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LINDELÖF'S THEOREM FOR HYPERBOLIC CATENOIDS

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Abstract. In this paper, we study the maximal stable domains on minimal and constant mean curvature 1 catenoids in hyperbolic space. In particular we investigate whether half-vertical catenoids are maximal stable domains (*Lindelöf's property*). Our motivation comes from Lindelöf's 1870 paper on catenoids in Euclidean space.

1. Introduction

In [8], L. Lindelöf determines which domains of revolution on the catenoid $\mathcal{C}$ in $\mathbb{R}^3$ are stable. More precisely, he gives the following geometric construction; see Figure 1 in Section 3. Take any point A on the generating catenary $C = \{(x, z) \in \mathbb{R}^2 \mid x = \cosh(z)\}$. Draw the tangent to C at point A and let I be the intersection point of the tangent with the axis $\{x = 0\}$. From I, draw the second tangent to C. It touches C at point B. Lindelöf's result states that the compact connected arc AB generates a stable-unstable domain on the catenoid $\mathcal{C}$ (see definitions in Section 2). As a consequence, *the upper-half of the catenoid, $\mathcal{C} \cap \{z > 0\}$, is a maximal stable domain among domains invariant under rotations.* We will refer to this property as *Lindelöf's property*.

In [2], we prove that minimal catenoids in $\mathbb{H}^n \times \mathbb{R}$ do not satisfy Lindelöf's property (Theorem 3.5, Assertion 2). In this paper, we consider minimal and embedded constant mean curvature 1 catenoids in $\mathbb{H}^3$. The global picture is as follows. Catenoid-cousins in $\mathbb{H}^3$, as their minimal counterparts in $\mathbb{R}^3$, satisfy Lindelöf's property (Theorem 4.16). This is not surprising in view of the local correspondence between minimal surfaces in $\mathbb{R}^3$ and surfaces with constant mean curvature 1 in $\mathbb{H}^3$. One may also observe that the Jacobi operators look the same, namely $-\Delta - |A_0|^2$, where A_0 is the second fundamental form for catenoids or its traceless analog for catenoid-cousins. On the other hand, catenoids in $\mathbb{H}^3$ divide into two families: a family of stable catenoids which foliate the space and a family of index 1 catenoids which intersect each other and have an envelope; the hyperbolic catenoids do not satisfy Lindelöf's property (Theorem 4.7, Proposition 4.8). We finally point out that among the examples we have studied, the hypersurfaces which do not satisfy Lindelöf's property are precisely those which are vertically bounded.

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2. Preliminaries

Let $M^2 \looparrowright \widehat{M}^3$ be an orientable minimal or constant mean curvature surface in an oriented Riemannian manifold $(\widehat{M}, \widehat{g})$. Let N_M be a unit normal field along M and let A_M be the second fundamental form with respect to N_M. Let $\widehat{\mathrm{Ric}}$ be the Ricci curvature of $\widehat{M}$. The second variation of the volume functional gives rise to the *Jacobi operator* (or stability operator) J_M of M (see [6]),

$$J_M := -\Delta_M - \big(|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)\big), \tag{2.1}$$

where Δ_M is the non-positive Laplacian on M for the induced metric.

Given a relatively compact regular domain Ω on the surface M, we let $\mathrm{Ind}(\Omega)$ denote the number of negative eigenvalues of J_M for the Dirichlet problem in Ω. The *index* of M is defined to be the supremum $\mathrm{Ind}(M) := \sup\{\mathrm{Ind}(\Omega) \mid \Omega \Subset M\} \leq +\infty$, taken over all relatively compact regular domains. Let $\lambda_1(\Omega)$ be the least eigenvalue of the operator J_M with Dirichlet boundary conditions in Ω. We call a relatively compact regular domain Ω *stable* if $\lambda_1(\Omega) > 0$, *unstable* if $\lambda_1(\Omega) < 0$, and *stable-unstable* if $\lambda_1(\Omega) = 0$. More generally, we say that a domain Ω is stable if any relatively compact subdomain is stable.

Properties 2.1. *We recall the following properties:*

(1) *Let Ω be a stable-unstable relatively compact domain. Then, any smaller domain is stable while any larger domain is unstable (monotonicity property of the Dirichlet eigenvalues).*
(2) *We refer to the solutions of the equation $J_M(u) = 0$ as* Jacobi fields *on M. Let $X_a : M^n \looparrowright (\widehat{M}^{n+1}, \widehat{g})$ be a one-parameter family of oriented immersions, with constant mean curvature H_a, with variation field $V_a = \frac{\partial X_a}{\partial a}$ and with unit normal N_a. If H_a does not depend on a, then the function $\widehat{g}(V_a, N_a)$ is a Jacobi field on M* ([1, Theorem 2.7]).
(3) *Let Ω be a relatively compact domain on a minimal or constant mean curvature manifold M. If there exists a positive function u on Ω such that $J_M(u) \geq 0$, then Ω is stable* ([5, Theorem 1]).

