ell) \in \mathbb{Z}^2$, we write $W(\mu) = U^k V^\ell$. Then linear combinations of $W(\mu)$ are dense in A_θ. Note that A_θ is simple and has the unique tracial state τ_θ given by $\tau_\theta(W(\mu)) = 0$ whenever $\mu \neq (0, 0)$.

For each matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}),$$

we define $\alpha_T(U) = U^a V^b$ and $\alpha_T(V) = U^c V^d$ and extend it to $U^k V^\ell$ by using the universality of A_θ and the relation $VU = e^{2\pi\theta\sqrt{-1}}UV$. The fact that α_T is onto follows easily from the relation $ad - bc = 1$, and then we obtain the automorphism α_T on A_θ.

We assume that T has real eigenvalues λ and λ^{-1} with $|\lambda| > 1$. For the value of other entropies of these automorphisms, we refer the reader to [12].

The irrational rotation algebra A_θ has tracial rank zero. That is shown by using the result of G. A. Elliott and D. E. Evans in [5], which is that A_θ is an A$\mathbb{T}$-algebra with real rank zero. The following results plays a key role.

Theorem 5.1 ([5, Theorem 1])**.** *There exists a continuous function $C : \mathbb{R}_+ \to \mathbb{R}_+$ with the following property. Let $s, s', t, t' \in \mathbb{Z}$ be such that*

$$\frac{t}{s} < \theta < \frac{t'}{s'}$$

with $st' - s't = 1$ and $s, s' > 0$. Then there exists a unital C^*-subalgebra $A_\theta(t/s, t'w/s')$ of A_θ, which is isomorphic to

$$M_s(C(\mathbb{T})) \oplus M_{s'}(C(\mathbb{T})),$$

such that

$$\max(\mathrm{dist}(U,u),\mathrm{dist}(V,v)) < C(\gamma)\max\left\{\frac{1}{s},\frac{1}{s'}\right\},$$

where $u \in A_\theta(t/s, t'/s')$ *is a unitary of the form*

$$w_s \oplus z_{s'} = \begin{bmatrix} 1 & & & \\ & \omega_s & & \\ & & \ddots & \\ & & & \omega_s^{s-1} \end{bmatrix} \oplus \begin{bmatrix} 0 & & & z \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \qquad (\omega_s = e^{2\pi\sqrt{-1}t/s}),$$

$v \in A_\theta(t/s, t'/s')$ *is a unitary of the form*

$$z_s \oplus w_{s'} = \begin{bmatrix} 0 & & & z \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & & & \\ & \omega_{s'} & & \\ & & \ddots & \\ & & & \omega_{s'}^{s'-1} \end{bmatrix} \qquad (\omega_{s'} = e^{2\pi\sqrt{-1}t'/s'}),$$

and

$$\gamma = \frac{\theta - t/s}{t'/s' - \theta}.$$

It is known that if a unital C^*-algebra A has real rank zero, then A has the weak (FU) property; i.e., any unitary $u \in U(A)_0$ can be approximated by unitaries with a finite spectrum. We need the following lemmas to control the cardinal number of the spectrum for our purpose. The proof is essentially as in [10].

Lemma 5.2 (cf. [10, Theorem 4.2.8])**.** *For any* $\varepsilon > 0$*, there is* $d \in \mathbb{N}$ *satisfying the following: If the unital* C^**-algebra* A *has real rank zero, then for any unitary* $u \in U(A)_0$*, there is a unitary* $u' \in A$ *with at most* d *spectrums such that* $\|u - u'\| < \varepsilon$*.*

Lemma 5.3 (cf. [10, Lemma 4.3.4])**.** *Let* A *be a unital simple* C^**-algebra with real rank zero and stable rank one, and* $u \in U(A)$*. Then for any non-zero projection* $e \in A$ *and* $\varepsilon > 0$*, there exists a non-zero projection* $p \in A$ *such that* (i) $[1-p] \le [e]$ *and* (ii) $\|u - u_1 \oplus u_2\| < \varepsilon$ *for some* $u_1 \in U(pAp)_0$ *and* $u_2 \in U((1-p)A(1-p))$*.*

Now we show that A_θ has tracial rank zero with a certain estimation of rank of finite dimensional C^*-subalgebras.

