POINCARÉ SERIES AND THE DIVISORS OF MODULAR FORMS

D. CHOI

Abstract. Recently, Bruinier, Kohnen and Ono obtained an explicit description of the action of the theta-operator on meromorphic modular forms $f$ on $SL_2(\mathbb{Z})$ in terms of the values of modular functions at points in the divisor of $f$. Using this result, they studied the exponents in the infinite product expansion of a modular form and recurrence relations for Fourier coefficients of a modular form. In this paper, we extend these results to meromorphic modular forms on $\Gamma_0(N)$ for an arbitrary positive integer $N > 1$.

1. Introduction

Let $N$ be a positive integer and $q := e^{2\pi iz}$, $z = x + iy$. If $f(z) = \sum_{n=-\infty}^{\infty} a(n)q^n$ is a meromorphic modular form on $\Gamma_0(N)$, then we define the theta-operator by

$$\theta f(z) := \frac{1}{2\pi i} \frac{d}{dz} f(z) = \sum_{n=-\infty}^{\infty} na(n)q^n.$$ 

Recently, Bruinier, Kohnen and Ono obtained in [6] an explicit description of the action of the theta-operator on meromorphic modular forms on $SL_2(\mathbb{Z})$ in terms of the values of modular functions at points in the divisor of $f$. Using this result, they also studied the exponents in the infinite product expansion of a modular form and recurrence relations for the Fourier coefficients of modular forms (see Theorems 3 and 5 in [6]). Ahlgren gave analogues of these results for meromorphic modular forms on $\Gamma_0(p)$ for $p \in \{2, 3, 5, 7, 13\}$ (see [2]). For general primes $p$, using eta-quotients, the author studied in [8] the action of the theta-operator on meromorphic modular forms on $\Gamma_0(p)$ and the exponents in their infinite product expansion. But, recurrence relations for the Fourier coefficients of the modular forms in [6] and [2] were not generalized in [8]. In this paper, we extend the results [2] to meromorphic modular forms on $\Gamma_0(N)$ for an arbitrary positive integer $N > 1$ and obtain recurrence relations for their Fourier coefficients.

To obtain our main theorems, we consider Poincaré series of weight 0 instead of eta-quotients. Since in general these Poincaré series are not meromorphic functions on the complex upper half plane, we cannot use the valence formula or the residue theorem as in [6], [2] and [8]. Thus, following the argument of [5], we use the regularized integral and Stokes' Theorem.
To state our results, we introduce some notation. The group $\Gamma_0(N)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$  

Let $C_N$ be the set of inequivalent cusps of $\Gamma_0(N)$ and $C_N^* = C_N \setminus \{\infty\}$. For a cusp $u$ of $\Gamma_0(N)$ let

$$\Gamma_0(N)_u := \{ \sigma \in \Gamma_0(N) \mid \sigma u = u \}.$$  

Here, $\sigma u := \frac{au + b}{cu + d}$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. We denote by $\mathcal{F}_N$ a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$. The modular curve $X_0(N)$ is defined as the quotient space of orbits under $\Gamma_0(N)$, $X_0(N) = \Gamma_0(N) \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that $f(z)$ is a meromorphic modular form of weight $k$ on $\Gamma_0(N)$. We consider the following functions: a meromorphic modular form $f_\theta(z)$ and a Poincaré series $j_{N,m}(z)$ of weight 0 and index $m$. For $N > 1$ let

$$f_\theta(z) := \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N - 1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N - 1} \cdot E_2(z).$$

Here, $E_2(z)$ is the usual normalized Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where $\sigma_k(n) := \sum_{d|n} d^k$ if $n \in \mathbb{N}$ and $\sigma_k(n) = 0$ if $n \not\in \mathbb{N}$.