3. Catenoids in $\mathbb{R}^3$

In this section, we briefly recall Lindelöf's results for Euclidean catenoids. We consider the family of catenoids given by the parametrization

$$X(a,t,\theta) = \big(a\cosh(\frac{t}{a})\cos\theta, a\cosh(\frac{t}{a})\sin\theta, t\big),\ t \in \mathbb{R}, a > 0,$$

and in particular the catenoid $\mathcal{C}$ given by $a = 1$. Let $N(t,\theta)$ be the unit normal to $\mathcal{C}$, pointing towards the axis. According to Properties 2.1, the functions

$$\begin{cases} v(t) = \tanh(t) = \langle \frac{\partial}{\partial z}, N(t,\theta)\rangle \ \text{ and} \\ e(t) = 1 - t\,\tanh(t) = -\langle \frac{\partial X_a}{\partial a}(1,t,\theta), N(t,\theta)\rangle \end{cases} \tag{3.1}$$

are Jacobi fields on $\mathcal{C}$.

Theorem 3.1. *Let ξ_0 be the positive zero of the function $e(t) = 1 - t\,\tanh(t)$.*

(1) *The domain $\mathcal{D}(-\xi_0, \xi_0) = X(1,]-\xi_0, \xi_0[, [0, 2\pi])$ is a stable-unstable domain on the catenoid $\mathcal{C}$.*

(2) *The domain* $\mathcal{D}(0,\infty) = X(1,]0,\infty[,[0,2\pi])$ *is a maximal stable rotation invariant domain on* $\mathcal{C}$. *More precisely, given any* $\alpha > 0$, *there exists some* $\beta(\alpha) > 0$ *such that the domain* $\mathcal{D}\big(\alpha,\beta(\alpha)\big) = X(1,]-\alpha,\beta(\alpha)[,[0,2\pi])$ *is stable-unstable; see Figure* 1.

(3) *The catenoid* $\mathcal{C} = X(1,]-\infty,\infty[,[0,2\pi])$ *has index* 1.

Sketch of the proof of Theorem 3.1. *Assertion 1.* Use the Jacobi field $e(t)$.

Assertion 2. The Jacobi field $v(t)$ is positive in $\mathcal{D}(0,\infty)$. It follows that this domain is stable (Properties 2.1). Take any $\alpha > 0$. The function $e(\alpha,t)$ defined by

$$e(\alpha,t) = v(\alpha)e(t) + e(\alpha)v(t) \tag{3.2}$$

is a Jacobi field and satisfies $e(\alpha,-\alpha) = 0$. Since $e(\alpha,\pm\infty) = -\infty$ and $\frac{\partial e}{\partial t}(\alpha,-\alpha) \neq 0$, the function $e(\alpha,\cdot)$ must have another zero $\beta(\alpha) \neq -\alpha$. Observe now that the function $e(\alpha,\cdot)$ cannot have two negative zeroes or two positive zeroes because the upper and lower half-catenoids $\mathcal{C} \cap \{z \gtrless 0\}$ are stable. Hence $e(\alpha,\cdot)$ has exactly one negative zero $-\alpha$ and one positive zero $\beta(\alpha)$. It follows that the domain $\mathcal{D}\big(\alpha,\beta(\alpha)\big)$ is stable-unstable. It also follows that $\mathcal{D}(0,\infty)$ is a maximal stable domain among rotation invariant domains.

Assertion 3. It follows from Assertion 1 that the index of $\mathcal{C}$ is at least 1. The horizontal half-catenoid $\mathcal{C} \cap \{x > 0\}$ is stable. This implies that an eigenfunction of the Jacobi operator associated with a negative eigenvalue must be invariant by rotations (see [2], Theorem 3.5 or [12]). Use Assertion 1 to conclude that there is only one negative eigenvalue. □