Lemma 5.4 (cf. [10, Theorem 4.3.5])**.** *For any* $\delta > 0$*, there exists* $d \in \mathbb{N}$ *satisfying the following: Let* $s, s', t, t' \in \mathbb{N}$ *be such that* $t/s < \theta < t'/s'$ *with* $st' - s't = 1$ *and* $s, s' > 0$*, and let* $u, v \in A_\theta(t/s, t'/s')$ *be unitaries as in Theorem* 5.1*. For any non-zero positive element* $a \in A_\theta$*, there exists a finite dimensional* C^**-subalgebra* $B_\theta(t/s, t'/s') \subset A_\theta$ *with the unit* p *such that*

(1) $\|pu^k v^\ell - u^k v^\ell p\| < (\max\{|k|/s', |\ell|/s\} + 1)\delta$,
(2) $pu^k v^\ell p \in_{(\max\{|k|/s', |\ell|/s\}+1)\delta} B_\theta(t/s, t'/s')$,
(3) $[1-p] \le [a]$,
(4) $\mathrm{rank}\ B_\theta(t/s, t'/s') \le d(s + s')$.

Proof. For any $\delta > 0$, we get $d \in \mathbb{N}$ by applying Lemma 5.2 for $\varepsilon = \delta/2$.

By the proof of [5, Theorem 1] the C^*-subalgebra of A_θ, generated by the system $\{e_{ij}\}$ of matrix units and the unitary z_{1s} in $e_{11}A_\theta e_{11}$, is isomorphic to $M_s(C(\mathbb{T}))$ with unit e, and the C^*-subalgebra of A_θ, generated by the system $\{e'_{ij}\}$ of matrix

units and the unitary $z'_{1s'}$ in $e'_{11}A_\theta e'_{11}$, is isomorphic to $M_{s'}(C(\mathbb{T}))$ with the unit $e' = 1 - e$. We set

$$z_s = \sum_{i=0}^{s-2} e_{2+i,1+i} + z_{1s}e_{1s} \quad \text{and} \quad w_s = \sum_{i=0}^{s-1} \omega_s^i e_{ii}.$$

Similarly we define $z'_{s'}$ and $w'_{s'}$, respectively. Then we have $u = w_s \oplus z'_{s'}$ and $v = z_s \oplus w'_{s'}$.

Let a be a non-zero positive element in A_θ. Since A_θ is simple with (SP), there exists a non-zero projection $e_0 \in \overline{aA_\theta a}$ such that $\max\{s, s'\}[e_0] \le [a]$. Moreover, there are mutually equivalent and mutually orthogonal non-zero projections $e_1, e_2 \in e_0A_\theta e_0$. By using (SP) again, there is a non-zero projection f_1 in $e_{11}A_\theta e_{11}$ such that $[f_1] \le [e_1]$. By applying Lemma 5.3, there is a non-zero projection q in $e_{11}A_\theta e_{11}$ such that $[e_{11} - q] \le [f_1]$ and

$$\|z_{1s} - z_1 \oplus z_2\| < \delta/2,$$

where $z_1 \in U(qA_\theta q)_0$ and $z_2 \in U((e_{11}-q)A_\theta(e_{11}-q))$. Then by applying Lemma 5.2 to the unitary $z_1 \in qA_\theta q$, there is a unitary $w_1 \in qA_\theta q$ with at most d spectrums such that

$$\|z_1 - w_1\| < \delta/2.$$

We define the unital abelian C^*-subalgebra $C = C^*(w_1) \subset qA_\theta q$ and the finite dimensional C^*-subalgebra $D = M_s(C) \subset eA_\theta e$ with the unit $r = \sum_{i=1}^s e_{i1}qe_{1i}$.