Let $I_v(z)$ be the usual modified Bessel functions as in [1] and $e(x) := e^{2\pi i x}$. For a positive integer $m$ we define the Poincaré series of weight 0 and index $m$ by

$$F_{N,m}(z,s) := \sum_{\gamma \in \Gamma_0(N) \backslash \Gamma_0(N)} \pi |m \Im(\gamma z)|^{1/2} I_{s-\frac{1}{2}}(2\pi m \Im(\gamma z)) e(-m \Re(\gamma z)),$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. Let $j_{N,m}(z)$ be the continuation of $F_{N,m}(z,s)$ as $s \to 1$ from the right. The function $j_{N,m}(z)$ is a weak Maass form on $\Gamma_0(N)$ (see Section 2 for details). For convenience, if $t$ is a cusp of $\Gamma_0(N)$, then $j_{N,m}(t)$ denotes the constant term of the Fourier expansion of $F_{N,m}(z,1)$ at $t$. We define a differential operator $\xi_0$ by

$$\xi_0(j_{N,m})(z) := 2i \frac{\partial}{\partial z} j_{N,m}(z)$$

and consider the integral

$$\int_{\mathcal{F}_N} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy.$$  

Note that in general $f_\theta(z)$ is not holomorphic on $\mathbb{H}$. Thus, we have to regularize the integral (1.3) and denote it by

$$\int_{\mathcal{F}_N}^{\text{reg}} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy$$

(see (5.1) in Section 3 for the exact definition). With this notation, we state our main theorem.
Theorem 1.1. Suppose that \( f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n \) is a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \) with a positive integer \( N, N > 1 \). Let \( \{c(n)\}_{n=1}^{\infty} \) be the complex numbers for which
\[
f(z) = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}.
\]

Then we have
\[
\sum_{d|m} d \cdot c(d) = \sum_{\tau \in \mathcal{F} \cup \mathcal{C}_N} \nu_{\tau}^{(N)}(f(z)) j_{N,m}(\tau) - \int_{\mathcal{F}_N} f_0(z) \cdot \xi_0(j_{N,m}(z)) dx dy + \frac{2Nk - 24h}{N - 1} \sigma_1(m) + \frac{24h - 2k}{N - 1} N \sigma_1(m/N).
\]

Here, \( \nu_{\tau}^{(N)}(f(z)) \) denotes the order of zero of \( f(z) \) at \( \tau \) on \( \mathbb{H} \).

Remark 1.2. Suppose that \( j_{N,m}(z) \) is holomorphic on \( \mathbb{H} \). From the definition of the differential operator \( \Omega \) we have
\[
\xi_0(j_{N,m}(z)) = 0.
\]

This implies that in Theorem 1.1
\[
\int_{\mathcal{F}_N} f_0(z) \cdot \xi_0(j_{N,m}(z)) dx dy = 0.
\]

Remark 1.3. From Remark 1.2 it may be an interesting question when \( j_{N,m}(z) \) is holomorphic on \( \mathbb{H} \). Since \( \xi_0(j_{N,m}(z)) \) is a cusp form, the results of [5] imply that \( j_{N,m}(z) \) is holomorphic on \( \mathbb{H} \) if and only if for every cusp form \( g(z) = \sum_{n=1}^{\infty} b(n)q^n \) of weight 2 on \( \Gamma_0(N) \) we have
\[
b(m) = 0.
\]

(1) Suppose that the genus of \( \Gamma_0(N) \) is zero. Then \( j_{N,m}(z) \) is holomorphic on \( \mathbb{H} \). Thus, when \( N \in \{2, 3, 5, 7, 13\} \), Theorem 1.1 recovers the result of Ahlgren [3].

