ARITHMETIC RIGIDITY

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Abstract. We prove an arithmetic analogue of rigidity results of Suslin and Beilinson, and then give some applications to countability of certain motivic cohomology groups of varieties over the complex numbers, assuming a finite generation of these groups for varieties over finitely generated fields.

Introduction

Let $K$ be a field and consider the Quillen $K$-group, $K_3(K)$. Denote by $K^M_3(K)$ the Milnor $K$-group of $K$ that is generated by symbols. The natural map

$$K^M_3(K) \to K_3(K)$$

is now known to be injective (it follows from results in [Su2] that the kernel is killed by 2). Let $K_3(K)^{ind}$ be the quotient of $K_3(K)$ by $K^M_3(K)$. This is called the indecomposable $K_3$. There is a regulator map, a real version of which was first considered by Borel [Bo] for number fields:

$$K_3^{ind}(\mathbb{C}) \to \mathbb{C}/(2\pi i)^2 \mathbb{Z}.$$ 

Beilinson showed that the image of this regulator is countable by a rigidity argument (see [Be], 2.3.4 and [M], §3, especially Corollary 3.6), which is what led to Conjecture 1 (see e.g. [MS] Conjecture 11.5). We have

$$K_3^{ind}(\overline{\mathbb{Q}}) = K_3^{ind}(\mathbb{C}),$$

so that $K_3^{ind}(\mathbb{C})$ is a countable abelian group. Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Unfortunately, it is not known whether the regulator map is injective.

There is an analogous situation for higher motivic cohomology. Let $X$ be a smooth projective variety over $\mathbb{C}$. Consider, for example, the motivic cohomology group $H^3(X, \mathbb{Z}(2))$, or, if you prefer, $H^1(X, K_2)$, where $K_2$ is the Zariski sheaf associated to the presheaf of Quillen $K_2$-groups. Using the identifications

$$Pic(X) \cong H^2(X, \mathbb{Z}(1)), \quad \mathbb{C}^* \cong H^1(X, \mathbb{Z}(1))$$

and the product structure on motivic cohomology, one gets a product map:

$$Pic(X) \otimes \mathbb{C}^* \to H^3(X, \mathbb{Z}(2)).$$
Let $H^3(X, \mathbb{Z}(2))^{\text{ind}}$ be the quotient of $H^3(X, \mathbb{Z}(2))$ by the image of the product map. Let $H^3(X, \mathbb{Z}(2))$ denote Deligne cohomology (see e.g. [EV]), and similarly $H^3(X, \mathbb{Z}(2))^{\text{ind}}$ as the quotient of $H^3(X, \mathbb{Z}(2))$ by the image of the product map:

$$H^1(X, \mathbb{Z}(1)) \otimes H^2(X, \mathbb{Z}(1)) \to H^3(X, \mathbb{Z}(2)).$$

Then Beilinson’s rigidity result (loc. cit.) also applies to the image of the regulator (cycle class) map:

$$H^3(X, \mathbb{Z}(2))^{\text{ind}} \to H^3(X, \mathbb{Z}(2))^{\text{ind}}$$

and shows that this image is countable, which leads to:

**Conjecture 2.** For $X$ smooth and projective over any field $K$, the group $H^3(X, \mathbb{Z}(2))^{\text{ind}}$ is countable.

Briefly, such rigidity arguments are as follows: if $Y$ is a model of $X$ over an algebraic closure $\bar{K}$ of some finitely generated field, then one shows using the proper base change theorem that the image of the regulator map all comes from the image of $H^3(Y, \mathbb{Z}(2))^{\text{ind}}$ and is therefore countable, since $\bar{K}$ is countable. In this note, we give an arithmetic analogue of such rigidity results, hence the name, “arithmetic rigidity”. As a consequence, we show that the images of suitable $\ell$-adic regulator maps are countable for indecomposable motivic cohomology groups on varieties over universal domains such as $\mathbb{C}$, and we show in some cases that if such motivic cohomology groups are finitely generated over any finitely generated ring over $\mathbb{Z}$, as is expected, then they are, in fact, countable over $\mathbb{C}$. At least in the case of indecomposable $K_3$, we strongly suspect that such results are known to some of the experts, but we could not find a reference. This follows easily from results of Suslin [Su2], Merkur’ev-Suslin [MS] and Levine [La]. In fact, the rigidity lemma below is a fairly simple but very useful generalization of a result of Suslin ([Su2], Corollary 2.7).