Let $k, \ell \in \mathbb{Z}$. We may assume that $\ell > 0$. Then we take natural numbers $\ell', \ell'' \ge 0$ such that $\ell = \ell's + \ell''$ and $\ell'' \le s - 1$. Note that z_s^ℓ is of the form

$$\begin{array}{cl} \begin{bmatrix} & & & z^{\ell'+1} & & \\ & & & & \ddots & \\ & & & & & z^{\ell'+1} \\ z^{\ell'} & & & & & \\ & \ddots & & & & \\ & & z^{\ell'} & & & \end{bmatrix} & \leftarrow (\ell''+1)\text{-th} \end{array}$$

Since $\ell' = \ell/s - \ell''/s \le \ell/s$, we obtain

$$\|rz_s^\ell - z_s^\ell r\| < (\ell'+1)\delta \le \left(\frac{\ell}{s} + 1\right)\delta,$$
$$rz_s^\ell r \in_{(\ell/s+1)\delta} D,$$
$$[e - r] = s[e_{11} - q] \le s[f_1] \le s[e_1],$$

and rank $D \le ds$. Similarly we obtain a finite dimensional C^*-subalgebra D' of $e'A_\theta e'$ with the unit r' such that

$$\|r'z_{s'}'^k - z_{s'}'^k r'\| < \left(\frac{|k|}{s'} + 1\right)\delta,$$
$$r'z_{s'}'^k r' \in_{(|k|/s'+1)\delta} D',$$
$$[e' - r'] \le s'[e_2],$$

and rank $D' \le ds'$.

Note that $w_s, w'_{s'}$ commute with r, r' and $w_s r \in D$, $w'_{s'} r' \in D'$. Now we set the finite dimensional C^*-subalgebra $B = D \oplus D' \subset A_\theta$ with the unit $p = r \oplus r'$. Then one can show that

$$\|pu^k v^\ell - u^k v^\ell p\| < \left(\max\left\{\frac{|k|}{s'}, \frac{|\ell|}{s}\right\} + 1\right)\delta,$$
$$pu^k v^\ell p \in_{(\max\{|k|/s', |\ell|/s\}+1)\delta} B,$$
$$[1-p] = [e-r] + [e'-r'] \leq s[e_1] + s'[e_2] \leq \max\{s, s'\}[e_0] \leq [a],$$

and rank $B \leq d(s+s')$. □

Now we give the estimate of the tracial topological approximation entropy. Let θ and T be given. Then we assume that there exist $K, L > 1$ and

$$\begin{bmatrix} t'_n & t_n \\ s'_n & s_n \end{bmatrix} \in SL(2, \mathbb{Z})$$

such that for any $n \in \mathbb{N}$,

(i) $t_n/s_n < \theta < t'_n/s'_n$,
(ii) $0 < s_n, s'_n \to \infty$,
(iii) $C(\gamma_n) \leq K$,
(iv) $L^{-1}|\lambda|^n \leq s_n, s'_n \leq L|\lambda|^n$,

where C is the non-negative-valued continuous function given in Theorem 5.1, λ and λ^{-1} are the eigenvalues of T with $|\lambda| > 1$ and

$$\gamma_n = \frac{\theta - t_n/s_n}{t'_n/s'_n - \theta}.$$

We remark that [5, Lemma 3] guarantees that there exists a sequence of matrices in $SL(2, \mathbb{Z})$ satisfying the above conditions (i), (ii) and (iii).