(2) Suppose that the genus of \( \Gamma_0(N) \) is one and that \( g(z) = \sum_{n=1}^{\infty} b(n)q^n \) is the unique normalized cusp form of weight 2 on \( \Gamma_0(N) \). For a prime \( p \) let \( \mathbf{F}_p \) denote a finite field \( \mathbb{Z}/p\mathbb{Z} \). There exists an elliptic curve \( E_g \) of conductor \( N', N'|N \), defined over \( \mathbb{Q} \) such that for all \( p \nmid N \)
\[
1 - a(p) + p = \sharp E_g(F_p).
\]

Note that \(|a(p)| \leq 2\sqrt{p}\) and \( g(z) \) is a Hecke eigenform. Thus, for an odd integer \( m, j_{N,m}(z) \) is holomorphic on \( \mathbb{H} \) if and only if \( E_g \) is supersingular at \( p \) for some prime \( p, p|m \). In [10] Elkies proved the existence of infinitely many supersingular primes for every elliptic curve defined over \( \mathbb{Q} \). This implies that there are infinitely many primes \( p \) such that \( j_{N,m}(z) \) is holomorphic on \( \mathbb{H} \) for every positive integer \( m, p|m \).

Using Theorem 1.1 we obtain a description of the action of the theta-operator on meromorphic modular forms on \( \Gamma_0(N) \).
Theorem 1.4. Suppose that \( f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n \) is a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \) with a positive integer \( N > 1 \). Then
\[
f_\Theta(z) + \sum_{m=1}^{\infty} \left( \int_{F_N}^{reg} f_\Theta(z) \cdot \xi_0(j_{N,m}(z))dxdy \right) q^m = \sum_{m=1}^{\infty} \sum_{\tau \in F_N \cup \mathcal{C}_N} \nu^1(N)(f(z))j_{N,m}(\tau) q^m.
\]

From Remark 1.3(2) and Theorem 1.4 we have immediately the following corollary.

Corollary 1.5. Suppose that the genus of \( \Gamma_0(N) \) is one and that \( f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n \) is a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \) with a positive integer \( N > 1 \). Then there are infinitely many primes \( p \) such that
\[
f_\Theta(z)|_{U_p} = \sum_{m=1}^{\infty} \sum_{\tau \in F_N \cup \mathcal{C}_N} \nu^1(N)(f(z))j_{N,p,m}(\tau) q^m.
\]

Remark 1.6. The formula in Corollary 1.5 does not contain the regularized integral. Thus, if \( f_\Theta(z)|_{U_p} \cdot E_{\ell-1} \) is a modular form for some prime \( \ell \), then we can study congruence for the values of \( j_{N,p,m} \) by using the argument of [6]. But, it seems difficult to check whether \( f_\Theta(z)|_{U_p} \cdot E_{\ell-1} \) is a modular form.

As another application of Theorem 1.1 we obtain universal recursion formulas for coefficients of meromorphic modular forms on \( \Gamma_0(N) \). For each \( n \geq 1 \), we define the polynomial
\[(1.5) \quad F^N_n(K, H, x_1, \cdots, x_n) := \sum_{m_1+2m_2+\cdots+(n-1)m_{n-1}=m, m_1, \cdots, m_{n-1} \geq 0} (-1)^{m_1+\cdots+m_{n-1}} \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!m_2!\cdots m_{n-1}!} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} - \frac{1}{n} \left( \frac{2NK - 24H}{N - 1} \right) \sigma_1(m) - \frac{1}{n} \left( \frac{24H - 2K}{N - 1} \right) N\sigma_1(m/N).
\]

Theorem 1.7. Suppose that
\[f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n\]
is a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \) with a positive integer \( N, N > 1 \). For each \( n \geq 1 \), the polynomial \( F^N_n(K, H, x_1, \cdots, x_n) \) is defined as in (1.5). Then we have
\[a(h+n) = F^N_n(k, h, a(h+1), \cdots, a(h+n-1)) - \frac{1}{n} \sum_{\tau \in F_N \cup \mathcal{C}_N} \nu^1(N)(f(z))j_{N,m}(\tau) + \frac{1}{n} \int_{F_N}^{reg} f_\Theta(z) \cdot \xi_0(j_{N,m}(z))dxdy.
\]