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1. **Notation and preliminaries**

Let $K$ be a field that is finitely generated over its prime subfield and let $L$ be a finitely generated, separable extension of $K$ in which $K$ is algebraically closed. We shall call such an $L$ a finitely generated regular extension of $K$. We denote by $K$ a separable closure of $K$ and by $\bar{L}$ a separable closure of $L$.

Let $\ell$ be a prime number different from the characteristic of $K$. We denote by $\mathbb{Z}/\ell^n\mathbb{Z}(r)$ the group of $\ell^n$-th roots of unity, twisted $r$ times à la Tate, and $\mathbb{Z}_\ell(r) = \lim_{\leftarrow} \mathbb{Z}/\ell^n\mathbb{Z}(r)$. If $M$ is a finitely generated $\mathbb{Z}_\ell$ module with continuous action of $\text{Gal}(\bar{K}/K)$, we set $M(r) := M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(r)$.

Let $X$ be a smooth projective variety over $K$. We denote by $\overline{X}$ the variety $X \times_K \bar{K}$. Consider the motivic cohomology groups $H^i(X, \mathbb{Z}(j))$. These may be taken in the sense of Bloch’s higher Chow groups [Bl] or Voevodsky [V]. Then there are integral and rational cycle class maps:

$$H^i(X, \mathbb{Z}(j)) \to H^i(X, \mathbb{Z}(j)_\ell(r)), $$

$$H^i(X, \mathbb{Q}(j)) \to H^i(X, \mathbb{Q}(j)_\ell(r)).$$
Here the groups on the right are the continuous étale cohomology groups in the sense of Jannsen [J]. These are the right derived functors of the functor

\[(\mathcal{F}_n) \mapsto \lim_{\leftarrow n} H^0(X, \mathcal{F}_n),\]

where \((\mathcal{F}_n)\) is an inverse system of sheaves of \(\mathbb{Z}/\ell^n\mathbb{Z}\)-modules on \(X\).

**Conjecture 3.** The rational cycle class map is injective for \(X\) as above.

There is a Hochschild-Serre spectral sequence:

\[E_{2}^{r,s} = H^r(K, H^s(X, \mathbb{Z}(j))) \Rightarrow H^{r+s}(X, \mathbb{Z}(j)).\]

We refer to the filtration on \(H^i(X, \mathbb{Z}(j))\) that one gets by pulling back the filtration given by the spectral sequence as the **Hochschild-Serre filtration**.

From the spectral sequence, we get a map

\[H^i(X, \mathbb{Z}(j))^0 \to H^1(K, H^{i-1}(\overline{X}, \mathbb{Z}(j))),\]

where

\[H^i(X, \mathbb{Z}(j))^0 = \ker[H^i(X, \mathbb{Z}(j)) \to H^i(\overline{X}, \mathbb{Z}(j))].\]

If \(i \neq 2j\), this last group is of finite index in \(H^i(X, \mathbb{Z}(j))\), as follows easily from a specialization argument and the Weil conjectures as proved by Deligne (see [CTR1], Theorem 1 for this argument). Thus

\[H^i(X, \mathbb{Q}(j))^0 = H^i(X, \mathbb{Q}(j))\]

if \(i \neq 2j\).

**Lemma 1.1** (Rigidity Lemma). With notation as above, let \(M\) be a finitely generated torsion free \(\mathbb{Z}_\ell\)-module with continuous action of \(G = \text{Gal}(\overline{K}/K)\). Make \(M\) into a \(G = \text{Gal}(\overline{L}/L)\)-module by making the kernel of the natural map \(G \to G\) act trivially. Assume:

(i) for all open subgroups \(H\) of finite index in \(G\), we have

\[M(-1)^H = 0\]

and

(ii) for any abelian variety \(A\) over \(K\), we have that

\[[T_\ell(A) \otimes \mathbb{Q}_\ell M(-1)]^G = 0,\]

where \(T_\ell(A)\) is the \(\ell\)-adic Tate module,

\[\lim_{\leftarrow n} A[\ell^n].\]

Then the natural map

\[H^1(K, M) \to H^1(L, M)\]

is an isomorphism.