Theorem 5.5. *Under the above assumptions, we have*

$$\mathrm{htr}(\alpha_T) = \mathrm{htr}_{\tau_\theta}(\alpha_T) \leq \log|\lambda|.$$

Proof. Recall that $\mathrm{htr}(\alpha_T) = \mathrm{htr}_{\tau_\theta}(\alpha_T)$ by Proposition 2.6. Hence it suffices to check that $\mathrm{htr}(\alpha_T) \leq \log|\lambda|$. Let $\omega \Subset A_\theta$, $0 \neq a \in (A_\theta)_+$, $\delta > 0$ and $n \in \mathbb{N}$. We will show that $\mathrm{htr}(\alpha_T, \omega; 2\delta, a) \leq \log|\lambda|$. Thanks to Proposition 4.2, we may assume that $\omega = \{W(\mu) \mid \mu \in \Sigma\}$ for some finite subset $\Sigma \subset \mathbb{Z}^2$. We put $C_\Sigma = \max\{|k| + |\ell| : \mu = (k, \ell) \in \Sigma\}$ and choose $m \in \mathbb{N}$ such that $C_\Sigma(8K + \delta) \leq \delta|\lambda|^m/L$. By Theorem 5.1 and Lemma 5.4, we obtain unitaries u_{n+m}, v_{n+m} in $A_\theta(t_{n+m}/s_{n+m}, t'_{n+m}/s'_{n+m})$ and a finite dimensional C^*-subalgebra $B_{n+m} = B_\theta(t_{n+m}/s_{n+m}, t'_{n+m}/s'_{n+m})$ with $1_{B_{n+m}} = p_{n+m}$.

Let ξ_1, ξ_2 be eigenvectors with respect to eigenvalues λ, λ^{-1} of T such that $\|\xi_1\|_2 = \|\xi_2\|_2 = 1$, where $\|\cdot\|_2$ is the Euclidean norm. For $\mu = \mu_1\xi_1 + \mu_2\xi_2 \in \Sigma$ and $j = 0, \ldots, n-1$, we have

$$\|T^j\mu\|_2 = \|\mu_1\lambda^j\xi_1 + \mu_2\lambda^{-j}\xi_2\|_2 \leq (|\mu_1| + |\mu_2|)|\lambda|^j \leq 2C_\Sigma|\lambda|^n.$$

For any $(k, \ell) = T^j\mu \in \bigcup_{i=0}^{n-1} T^i\Sigma$, thanks to Theorem 5.1, we have

$$\begin{aligned} \|U^k V^\ell - u^k_{n+m} v^\ell_{n+m}\| &< (|k| + |\ell|)C(\gamma_{n+m}) \max\{1/s_{n+m}, 1/s'_{n+m}\} \\ &\leq 4C_\Sigma|\lambda|^n K \max\{1/s_{n+m}, 1/s'_{n+m}\}. \end{aligned}$$

Moreover by Lemma 5.4, we obtain

$$\begin{aligned}\|p_{n+m}u^k_{n+m}v^\ell_{n+m} - u^k_{n+m}v^\ell_{n+m}p_{n+m}\| &< (\max\{|k|/s'_{n+m}, |\ell|/s_{n+m}\} + 1)\delta \\ &\le \max\{|k|, |\ell|\}\max\{1/s'_{n+m}, 1/s_{n+m}\}\delta + \delta \\ &\le C_\Sigma|\lambda|^n \max\{1/s_{n+m}, 1/s'_{n+m}\}\delta + \delta,\end{aligned}$$

and $p_{n+m}u^k_{n+m}v^\ell_{n+m}p_{n+m} \in_{C_\Sigma|\lambda|^n \max\{1/s_{n+m},1/s'_{n+m}\}\delta+\delta} B_{n+m}$. Hence U^kV^ℓ are almost contained in B_{n+m} with error at most

$$\begin{aligned}C_\Sigma(8K+\delta)|\lambda|^n \max\{1/s_{n+m}, 1/s'_{n+m}\} + \delta &\le \frac{\delta}{L}|\lambda|^{n+m}\max\{1/s_{n+m}, 1/s'_{n+m}\} + \delta \\ &\le 2\delta.\end{aligned}$$

Since $\operatorname{rank} B_{n+m} \le d(s_{n+m} + s'_{n+m}) \le 2dL|\lambda|^{n+m}$, we have $\operatorname{htr}(\alpha_T, \omega; 2\delta, a) \le \log|\lambda|$. □