Example 1.8. Let \( E \) be an elliptic curve of conductor \( N \) defined over \( \mathbb{Q} \) and let \( \mathbb{F}_p \) denote a finite field \( \mathbb{Z}/p\mathbb{Z} \) for a prime \( p \). It is known that there is a normalized
Hecke eigenform $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ of weight 2 on $\Gamma_0(N)$ such that for all $p \nmid N,$

$$1 - a(p) + p = \sharp E(F_p).$$

From Theorem 1.7 we have

(1.6) \quad \quad (1.6)

$$a(1 + n) = F_n^{(N)}(2, 1, a(2), \cdots, a(n))$$

$$-\frac{1}{n} \sum_{\tau \in F_N \cup C_N} \nu^{(N)}(f(z)) j_{N,m}(\tau) + \frac{1}{n} \int_{F_N} f_0(z) \cdot \xi_0(j_{N,m}(z)) dx dy.$$ 

By the definition of the Hecke operator, we obtain from (1.6) recursive relations for \(\sharp E(F_p).\) For example, if \(N = 11\) and \(m = 1,\) then

$$\int_{F_N} f_0(z) \cdot \xi_0(j_{N,m}(z)) dx dy = 0.$$ 

Remark 1.9. Our results (Theorems 1.1, 1.4 and 1.7) can be immediately extended to congruence subgroups \(\Gamma \subset \Gamma_0(N)\) for \(N > 1.\)

This paper is organized as follows. In section 2 we recall the definition of Poincaré series of weight zero on \(\Gamma_0(N).\) In section 3 we define the regularized integral of a meromorphic modular form of weight 2. In sections 4 and 5 we give the proofs of the main theorems.

2. Poincaré series of weight 0 on \(\Gamma_0(N)\)

In this section we consider non-holomorphic Poincaré series of weight 0. For details we refer to [11], [13], [4], [7].

Let \(I_s(z)\) and \(K_s(z)\) be the usual modified Bessel functions as in [1]. We define for \(s \in \mathbb{C}\) and \(y \in \mathbb{R}\setminus\{0\}:

$$I_s(y) := \sqrt{\pi |y|} I_{s-\frac{1}{2}}(|y|),$$

$$K_s(y) := \sqrt{\pi |y|} K_{s-\frac{1}{2}}(|y|).$$

Note that \(I_s(y)\) and \(K_s(y)\) are holomorphic in \(s.\) If \(s = 1,\) then we have

$$I_1(y) = \sinh(|y|),$$

$$K_1(y) = e^{-|y|},$$

$$2I_1(y) + K_1(y) = e^{|y|}.$$ 

For a cusp \(u\) let \(\sigma_u\) denote a matrix in \(SL_2(\mathbb{R})\) such that

$$\sigma_u^{-1} \infty = u \quad \text{and} \quad \sigma_u \Gamma_0(N) \sigma_u^{-1} = \{ \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \mid n \in \mathbb{Z} \}.$$ 

If \(t_1\) and \(t_2\) are equivalent cusps of \(\Gamma_0(N),\) i.e., \(\gamma t_1 = t_2\) for some \(\gamma \in \Gamma_0(N),\) then we write \(t_1 \sim t_2.\) For a positive integer \(m\) we define the Poincaré series of weight 0 and index \(m\) by

(2.1) \quad \quad (2.1)

$$F_{N,m}(z, s) := \sum_{\gamma \in \Gamma_0(N) \setminus \Gamma_0(N)} I_s(2\pi m \text{ Im}(\gamma z)) e(-m \text{ Re}(\gamma z)), $$

where \(z \in \mathbb{H}\) and \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1.\) The special value \(F_{N,m}(z, 1)\) satisfies the following properties.
Theorem 2.1. The function $F_{N,m}(z,1)$ is a harmonic weak Maass form of weight 0 on $\Gamma_0(N)$. Moreover, $F_{N,m}(z,1)$ has the following properties at each cusp $t$:

(2.2) $F_{N,m}(z,1) = q^{-m} + \sum_{n>0} b_n(n,1)q^n + \sum_{n>0} b_n(-n,1)e(nz)$ if $t \sim \infty$,

(2.3) $\lim_{\text{Im } z \to \infty} F_{N,m}(\sigma_t z,1) = j_{N,m}(t)$ if $t \sim \infty$.