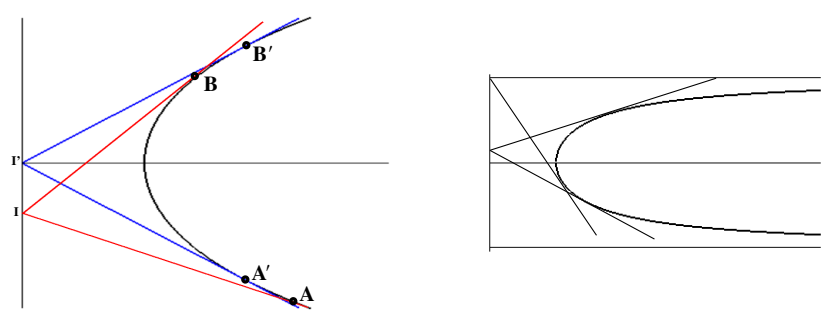

FIGURE 1. Catenaries $n \geq 2$ and $n \geq 3$

Remarks. (1) Using the function $e(\alpha,t)$ defined by (3.2), one can recover Lindelöf's tangent construction; see Figure 1.

(2) The function $e(\alpha,\cdot)$ is a Jacobi field arising from the variation of the one-parameter family of catenaries passing through the point $(\cosh(\alpha),-\alpha)$.

(3) It turns out that the above proof and tangent construction work for n-dimensional catenoids in $\mathbb{R}^{n+1}$ as well. When $n \geq 3$, these catenoids have index 1 ([12, 3]) and bounded height. The tangent construction ([3]) shows that they do not satisfy Lindelöf's property; see Figure 1.

4. Catenoids and catenoid cousins in $\mathbb{H}^3$

In the sequel, we use subscripts to denote the derivative with respect to a given variable. For example, if $y_0(a,s)$ is a function of the variables a and s, then $y_{0,a}(a,s) = \frac{\partial}{\partial a} y(a,s)$ and $y_{0,s}(a,s) = \frac{\partial}{\partial s} y(a,s)$.

4.1. **Preliminaries.** We work in the half-space model for the hyperbolic space,

$$\mathbb{H}^3_{\{x_1,x_2,x_3\}} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 > 0\}, \qquad g_h = x_3^{-2}\big(dx_1^2 + dx_2^2 + dx_3^2\big).$$

In the hyperbolic plane $\mathbb{H}^2_{\{x_1,x_3\}} = \{(x_1,x_3) \in \mathbb{R}^2 \mid x_3 > 0\}$, we consider the Fermi coordinates (u,v) associated with the geodesic $s \to (0,e^s)$. They are defined by $\mathbb{R}^2 \ni (u,v) \to \big(x_1 = e^v \tanh(u), x_3 = e^v/\cosh(u)\big)$. Given a function f, we consider the curve $t \to (t, f(t))$ in the $\{u,v\}$-plane and the associated rotation surface $F : M \looparrowright \mathbb{H}^3_{\{x_1,x_2,x_3\}}$,

$$F(t,\theta) = \Big(e^{f(t)} \tanh(t)\, \omega_\theta\,,\ \frac{e^{f(t)}}{\cosh(t)}\Big),$$

where we write $\omega_\theta = (\cos\theta, \sin\theta)$ for short.

The principal directions of curvature are the tangents to the generating curve and to the horizontal circle. Using [10] Theorem 2.6.18, we easily compute the respective principal curvatures

$$\begin{cases} k_g(t) &= \dfrac{f_{tt}(t)\cosh(t) + 2f_t(t)\sinh(t) + f_t^3(t)\cosh^2(t)\sinh(t)}{\big(1+\cosh^2(t)f_t^2(t)\big)^{3/2}}\,, \\ k_c(t) &= \dfrac{f_t(t)\cosh^2(t)}{\sinh(t)\big(1+\cosh^2(t)f_t^2(t)\big)^{1/2}} \end{cases}$$

and the mean curvature

$$H(t)\sinh(2t) = \frac{d}{dt}\,\frac{f_t(t)\sinh(t)\cosh^2(t)}{\big(1+\cosh^2(t)f_t^2(t)\big)^{1/2}} \tag{4.1}$$

of the immersion F. Integrating equation (4.1) when $H = 0$ or $H = 1$ in the next sections, we will find graphs $\varphi(a,t) = (t, \lambda(a,t)), t \geq a$, in the plane $\mathbb{H}^2_{\{u,v\}}$. They extend by symmetry with respect to the u-axis as smooth curves with an arc-length parametrization of the form $\Phi(a,s) = \big(y(a,s), \Lambda(a,s)\big)$, where $y(a,s)$ is a smooth even function of s and $\Lambda(a,s)$ a smooth odd function of s, such that $\Lambda(a,s) = \lambda\big(a, y(a,s)\big)$ for $s \geq 0$.