**Remark 1.2.** (i) Assumption (ii) of Lemma 1.1 is satisfied if \(M\) is such that \(V = M \otimes \mathbb{Q}_\ell\) is a Galois representation of pure weight different from \(-1\).

To see this, let \(A\) be an abelian variety over \(K\) and consider

\[T_\ell(A) \otimes \mathbb{Z}_\ell M(-1).\]

By the Riemann hypothesis for abelian varieties as proved by Weil and a specialization argument, \(T_\ell(A)\) is of pure weight \(-1\). Thus, if \(M\) is of pure weight different from \(-1\), then \(M(-1)\) is of pure weight different from 1,
and the tensor product above is of weight different from 0, and hence has no $G$-invariants.

(ii) We shall need an analogous version of rigidity, where one takes an algebra $A$ that is finitely generated over the base field $K$. The proof is similar but easier than the case of a field $L$ that is finitely generated over $K$, and we omit it here.

**Corollary 1.** Let $\Omega$ be an uncountable algebraically closed field containing $K$. Then if $M$ satisfying the hypotheses of the lemma is a torsion free quotient of the étale cohomology group $H^{i-1}(\overline{X}, \mathbb{Z}_\ell(j))$ with $i - 1 - 2j \neq -1, -2$, the image of the map

$$(*) \quad H^i(X_{\Omega}, \mathbb{Z}(j))^0 \to \lim_{[K':K]<\infty} \lim_{L_f, g/K'} H^j(L, M)$$

is countable. Here the outside limit is taken over all finite separable extensions of $K$ and the inside limit is taken over all finitely generated regular extensions of $K'$.

**Proof of the corollary assuming the Rigidity Lemma.** Since we are dealing with $\ell$-adic cohomology for $\ell \neq \text{char}(K)$, we reduce to the case where $\Omega$ is separably closed. We will use Remark 1.2 (ii) above. Let $K'$ be a finite separable extension of $K$ and $L$ a finitely generated regular field extension of $K'$. Any element of $H^j(X_L, \mathbb{Z}(j))^0$ comes from an element of $H^j(X_L, \mathbb{Z}(j))^0$, for some finitely generated $K'$-algebra, $A$. Localizing, if necessary, we may assume that $A$ is regular, and replacing $K'$ by a finite separable extension, if necessary, we may assume that the natural map $K' \to A$ has a section. Consider the following obvious commutative diagram:

$$
\begin{array}{ccc}
H^i(X_{K'}, \mathbb{Z}(j))^0 & \to & H^j(K', M) \\
\downarrow & & \downarrow \\
H^i(X_A, \mathbb{Z}(j))^0 & \to & H^j(A, M).
\end{array}
$$

The vertical maps have splittings given by the section $A \to K'$. From the diagram and the rigidity lemma, we see that the image of $H^i(X_A, \mathbb{Z}(j))^0$ in $H^j(A, M)$ comes from the image of $H^i(X_{K'}, \mathbb{Z}(j))^0$ in $H^j(K', M)$. Note that this image is countable, since $K'$ is finitely generated over the prime subfield. Taking the limit over all such finite extensions $K'$ of $K$ and all regular finitely generated field extensions $L/K'$, we see that the image of $(*)$ is a countable union of countable groups, so is countable, as claimed. 

**Example 1.3.**

(i) Let $L$ be a field that is finitely generated over $\mathbb{Q}$ and consider the motivic cohomology group $H^1(L, \mathbb{Z}(2))$. There is a regulator (cycle class) map

$$H^1(L, \mathbb{Z}(2)) \to H^1(L, \mathbb{Z}_\ell(2)).$$

It is easy to see that the Galois module $\mathbb{Z}_\ell(2)$ satisfies the hypotheses of the rigidity lemma (in this case, the rigidity is due to Suslin (see [Su2, Corollary 2.7])). If $K$ is the algebraic closure of $\mathbb{Q}$ in $L$, then we get an isomorphism:

$$H^1(K, M) \to H^1(L, M).$$

(ii) Let $X$ be a smooth projective variety over $K$ and let $M = T_\ell(Br(X))(1)$, the $\ell$-Tate-module of the Brauer group of $X$, twisted by one. Suppose that $T_\ell(Br(X))^H = 0$ for all open subgroups $H$ of $G = \text{Gal}(\overline{K}/K)$. Then assumption (i) of the lemma is satisfied. This is a consequence of the Tate conjecture for divisors for $X_L$ over every finite extension $L$ of $K$ together
with semi-simplicity of the action of $G$ (if $K$ is a finite field or a number field, one can avoid assuming semi-simplicity). By Remark 1.2 (ii) above, assumption (ii) of the Rigidity Lemma is satisfied. Thus the Rigidity Lemma applies to such an $M$. Now it is not hard to see that there is a map

$$H^3(X_L, \mathbb{Z}(2))^{ind} \to H^1(L, M),$$

so the corollary above applies. This applies to e.g. abelian varieties, K3 surfaces, etc.

**Proof of the Rigidity Lemma.** We have the inflation-restriction sequence,

$$0 \to H^1(K, M) \to H^1(L, M) \to H^1(KL, M)^G \to \cdots .$$

Using the hypotheses, we show that the group on the right is zero. Our argument is actually very similar to that of Suslin ([Su2], Corollary 2.7). Let $Y$ be a normal, projective, geometrically connected model of $L$ over $K$, and consider the exact sequence,

$$0 \to \mathbb{K}(Y)^* / \mathbb{K}^* \to \text{Div}(Y) \to \text{Pic}(Y) \to 0.$$

Let $M_n = M / \ell^n M$ and recall the identification from Kummer theory:

$$H^1(KL, \mathbb{Z}/\ell^n(1)) = (KL)^* / (KL)^* \ell^n.$$

Tensoring the sequence with $M_n(-1)$ and using the fact that $\text{Div}(Y)$ is torsion free, we get the sequence

$$0 \to \text{Tor}^1(\text{Pic}(Y), M_n(-1)) \to (KL)^* \otimes M_n(-1) \xrightarrow{\ell} \text{Div}(Y) \otimes M_n(-1) \to \text{Pic}(Y) \otimes M_n(-1).$$

Taking projective limits and $G$-invariants, we get the exact sequence

$$(\ast) \quad 0 \to \lim_{\leftarrow n} \text{Tor}^1(\text{Pic}(Y), M_n(-1))^G \to \lim_{\leftarrow n} (KL)^* \otimes M_n(-1))^G \to \lim_{\leftarrow n} \text{Div}(Y) \otimes M_n(-1))^G.$$

Note that all of the terms of the projective systems satisfy the Mittag-Leffler property, but since the groups in the system on the left are finite, we really don’t need this to get the exactness of $(\ast)$.

We claim that the right and left terms of $(\ast)$ are zero using, respectively, hypotheses (i) and (ii). We deal with the right term first. For each irreducible codimension 1 subvariety $Z$ of $X$, let $k_Z$ be the algebraic closure of $K$ in the function field of $Z$ and let $H_Z$ be the absolute Galois group of $K_Z$. Let $W_Z$ be an irreducible subvariety of $Y$ lying over $Z$. Then, by Shapiro’s lemma and hypothesis (i), we have

$$\lim_{n} (\text{Div}(Y) \otimes M_n(-1))^G \subseteq \prod_{Z} \left( \bigoplus_{W \to Z} M(-1)_W \right)^G = \prod_{Z} \bigoplus_{W_Z} M(-1)^{H_Z} = 0.$$ 

As for the left term of the exact sequence $(\ast)$, an easy computation shows that

$$\lim_{n} \text{Tor}^1(\text{Pic}(Y), M_n(-1))^G = \text{Tor}(\text{Pic}(Y)) \otimes M(-1))^G = 0,$$

by assumption (ii). This completes the proof of the Rigidity Lemma. □
2. Countability

Let $K$ be a field that is finitely generated over $\mathbb{Q}$. There is a finitely generated $\mathbb{Q}$-algebra $A$ that is regular and has fraction field $K$. Let $Y = \text{Spec}(A)$. Then there is an exact sequence of motivic cohomology groups,

$$\cdots \rightarrow H^1(Y, \mathbb{Z}(2)) \rightarrow H^1(K, \mathbb{Z}(2)) \rightarrow \lim_{\rightarrow} Z_n H^2(Y, \mathbb{Z}(2)) \rightarrow \cdots,$$

where the limit is taken over all codimension one subschemes $Z$ of $Y$. Standard conjectures on the finite generation of motivic cohomology groups of schemes finitely generated over $\mathbb{Z}$ would imply that the left group is finitely generated. An easy induction argument using purity and the vanishing of $H^0(W, \mathbb{Z}(1))$ for $W$ smooth shows that the right group is zero. Thus we expect that $H^1(K, \mathbb{Z}(2))$ is finitely generated, although we have no idea how to prove this at present. Note that $H^1(K, \mathbb{Z}(2)) \cong K_{3}^{\text{ind}}(K)$.