Corollary 5.6. *Suppose that an irrational number θ has a continued fraction expansion $[a_0, a_1, a_2, \dots]$ where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ are the unique choices so that*

$$\theta = \lim_{n\to\infty}[a_0, a_1, \dots, a_n] = \lim_{n\to\infty} a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

If the sequence $\{a_n\}_{n\ge 0}$ is bounded, then we have

$$\operatorname{htr}(\alpha_T) = \operatorname{htr}_{\tau_\theta}(\alpha_T) \le \log|\lambda|.$$

In particular, if θ is an algebraic number of degree 2, then the above inequality holds.

Proof. The rational approximations $b_n/c_n = [a_0, a_1, \dots, a_n]$ are determined by the recursion formulae

$$\begin{aligned}b_0 &= a_0, & c_0 &= 1, \\ b_1 &= a_1a_0 + 1, & c_1 &= a_1, \\ b_n &= b_{n-1}a_n + b_{n-2}, & c_n &= c_{n-1}a_n + c_{n-2} \quad \text{for } n \ge 2.\end{aligned}$$

In particular $\{b_n/c_n\}$ is an alternating sequence satisfying

$$\begin{bmatrix} b_{2n+1} & b_{2n} \\ c_{2n+1} & c_{2n} \end{bmatrix} \in SL(2, \mathbb{Z})$$

and

$$\frac{b_{2n-2}}{c_{2n-2}} < \frac{b_{2n}}{c_{2n}} < \theta < \frac{b_{2n+1}}{c_{2n+1}} < \frac{b_{2n-1}}{c_{2n-1}}.$$

Note that

$$\begin{aligned}\gamma_n = \frac{\theta - b_{2n}/c_{2n}}{b_{2n+1}/c_{2n+1} - \theta} &\leq \frac{b_{2n+1}/c_{2n+1} - b_{2n}/c_{2n}}{b_{2n+1}/c_{2n+1} - b_{2n+3}/c_{2n+3}} \\ &\leq \frac{b_{2n+1}c_{2n} - b_{2n}c_{2n+1}}{c_{2n}c_{2n+1}} \cdot \frac{c_{2n+1}c_{2n+3}}{b_{2n+1}c_{2n+3} - b_{2n+3}c_{2n+1}} \\ &= \frac{c_{2n+3}}{c_{2n}} \frac{1}{b_{2n+1}c_{2n+3} - b_{2n+3}c_{2n+1}} \\ &= \frac{c_{2n+3}}{c_{2n}} \frac{1}{a_{2n+3}}.\end{aligned}$$

Similarly we have

$$\gamma_n \geq \frac{c_{2n+1}}{c_{2n+2}} a_{2n+2}.$$

Since for any $m \geq 2$,

$$\frac{c_{m+1}}{c_m} = a_{m+1} + \frac{c_{m-1}}{c_m} < a_{m+1} + 1,$$

if $\{a_n\}$ is bounded, then there is $M > 1$ such that $c_{m+1}/c_m \leq M$ for $m \in \mathbb{N}$. Hence there is $D > 1$ such that $D^{-1} \leq \gamma_n \leq D$ for any $n \geq 1$. Take $k_n \in \mathbb{N}$ such that $c_{k_n-1} < |\lambda|^n \leq c_{k_n}$. Then

$$\frac{1}{M} \leq \frac{c_{k_n-1}}{c_{k_n}} < \frac{|\lambda|^n}{c_{k_n}} \leq 1.$$

Namely $|\lambda|^n \leq c_{k_n} \leq M|\lambda|^n$. If k_n is even, then

$$|\lambda|^n \leq c_{k_n} \leq c_{k_n+1} \leq M c_{k_n} \leq M^2|\lambda|^n,$$

and hence it suffices to put

$$\begin{bmatrix} t'_n & t_n \\ s'_n & s_n \end{bmatrix} = \begin{bmatrix} b_{k_n+1} & b_{k_n} \\ c_{k_n+1} & c_{k_n} \end{bmatrix}.$$