The corresponding constant mean curvature rotation surfaces in $\mathbb{H}^3$ are given by the parametrizations

$$Y(a,s,\theta) = \Big(e^{\Lambda(a,s)} \tanh(y(a,s))\, \omega_\theta\,,\ \frac{e^{\Lambda(a,s)}}{\cosh(y(a,s))}\Big). \tag{4.2}$$

The Killing field associated with the hyperbolic translations along the vertical geodesic $t \mapsto (0,0,e^t)$ in $\mathbb{H}^3_{\{x_1,x_2,x_3\}}$ is just the position vector. The *vertical Jacobi field* is the function $v_Y(a,s) = g_h(Y, N_Y)$, where N_Y is the unit normal vector to the immersion Y. The following results are straightforward.

Property 4.1. *The vertical Jacobi field $v_Y(a,s) = g_h(Y,N_Y)$ is an odd function of s given by $v_Y(a,s) = \cosh(y(a,s))y_s(a,s)$.*

Property 4.2. *The* variation Jacobi field $e_Y(a,s)$ *is an even function of* s *given by* $e_Y(a,s) = g_h(Y_a, N_Y) = \cosh(y(a,s))\big(\Lambda_a y_s - \Lambda_s y_a\big)$.

4.2. **Minimal catenoids in $\mathbb{H}^3$.** When $H = 0$, equation (4.1) yields the solution curves $\{C_{0,a}\}_{a>0}$,

$$\lambda_0(a,t) = \sinh(2a)\int_a^t \frac{d\tau}{\cosh(\tau)\big(\sinh^2(2\tau) - \sinh^2(2a)\big)^{1/2}}, \tag{4.3}$$

which are defined for $t \geq a$ (here, the lower index 0 refers to the value of H). Notice that this parametrization only covers a half-catenary and that we work up to a v-translation in $\mathbb{H}^2_{\{u,v\}}$, *i.e.* up to a hyperbolic translation with respect to the vertical geodesic in $\mathbb{H}^2_{\{x_1,x_3\}}$. The arc-length parameter along the curve is given by

$$\begin{cases} S_0(a,t) = \int_a^t \sinh(2\tau)\big(\cosh^2(2\tau) - \cosh^2(2a)\big)^{-1/2}\, d\tau, \\ \cosh(2a)\cosh\big(2S_0(a,t)\big) = \cosh(2t), t \geq a. \end{cases} \tag{4.4}$$

Proposition 4.3. *For* $s \in \mathbb{R}$, *define the functions* $y_0(a,s)$ *and* $\Lambda_0(a,s)$ *by the formulas*

$$\begin{cases} y_0(a,s) &= a + \displaystyle\int_0^s \frac{\cosh(2a)\sinh(2t)}{\big(\cosh^2(2a)\cosh^2(2t) - 1\big)^{1/2}}\, dt \text{ and} \\ \Lambda_0(a,s) &= \sqrt{2}\sinh(2a)\displaystyle\int_0^s \frac{\big(\cosh(2a)\cosh(2t) - 1\big)^{1/2}}{\big(\cosh^2(2a)\cosh^2(2t) - 1\big)}\, dt. \end{cases} \tag{4.5}$$

(1) *The function* y_0 *is an even function of* s, *and* Λ_0 *is an odd function of* s.
(2) *For* $s \geq 0$, *the function* $y_0(a,\cdot)$ *is the inverse function of the function* $S_0(a,\cdot)$. *In particular,* $\cosh\big(2y_0(a,s)\big) = \cosh(2a)\cosh(2s)$.
(3) *For* $s \geq 0$, *we have* $\Lambda_0(a,s) = \lambda_0(a, y_0(a,s))$.
(4) *For* $s \in \mathbb{R}$, *the functions* $s \mapsto \big(y_0(a,s), \Lambda_0(a,s)\big)$ *are arc-length parametrizations of the family of catenaries* $\{C_{0,a}\}_{a>0}$.

Proof. The proof is straightforward. □

For later reference, we introduce the function

$$J_0(a,t) = \sinh(2a)(\cosh(2a)\cosh(2t) + 1)^{-1}(\cosh(2a)\cosh(2t) - 1)^{-1/2}, \tag{4.6}$$

so that $\Lambda_0(a,s) = \sqrt{2}\int_0^s J_0(a,t)\, dt$. We compute $\frac{\partial J_0}{\partial a}(a,t)$ and find

$$\begin{cases} I_0(a,t) &= \dfrac{\partial J_0}{\partial a}(a,t) = \dfrac{n(\cosh(2a), \cosh(2t))}{d(\cosh(2a), \cosh(2t))}, \text{ where} \\ n(A,T) &= A(3 - A^2)T^2 + (A^2 - 1)T - 2A, \\ d(A,T) &= (AT + 1)^2(AT - 1)^{3/2}. \end{cases} \tag{4.7}$$

We note that $n(A,T)$ is a polynomial of degree 2 in T.