Recall a theorem of Levine ([Le], Theorem 4.12) and of Merkur’ev-Suslin ([MS]) that says there is an isomorphism

$$\lim_{\leftarrow} n K_{3}^{\text{ind}}/\ell^n \rightarrow H^1(K, \mathbb{Z}_{\ell}(2)).$$

Theorem 2.1. Assume that for every field $K$ that is finitely generated over $\mathbb{Q}$, $H^1(K, \mathbb{Z}(2))$ is a finitely generated abelian group. Then the group $H^1(C, \mathbb{Z}(2))$ is countable.

Proof. We have that $H^1(C, \mathbb{Z}(2)) = \lim_{[K:\mathbb{Q}] < \infty \text{ Lf.g.}/K} \lim_{\leftarrow} H^1(L, \mathbb{Z}(2))$, where the limit is taken over all finite extensions $K/\mathbb{Q}$ and all finitely generated regular field extensions $L$ of $K$. From the theorem of Levine and Merkur’ev-Suslin just mentioned and the assumption of finite generation of $H^1(L, \mathbb{Z}(2))$, we have that the Chern class map

$$H^1(L, \mathbb{Z}(2)) \rightarrow H^1(L, \mathbb{Z}_{\ell}(2))$$

has kernel that is torsion prime to $\ell$. By the rigidity lemma and its corollary, the image of the injective map

$$H^1(C, \mathbb{Q}(2)) \rightarrow \lim_{[K:\mathbb{Q}] < \infty \text{ Lf.g.}/K} \lim_{\leftarrow} H^1(K, \mathbb{Q}_{\ell}(2))$$

is countable, and hence the group on the left is countable. Finally, one knows that the torsion of $H^1(C, \mathbb{Z}(2))$ is countable by Suslin’s rigidity theorem ([Su1]) (also proved by Gabber and Gillet-Thomason [GT]). This completes the proof of the theorem. \qed

Remark 2.2. The rigidity part of this result was proved by Suslin [Su2], who also noticed the remark about finite generation implying countability. Since this last part was not in the literature, we included it here.

Before stating the next result, we define $H^3(X, \mathbb{Q}(2))^{\text{ind}}$ for $X$ smooth and projective over a field $K$. Consider the product map

$$[\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}] \otimes_{\mathbb{Q}} [K^* \otimes_{\mathbb{Z}} \mathbb{Q}] \rightarrow H^3(X, \mathbb{Q}(2))$$

(here we use the identifications $\text{Pic}(X) = H^2(X, \mathbb{Z}(1)), K^* = H^1(X, \mathbb{Z}(1))$).

Let $H^3(X, \mathbb{Q}(2))^{\text{dec}}$ be the image of this map. Then we define $H^3(X, \mathbb{Q}(2))^{\text{ind}}$ to be the quotient $H^3(X, \mathbb{Q}(2))/H^3(X, \mathbb{Q}(2))^{\text{dec}}$. If $C$ is a curve over $K$, we have that $H^3(C, \mathbb{Q}(2))^{\text{ind}} = 0$, as follows easily from the Gersten-Quillen complex which
computes the cohomology of $K_2$ (note that we have $H^3(C, \mathbb{Z}(2)) = H^1(C, K_2)$). If $K$ is algebraically closed, we can define $H^3(X, \mathbb{Z}(2))^{\text{ind}}$ in a similar way. To define $H^3(X, \mathbb{Z}(2))^{\text{ind}}$ when $K$ is not algebraically closed is not difficult but is a bit tedious, as we have to take norms from finite extensions of $K$. However, this is not necessary for what we want to prove here.