Proof. Let

(2.4) $I_\gamma(z,s) := \sum_{\beta \in \Gamma_0(N)_{\infty}} I_{\delta}(2\pi m \text{ Im}(\gamma \beta z)) e(-m \text{ Re}(\gamma \beta z))$.

If we define

$\delta_{t,\infty} := 1$ if $t \sim \infty$,

$\delta_{t,\infty} := 0$ if $t \sim \infty$,

then

$F_{N,m}(\sigma_t z,s) = \delta_{t,\infty} \cdot I_{\delta}(2\pi m \text{ Im}(z)) e(-m \text{ Re}(z)) + \sum_{\gamma \in \Gamma_0(N)_{\infty} \setminus \Gamma_0(N)_{\sigma_t} \Gamma_0(N)_{\infty}} I_\gamma(z,s)$.

Considering the Fourier expansion of

$\sum_{\gamma \in \Gamma_0(N)_{\infty} \setminus \Gamma_0(N)_{\sigma_t} \Gamma_0(N)_{\infty}} I_\gamma(z,s)$,

we can obtain (2.2) and (2.3). For the details of the proof, see §1.9 in [4] or [11, 13].

3. Regularized integration

Suppose that $g(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. Let $S(g)$ be the set of the singular points of $g(z)$ on $\mathcal{F}_N$. We define an $\varepsilon$-disk $B(t,\varepsilon)$ at $t$ by

$B(t,\varepsilon) := \begin{cases} \{ z \in \mathbb{H} \mid |z-t| < \varepsilon \} & \text{if } t \in \mathcal{F}_N, \\ \{ z \in \mathcal{F}_N \mid \text{Im}(\sigma_t z) > \varepsilon \} & \text{if } t \in \mathcal{C}_N. \end{cases}$

For sufficiently small positive $\varepsilon$, let $\mathcal{F}_N(g,\varepsilon)$ denote a punctured fundamental domain for $\Gamma_0(N)$ defined as

$\mathcal{F}_N(g,\varepsilon) = \mathcal{F}_N - \bigcup_{t \in S(g) \cup \mathcal{C}_N} B(t,\varepsilon)$.

We define the regularized integral of $g(z) \cdot \xi_0(j_{N,m}(z))dz$ on $\mathcal{F}_N$ by

(3.1) $\int_{\mathcal{F}_N} g(z) \cdot \xi_0(j_{N,m}(z))dz := \lim_{\varepsilon \to 0} \int_{\mathcal{F}_N(g,\varepsilon)} g(z) \cdot \xi_0(j_{N,m}(z))dz$.

Note that $g(z) \cdot j_{N,m}(z)dz$ is a $\Gamma_0(N)$-invariant 1-form on $\mathbb{H}$. We define

$\gamma(t,\varepsilon) := \begin{cases} \{ z \in \mathbb{H} \mid |z-t| = \varepsilon \} & \text{if } t \in \mathbb{H}, \\ \{ z \in \mathcal{F}_N \mid \text{Im}(\sigma_t z) = \varepsilon \} & \text{if } t \text{ is a cusp of } \Gamma_0(N). \end{cases}$

Then

$\partial \mathcal{F}_N(g,\varepsilon) = \bigcup_{t \in S(g) \cup \mathcal{C}_N} \gamma(t,\varepsilon)$. 
For \( t \in \mathbb{H} \) let \( \text{Res}_t(g) \) be the residue of \( g \) at \( t \) on \( \mathbb{H} \). Using the argument of Lemma 3.1 and Proposition 3.5 in [5], we prove the following lemma.