Lemma 4.4. *Let* $a_1 > 0$ *be such that* $\cosh^2(2a_1) = \frac{11+8\sqrt{2}}{7} \approx 3.1876$, i.e. $a_1 \approx 0.5915$. *For* $a \geq a_1$ *and for all* t, *we have* $n(\cosh(2a), \cosh(2t)) \leq 0$.

To the above family $\{C_{0,a}\}_{a>0}$ of catenaries corresponds to a family $\{\mathcal{C}_{0,a}\}_{a>0}$ of catenoids in $\mathbb{H}^3$ with the arc-length parametrization $Y_0(a,s,\theta)$,

$$Y_0(a,s,\theta) = \Big(e^{\Lambda_0}\tanh(y_0)\,\omega_\theta\,,\, e^{\Lambda_0}/\cosh(y_0)\Big), \tag{4.8}$$

where the functions $\Lambda_0(a,s)$ and $y_0(a,s)$ are given by Proposition 4.3.

Catenoids in $\mathbb{H}^3$ have been considered in [9, 4] and more recently in [11]. As pointed out by these authors, among the family $\{\mathcal{C}_{0,a}\}$ of catenoids in $\mathbb{H}^3$ there are stable and index one catenoids. We now give a precise analysis of this phenomenon, and we also consider Lindelöf's property for these catenoids.

4.2.1. *Jacobi fields on $\mathcal{C}_{0,a}$.* We introduce Jacobi fields on $\mathcal{C}_{0,a}$, keeping Properties 4.1 and 4.2 in mind. The *variation Jacobi field* $e_0(a,s)$ is given by

$$e_0(a,s) = -g_h(Y_{0,a}(a,s,\theta), N_0(a,s,\theta)) = -\cosh(y_0)\big(\Lambda_{0,a}y_{0,s} - \Lambda_{0,s}y_{0,a}\big).$$

We have

$$\begin{aligned} e_0(a,s) = \quad & \sinh^2(2a)\cosh(2s)\big(\cosh^2(2a)\cosh^2(2s)-1\big)^{-1} \\ & -\frac{\cosh(2a)\sinh(2s)}{\big(\cosh(2a)\cosh(2s)-1\big)^{1/2}}\int_0^s I_0(a,t)\,dt, \end{aligned} \tag{4.9}$$

where $I_0(a,t)$ is defined by (4.7). The *vertical Jacobi field* $v_0(a,s)$ is defined by

$$v_0(a,s) = \sqrt{2}\cosh(y_0)y_{0,s} = \frac{\cosh(2a)\sinh(2s)}{\big(\cosh(2a)\cosh(2s)-1\big)^{1/2}}. \tag{4.10}$$

Define

$$f_0(a,s) = \sinh^2(2a)\cosh(2s)\big(\cosh^2(2a)\cosh^2(2s)-1\big)^{-1} \tag{4.11}$$

as an even function of s which goes to 0 at infinity. In view of equations (4.9), (4.10) and (4.11), we have the relations

$$e_0(a,s) = f_0(a,s) - v_0(a,s)\int_0^s I_0(a,t)\,dt. \tag{4.12}$$

Observe that the integral

$$E_0(a) := \int_0^\infty I_0(a,t)\,dt \tag{4.13}$$

exists for all values of a.

4.2.2. *Stable domains on $\mathcal{C}_{0,a}$.* We can now investigate the stability properties of the catenoids $\mathcal{C}_{0,a}$ in $\mathbb{H}^3$.

Lemma 4.5. *The half-catenoids $\mathcal{D}_{0,a,\pm} = Y_0(a,\mathbb{R}^\bullet_\pm,[0,2\pi])$ are stable. As a consequence, a Jacobi field $w(a,s)$, which depends only on the radial variable s on $\mathcal{C}_{0,a}$, can have at most one zero on $\mathbb{R}^\bullet_+$ and on $\mathbb{R}^\bullet_-$.*

Proof. Use Property 2.1 (3) and the fact that $v_0(a,s)$ is a Jacobi field which only vanishes at $s=0$. □

Lemma 4.6. *The catenoid $\mathcal{C}_{0,a}$ has index at most* 1.