If k_n is odd, then

$$\begin{bmatrix} t'_n & t_n \\ s'_n & s_n \end{bmatrix} = \begin{bmatrix} b_{k_n} & b_{k_n+1} \\ c_{k_n} & c_{k_n+1} \end{bmatrix}$$

satisfies the desired conditions. □

Remark 5.7. Another proof of the Elliott-Evans result was given in [6], using Rieffel's imprimitivity bimodules [13] instead of cancelation in continuous fields. Although this proof is more explicit, it only provides the weaker estimate

$$\max(\mathrm{dist}(U, u), \mathrm{dist}(V, v)) < C \max\left\{\sqrt{\frac{1}{s}}, \sqrt{\frac{1}{s'}}\right\}$$

in Theorem 5.1, resulting in the weaker bound $\mathrm{htr}(\alpha_T) \leq 2\log|\lambda|$ in Corollary 5.6.

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References

[1] Brown, N. P. *Topological entropy in exact C^*-algebras.* Math. Ann. **314** (1999), no. 2, 347–367. MR1697449 (2000g:46093)

[2] Blackadar, B. *Comparison theory for simple C^*-algebras.* Operator algebras and applications, Vol. 1, pp. 21–54, London Math. Soc. Lecture Note Ser., 135, Cambridge Univ. Press, Cambridge, 1988. MR996438 (90g:46078)

[3] Connes, A.; Narnhofer, H.; Thirring, W. *Dynamical entropy of C^*-algebras and von Neumann algebras.* Comm. Math. Phys. **112** (1987), no. 4, 691–719. MR910587 (89b:46078)

[4] Connes, A.; Størmer, E. *Entropy for automorphisms of II_1 von Neumann algebras.* Acta Math. **134** (1975), no. 3-4, 289–306. MR0454657 (56:12906)

[5] Elliott, G. A.; Evans, D. E. *The structure of the irrational rotation C^*-algebra.* Ann. of Math. (2) **138** (1993), no. 3, 477–501. MR1247990 (94j:46066)

[6] Elliott, G. A.; Lin, Q. *Cut-down method in the inductive limit decomposition of non-commutative tori.* J. London Math. Soc. (2) **54** (1996), no. 1, 121–134. MR1395072 (98e:46067)

[7] Golodets, V. Y.; Neshveyev, S. V. *Entropy of automorphisms of II_1-factors arising from the dynamical systems theory.* J. Funct. Anal. **181** (2001), no. 1, 14–28. MR1818110 (2002i:46069)

[8] Hu, S.; Lin, H.; Xue, Y. *The tracial topological rank of C^*-algebras. II.* Indiana Univ. Math. J. **53** (2004), no. 6, 1578–1603. MR2106337 (2005i:46063)

[9] Lin, H. *The tracial topological rank of C^*-algebras.* Proc. London Math. Soc. (3) **83** (2001), no. 1, 199–234. MR1829565 (2002e:46063)

[10] Lin, H. *An introduction to the classification of amenable C^*-algebras.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001. MR1884366 (2002k:46141)

[11] Lin, H.; Phillips, N. C. *Crossed products by minimal homeomorphisms.* Preprint, 2004, arXiv:math.OA/0408291.

[12] Neshveyev, S.; Størmer, E. *Dynamical entropy in operator algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **50**. Springer-Verlag, Berlin, 2006. MR2251116 (2007m:46108)

[13] Rieffel, M. A. *Projective modules over higher-dimensional noncommutative tori.* Canad. J. Math. **40** (1988), no. 2, 257–338. MR941652 (89m:46110)

[14] Voiculescu, D. *Dynamical approximation entropies and topological entropy in operator algebras.* Comm. Math. Phys. **170** (1995), no. 2, 249–281. MR1334396 (97b:46082)

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