**Lemma 3.1.** Let \( g(z) := \sum_{n=0}^{\infty} a(n) q^n \) be a meromorphic modular form of weight 2 on \( \Gamma_0(N) \). Suppose that \( g(z) \) is holomorphic at each cusp and that every pole of \( g(z) \) is a simple pole. Then we have

\[
\lim_{\varepsilon \to 0} \int_{F_N(g, \varepsilon)} g(z) \cdot \xi_0(j_{N,m}(z)) dxdy
= b_m(1,0)a(0) + a(m) + \sum_{t \in \mathbb{C}_N} \alpha_t g(t) j_{N,m}(t) + \sum_{t \in S(f)} \frac{2\pi i}{t} \text{Res}_t(g) j_{N,m}(t).
\]

**Proof.** Note that

\[
(g \cdot j_{N,m} dz) - \frac{\partial}{\partial z}(g \cdot j_{N,m} dz) = g(z) \cdot j_{N,m} dx.
\]

Thus, by Stokes' Theorem, we have

\[
\int_{F_N(g, \varepsilon)} g(z) \cdot \xi_0(j_{N,m}(z)) dxdy = \sum_{t \in S(g)} \frac{1}{l_t} \int_{\gamma(t, \varepsilon)} g(z) j_{N,m}(z) dz + \sum_{t \in \mathbb{C}_N} \int_{\gamma(t, \varepsilon)} g(z) j_{N,m}(z) dz.
\]

Note that if \( t \in \mathbb{C}_N \) and \( \varepsilon \) is sufficiently small, then

\[
\int_{\gamma(t, \varepsilon)} g(z) j_{N,m}(z) dz = \alpha_t \int_{-\frac{1}{2}}^{\frac{1}{2}} g \left( \sigma_t^{-1} \left( x + i \frac{1}{\varepsilon} \right) \right) j_{N,m} \left( \sigma_t^{-1} \left( x + i \frac{1}{\varepsilon} \right) \right) dx.
\]

Here, \( \alpha_t \) is a non-zero constant, and \( \alpha_t = 1 \) for a cusp \( t \sim \infty \).

Following the argument of Proposition 3.5 in [5], for a cusp \( t \sim \infty \) we have

\[
\lim_{\varepsilon \to 0} \int_{\gamma(t, \varepsilon)} g(z) j_{N,m}(z) dz = b_m(1,0)a(0) + a(m),
\]

and for \( t \neq \infty \),

\[
\lim_{\varepsilon \to 0} \int_{\gamma(t, \varepsilon)} g(z) j_{N,m}(z) dz = \alpha_t g(t) j_{N,m}(t).
\]

From now on, we consider

\[
\lim_{\varepsilon \to 0} \int_{\gamma(t, \varepsilon)} g(z) j_{N,m}(z) dz
\]

for \( t \in S(g) \). Suppose that \( g(z) \) has the Laurent series at \( t \):