Proof. The fact that the index of $\mathcal{C}_a$ is at most 1 has been proved by [11] using the same method as in [12]. Alternatively, one could use Jacobi fields associated to geodesics orthogonal to the axis of the catenoids to prove that half-catenoids $Y_0(a,\mathbb{R},]\varphi,\varphi+\pi[)$ are stable and to show that negative eigenvalues of the Jacobi operator $J_{\mathcal{C}_{0,a}}$ on domains of revolution are necessarily associated with eigenfunctions depending only on the parameter s; see [2], Theorem 3.5. □

We can now state the main theorem of this section. Recall that the number $E_0(a)$ is defined by (4.13) and that the Jacobi fields $v_0(a,s)$ and $e_0(a,s)$ are given respectively by (4.10) and (4.9), with the relation (4.12).

Theorem 4.7. *Let $\{\mathcal{C}_{0,a}\}_{a>0}$ be the family of catenoids in $\mathbb{H}^3$ given by* (4.8).

(1) *The index of the catenoid $\mathcal{C}_{0,a}$ depends on the value of the integral $E_0(a)$ defined by* (4.13). *More precisely, if $E_0(a) \leq 0$, then the catenoid $\mathcal{C}_{0,a}$ is stable, and if $E_0(a) > 0$, then the catenoid $\mathcal{C}_{0,a}$ has index* 1.
(2) *When $\mathcal{C}_{0,a}$ has index* 1, *there exists a positive number $z(a)$ such that the domain $\mathcal{D}_{0,z(a)} = Y_0(a,] - z(a), z(a)[, [0,2\pi])$ is stable-unstable.*
(3) *When $\mathcal{C}_{0,a}$ has index* 1, *there exists some $\ell(a)$, $0 < \ell(a) < z(a)$, such that $\mathcal{D}_{0,\ell(a)} = Y_0(a,] - \ell(a), \infty[, [0,2\pi])$ is a maximal stable, rotation invariant domain.*
(4) *The catenoids $\{\mathcal{C}_{0,a}\}_{a>0}$ do not satisfy Lindelöf's property.*
(5) *There exist two numbers $0 < a_2 < a_1$ such that for all $a > a_1$, the catenoids $\mathcal{C}_{0,a}$ are stable, and for all $a < a_2$, the catenoids $\mathcal{C}_{0,a}$ have index* 1.

Proof. Assertion 1. As stated in Lemma 4.5, the function $e_0(a,s)$ can have at most one zero on $]0,\infty[$ and at most one zero on $]-\infty,0[$. Observe that the function $e_0(a,s)$ is even and that $e_0(a,0) = 1$. To determine whether e_0 has a zero, it suffices to look at its behaviour at infinity. If $E_0(a) > 0$, the function $e_0(a,s)$ tends to $-\infty$ at infinity so that it has exactly two symmetric zeroes in $\mathbb{R}$. This implies that the index of $\mathcal{C}_{0,a}$ is at least 1. Using Lemma 4.6, we conclude that $\mathcal{C}_{0,a}$ has index 1. If $E_0(a) < 0$, the function $e_0(a,s)$ tends to $+\infty$ at infinity so that it is always positive and the catenoid $\mathcal{C}_{0,a}$ is stable. Assume now that $E_0(a) = 0$. We then have the relation

$$e_0(a,s) = f_0(a,s) + v_0(a,s)\int_s^\infty I_0(a,t)\,dt.$$

Using equation (4.7), we see that $I_0(a,t)$ is positive for t large enough provided that $\cosh^2(2a) \leq 3$. In that case, it follows that $e_0(a,s)$ is positive at infinity and hence that $\mathcal{C}_{0,a}$ is stable. If $E_0(a) = 0$ and $\cosh^2(2a) > 3$, we need to look at the behaviour of $e_0(a,s)$ at infinity more precisely. When s tends to $+\infty$, we have

$$f_0(a,s) \sim 2\tanh^2(2a)e^{-2s}, \quad v_0(a,s) \sim \sqrt{\frac{\cosh(2a)}{2}}e^s, \qquad \text{and}$$

$$\int_s^\infty I_0(a,t)\,dt \sim \frac{2^{3/2}(3-\cosh^2(2a))}{3\cosh^{5/2}(2a)}e^{-3s}.$$

It follows that $e_0(a,s) \sim \frac{4}{3}e^{-2s}$ is positive at infinity and hence that $\mathcal{C}_{0,a}$ is stable.

Assertion 2. Saying that $\mathcal{C}_{0,a}$ has index 1 is equivalent to saying that $E_0(a) > 0$ and hence that e_0 has two symmetric zeroes.