**Theorem 2.3.** Let $X$ be a smooth projective, geometrically connected variety over $\mathbb{C}$. Assume that

(i) for any smooth proper model $\mathcal{X}$ of $X$ over a ring $A$ that is finitely field over $\mathbb{Z}$, the group $H^3(\mathcal{X}, \mathbb{Z}(2))$ is finitely generated (as is expected),

(ii) the Tate conjecture for divisors is true for any model $Y$ of $X$ over a field $K$ that is finitely generated over $\mathbb{Q}$, and the absolute Galois group of $K$ acts semi-simply on $H^2(Y_K, \mathbb{Q}_\ell(1))$.

Then $H^3(X, \mathbb{Z}(2))^{\text{ind}}$ is countable.

**Proof.** The proof is very similar to the proof of Theorem 2.1, and we only sketch it here. Let $Y$ and $K$ be as in the statement of the theorem. Let $\overline{Y} = Y \times_K \overline{K}$, let $Br(\overline{Y})$ be the Brauer group of $\overline{Y}$, and let

$$V_\ell Br(\overline{Y}) = (\lim_{\leftarrow n} Br(\overline{Y})[\ell^n]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

be its $\ell$-adic Tate vector space. Note that for any algebraically closed field $M$ containing $\overline{K}$, we have

$$V_\ell Br(\overline{Y}) \cong V_\ell Br(Y_M).$$

Consider the exact sequence

$$0 \to NS(\overline{Y}) \otimes \mathbb{Q}_\ell(1) \to H^2(\overline{Y}, \mathbb{Q}_\ell(2)) \to V_\ell Br(\overline{Y})(1) \to 0$$

that comes from taking cohomology of the Kummer exact sequence of sheaves on $Y$ and Tate-twisting by 1. Then for any field $L$ that is finitely generated over $K$, we have a map

$$H^3(Y_L, \mathbb{Q}(2))^{\text{ind}} \to H^1(L, V_\ell Br(\overline{Y}))(1).$$

The hypotheses ensure that the rigidity lemma applies to the $Gal(\overline{K}/K)$-representation $V_\ell Br(\overline{Y})(1)$ (see Example 1.3 (ii)), and so by Corollary 1, the image of the map

$$H^3(X, \mathbb{Q}(2))^{\text{ind}} = \lim_{L, f.g./K} H^3(Y_L, \mathbb{Q}(2))^{\text{ind}} \to \lim_{L, f.g./K} H^1(L, V_\ell Br(\overline{Y}))(1)$$

is countable. We claim that the other graded quotients of $H^3(X, \mathbb{Q}(2))^{\text{ind}}$ for the Hochschild-Serre filtration (see §1) do not give any additional contribution. To see
this for $F^2$, take a generic curve $C$ on $Y$, e.g. by taking the (complete) $(d - 1)$-fold intersection of smooth hyperplane sections, where $d = \dim(Y)$. Consider the diagram

$$
\begin{array}{c}
F^2H^3(Y, \mathbb{Q}(2)) \rightarrow H^2(K, H^1(\mathbb{Y}, \mathbb{Q}_\ell(2))) \\
\downarrow \\
F^2H^3(C, \mathbb{Q}(2)) \rightarrow H^2(K, H^1(C, \mathbb{Q}_\ell(2))).
\end{array}
$$

By the weak Lefschetz theorem and Poincaré complete reducibility (see [CTR2], §4 for this argument), the right vertical arrow is injective. But $H^3(C, \mathbb{Q}(2))^{\text{ind}} = 0$ by the remark preceding the statement of this theorem, and hence the top horizontal arrow is zero after passing to the indecomposable quotients. This proves the claim for $F^2$. As for $F^3$, an argument similar to that in (R), Proposition 2.2) shows that $F^3 = F^3 + j$ for any $j \geq 0$. By hypothesis, the filtration is separated, and hence $F^3 = 0$. Thus we have an injection

$$H^3(Y, \mathbb{Q}(2))^{\text{ind}} \rightarrow H^1(K, V_\ell(\text{Br}(\mathbb{Y}))(1)).$$

Since the group on the left is countable and the image of these maps as $Y$ ranges over models of $X$ over finitely generated fields is rigid, we see that $H^3(X, \mathbb{Q}(2))^{\text{ind}}$ is countable. Now by (CTR1), Theorem 2.1) the torsion of $H^3(X, \mathbb{Z}(2))$ is countable, and this completes the proof of the theorem. □

References


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