\[
g(z) = \sum_{n=-1}^{\infty} a_n(t)(z - t)^n.
\]
Then we have
\[
\int_{\gamma(t, \epsilon)} g(z)j_{N,m}(z)dz = \frac{1}{l_{\tau}} \int_0^1 g(t + \epsilon e^{2\pi i u})j_{N,m}(t + \epsilon e^{2\pi i u})2\pi i \epsilon e^{2\pi i u}du \\
= \frac{2\pi i}{l_{\tau}} \int_0^1 \left( \sum_{n=0}^{\infty} a_{n-1}(t)(\epsilon e^{2\pi i u})^n \right) j_{N,m}(t + \epsilon e^{2\pi i u})du \\
= \frac{2\pi i}{l_{\tau}} \sum_{n=0}^{\infty} a_{n-1}(t)\epsilon^n \int_0^1 e^{2\pi i nu}j_{N,m}(t + \epsilon e^{2\pi i u})du.
\]
Thus, we get
\[
\lim_{\epsilon \to 0} \int_{\gamma(t, \epsilon)} g(z)j_{N,m}(z)dz = \lim_{\epsilon \to 0} \frac{2\pi i}{l_{\tau}} \sum_{n=0}^{\infty} a_{n-1}(t)\epsilon^n \int_0^1 e^{2\pi i nu}j_{N,m}(t + \epsilon e^{2\pi i u})du \\
= \frac{2\pi i}{l_{\tau}} a_{-1}(t) \int_0^1 j_{N,m}(t)du = \frac{1}{l_{\tau}} 2\pi i a_{-1}(t)j_{N,m}(t) \\
= \frac{2\pi i}{l_{\tau}} \text{Res}(g) j_{N,m}(t).
\]
This completes the proof. \qed

4. Proof of Theorem

Suppose that \( g \) is a meromorphic modular form of weight 2 on \( \Gamma_0(N) \). For \( \tau \in \mathbb{H} \), let \( Q_\tau \) be the image of \( \tau \) under the canonical map from \( \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \) to \( X_0(N) \). The residue of \( g \) at \( Q_\tau \) on \( X_0(N) \), denoted by \( \text{Res}_{Q_\tau} gdz \), is well defined since we have the canonical correspondence between a meromorphic modular form of weight 2 on \( \Gamma_0(N) \) and a meromorphic 1-form on \( X_0(N) \). Suppose that \( t \) is a cusp of \( \Gamma_0(N) \). Let \( \sigma_t^* \) be a matrix in \( SL_2(\mathbb{Z}) \) such that \( \sigma_t^* \infty = t \). Then there exists \( \alpha_t \) such that
\[
\sigma_t^{s-1}\Gamma_0(N)s_{\alpha_t} = \left\{ \pm \begin{pmatrix} 1 & k\alpha_t \\ 0 & 1 \end{pmatrix} \bigg| k \in \mathbb{Z} \right\},
\]
where \( \Gamma_0(N) \), denotes the stabilizer of the cusp \( t \) in \( SL_2(\mathbb{Z}) \). For convenience we define \( g(t) \) by the constant term of the Fourier expansion of \( g(z) \) at \( t \). If \( \text{Res}_\tau g \) denotes the residue of \( g \) at \( \tau \) on \( \mathbb{H} \), then we obtain
\[
\text{Res}_{Q_\tau} gdz = \frac{1}{l_{\tau}} \text{Res}_\tau g \text{ if } \tau \in \mathbb{H},
\]
\[
\text{Res}_{Q_\tau} gdz = \alpha_t g(t) \text{ if } t \in \mathcal{C}_N.
\]
Here, \( l_{\tau} \) is the order of the isotropy group at \( \tau \). In particular, if \( f \) is a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \) and \( g = \frac{\theta f}{\tau} \), then the residue of \( g \) at each point on \( X_0(N) \) is determined by the order of its zero or pole. If we denote by \( \text{ord}_\tau(f) \) the order of the zero or pole of \( f \) at \( \tau \) as a complex function on \( \mathbb{H} \), then
\[
\nu_t^{(N)}(f) = \frac{1}{l_{\tau}} \text{ord}_\tau(f).
\]
Note that the constant term of the Fourier expansion of \( \frac{\theta f}{\tau} \) at a cusp \( t \) is equal to the order of its zero or pole at the cusp. Thus, for each \( t \in \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \) we have
\[
2\pi i \cdot \text{Res}_{Q_t} \frac{\theta f}{\tau} = \nu_t^{(N)}(f).
\]
Proof of Theorem. We begin by stating a lemma which was proved by Eholzer and Skoruppa in [9].
Lemma 4.1 ([9]). Suppose that \( f = \sum_{n=h}^{\infty} a(n)q^n \) is a meromorphic modular function in a neighborhood of \( q = 0 \) and that \( a(h) = 1 \). Then there are uniquely determined complex numbers \( c(n) \) such that
\[
f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},
\]
where the product converges in a small neighborhood of \( q = 0 \). Moreover, the following identity is true:
\[
\frac{\theta f}{f} = h - \sum_{n \geq 1} \sum_{d|n} c(d) dq^n.
\]
Recall that
\[
\frac{\partial f}{f} = \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N - 1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N - 1} \cdot E_2(z).
\]
The function \( f_\theta(z) \) is a meromorphic modular form of weight 2 on \( \Gamma_0(N) \). Note that \( f_\theta(z) \) is holomorphic at each cusp of \( \Gamma_0(N) \). Moreover, the function
\[
\frac{k/12 - h}{N - 1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N - 1} \cdot E_2(z)
\]
is holomorphic on \( \mathbb{H} \). Thus, we have
\[
\text{Res}_\tau f_\theta = \text{Res}_\tau \frac{\theta f}{f}
\]
for \( \tau \in \mathbb{H} \). Using Lemma 3.1 and 4.1, we complete the proof. \( \square \)