Assertion 3. Given any $\alpha > 0$, we introduce the Jacobi field $e_0(a,\alpha,s)$,

$$e_0(a,\alpha,s) = v_0(a,\alpha)e_0(a,s) + e_0(a,\alpha)v_0(a,s). \tag{4.14}$$

This Jacobi field vanishes at $s = -\alpha < 0$ so that it cannot vanish elsewhere in $]-\infty,0[$ and can at most vanish once in $]0,\infty[$. Using equations (4.14) and (4.12), we can write

$$e_0(a,\alpha,s) = v_0(a,\alpha)f_0(a,s) + v_0(a,s)\big[e_0(a,\alpha) - v_0(a,\alpha)\int_0^s I_0(a,t)\,dt\big].$$

We have $e_0(a,\alpha,-\alpha) = 0$ and $e_0(a,\alpha,0) = v_0(a,\alpha) > 0$ so that $e_0(a,\alpha,\cdot)$ vanishes in $]0,\infty[$ if and only if $e_0(a,\alpha) - v_0(a,\alpha)E_0(a) < 0$ (recall that $E_0(a) = \int_0^\infty I_0(a,t)\,dt$). If $\mathcal{C}_{0,a}$ is stable, then clearly $e_0(a,\alpha,\cdot)$ cannot vanish twice in $\mathbb{R}$. Assume that $\mathcal{C}_{0,a}$ has index 1 or, equivalently, that $E_0(a) > 0$. In that case, $e_0(a,\cdot)$ has exactly one positive zero $z(a)$. (i) For $\alpha > z(a)$, $e_0(a,\alpha) < 0$ so that $e_0(a,\alpha) - v_0(a,\alpha)E_0(a) < 0$ and $e_0(a,\alpha,\cdot)$ has a positive zero β (which must satisfy $\beta < z(a)$). (ii) For $\alpha = z(a)$, $e_0(a,\alpha,s) = v_0(a,\alpha)e_0(a,s)$ has two zeroes $\pm z(a)$. (iii) For $0 < \alpha < z(a)$, we can argue as follows. Consider the Jacobi field $w(a,t) = e_0(a,t) - E_0(a)v_0(a,t)$. At $t = 0$ we have $w(a,0) = 1$ and at $t = z(a)$ we have $w(a,z(a)) < 0$, because $e_0(a,z(a)) = 0$, $E_0(a) > 0$ and $v_0(a,z(a)) > 0$. It follows that $w(a,t)$ has a unique zero in $]0,z(a)[$ and hence that there exists a value $\ell(a) > 0$ such that $\mathcal{D}_{0,\ell(a)} = Y_0(a,]-\ell(a),\infty[,[0,2\pi])$ is a maximal stable rotation invariant domain.

Assertion 4. This follows immediately from the previous assertion.

Assertion 5. The first part of the assertion follows from Lemma 4.4, which implies that $e(a,s)$ never vanishes when $a > a_1$. To prove the second part of Assertion 3, we can use either the fact that $E_0(a)$ tends to $+\infty$ when a tends to zero from above or the criteria given in [4] (Corollary 5.13, p. 708) or [11] (Corollary 4.2). □

We have the following geometric interpretation of Theorem 4.7.

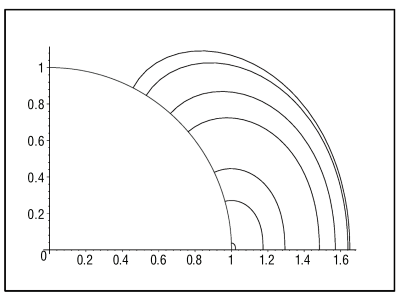

FIGURE 2. Foliating

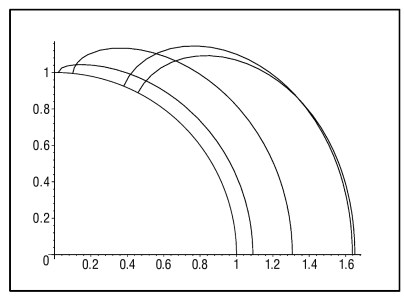

FIGURE 3. Intersecting

Proposition 4.8. *Geometric interpretation.*

(1) *Let $\mathcal{S}$ be an open interval on which $E_0 < 0$ (hence the catenoid $\mathcal{C}_{0,a}$ is stable for all $a \in \mathcal{S}$). For $a \in \mathcal{S}$, the catenaries $C_{0,a}$ locally foliate the hyperbolic plane $\mathbb{H}^2_{\{x_1,x_3\}}$; see Figure 2 (half-catenaries).*