5. Proofs of Theorems 1.4 and 1.7

Proof of Theorem 1.4. Note that
\[
f_\theta(z) = \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N - 1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N - 1} \cdot E_2(z)
\]
\[
= \sum_{n=1}^{\infty} \left( - \sum_{d|n} c(d) d + \frac{2Nk - 24h}{N - 1} \sigma_1(n) + \frac{24h - 2k}{N - 1} N \sigma_1(n/N) \right) q^n.
\]
This completes the proof by Theorem 1.1. \( \square \)

Proof of Theorem 1.7. We follow the argument of Theorem 3 in [8] (or Theorem 4 in [2]). Suppose that \( f(z) \) has the \( q \)-expansion of the form
\[
f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}
\]
as in Lemma 4.1. Let
\[
b(n) := \sum_{d|n} c(d) d.
\]
Lemma 4.1 implies that
\[
\left( q^h + \sum_{n=h+1}^{\infty} a(n)q^n \right) \left( h - \sum_{n=1}^{\infty} b(n)q^n \right) = h q^h + \sum_{n=h+1}^{\infty} na(n)q^n.
\]
Thus, for $n \geq 1$ we have

$$na(h + n) = -b(1)a(h + n - 1) - b(2)a(h + n - 2) - \cdots - b(n).$$

From this recurrence we obtain

$$b(n) = -na(h + n)$$

$$\sum_{m_1+2m_2+\cdots+(n-1)m_{n-1}=0} \frac{(-1)^{m_1+\cdots+m_{n-1}} \prod_{i=1}^{n-1} (m_1 + \cdots + m_{n-1} - 1)!}{m_1!m_2!\cdots m_{n-1}!} a(h + 1)^{m_1} \cdots a(h + n - 1)^{m_{n-1}}$$

(see (2.11) and Example 20 in §1.2 of [12]). Note that by Theorem 1.1

$$b(n) = \sum_{\tau \in F_N \cup C_N} \mathbf{\nu}(\tau) (f_N)(\tau) - \int_{F_N} f_0(z) \cdot \xi_0(j_N,m_N(z)) dxdy$$

$$+ \frac{2Nk - 24h}{N-1} \sigma_1(m) + \frac{24h - 2k}{N-1} N \sigma_1(m/N).$$

This completes the proof. \qed

### Acknowledgements

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-331-C00005), and the author wishes to express his gratitude to KIAS for its support through the Associate Membership program. The author also thanks the referee for useful comments.

### References


School of Liberal Arts and Sciences, Korea Aerospace University, 200-1, Hwajeon-dong, Goyang, Gyeonggi 412-791, Korea

E-mail address: choija@kau.ac.kr