(2) *Let $\mathcal{U}$ be an open interval on which $E_0 > 0$ (hence the catenoid $\mathcal{C}_{0,a}$ has index 1 for all $a \in \mathcal{U}$). For $a, b \in \mathcal{U}$, the catenaries $C_{0,a}$ and $C_{0,b}$ intersect exactly at two points in $\mathbb{H}^2_{\{x_1,x_3\}}$ and the family $\{C_{0,a}\}_{a\in\mathcal{U}}$ has an envelope. Furthermore, the points at which $\mathcal{C}_{0,a}$ touches the envelope correspond to the stable-unstable domain $\mathcal{D}_{0,z(a)}$; see Figure 3 (half-catenaries).*

Proof. Define the v-height function of the catenoid $\mathcal{C}_{0,a}$ in $\mathbb{H}^2_{\{u,v\}}$ by

$$V_0(a) = \lim_{t\to\infty} \lambda_0(a,t) = \lim_{s\to\infty} \Lambda_0(a,s). \tag{4.15}$$

□

Lemma 4.9. *Let $a_2 > a_1 > 0$ be two values of the parameter a. The catenaries C_{0,a_1} and C_{0,a_2} intersect at most at two symmetric points, and they do so if and only if $V_0(a_2) > V_0(a_1)$.*

Proof. To prove the lemma, consider the difference $w(t) := \lambda_0(a_2,t) - \lambda_0(a_1,t)$ for $t \geq a_2 > a_1$. A straightforward computation shows that this function increases from the negative value $-\lambda_0(a_1,a_2)$ (achieved for $t = a_2$) to $V_0(a_2) - V_0(a_1)$ (the limit at $t = \infty$). It follows that w has at most one zero and does so if and only if $V_0(a_2) - V_0(a_1) > 0$. The proposition follows from the fact that $V_0(a) = \sqrt{2}\int_0^\infty J_0(a,t)\,dt$ and that $V_0'(a) = \sqrt{2}E_0(a)$, where $E_0(a)$ is defined by (4.13). □

Observation 1. One can also define the x-height function of the catenoid $\mathcal{C}_{0,a}$ by $X_0(a) = \lim_{s\to\infty} e^{\Lambda_0(a,s)}\tanh(y_0(a,s)) = e^{\sqrt{2}\int_0^\infty J_0(a,t)\,dt}$, where $J_0(a,t)$ is defined by (4.6). Then, the critical points of $X_0(a)$ correspond to the zeroes of the function $E_0(a)$.

Observation 2. We point out that Theorem 4.7, Assertion 1, provides a criterion (the sign of $E_0(a)$) to determine whether the catenoid $\mathcal{C}_{0,a}$ has index 0 or 1, whereas [9, 4, 11] only give sufficient conditions to insure that the index is 0 or 1. Numerical computations show that the following result holds (see [3] and the graphs in Figures 4 and 5).

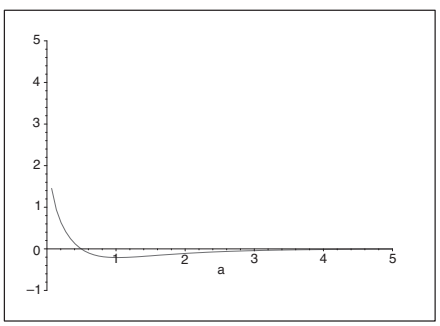

FIGURE 4. Graph of $E_0(a)$

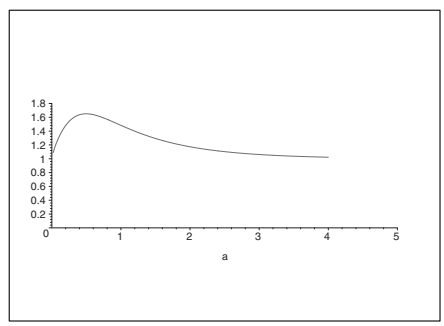

FIGURE 5. Graph of $X_0(a)$

Proposition 4.10. *The function $E_0(a)$ has exactly one zero $a_0 \approx 0.4955$ on $]0,\infty[$, which is also the unique critical point of the x-height function $X_0(a)$. It is positive for $a < a_0$ and negative for $a > a_0$.*

4.3. Catenoid cousins in $\mathbb{H}^3$.

4.3.1. *Basic formulas.* We now consider *catenoid cousins*, *i.e.* rotation surfaces with constant mean curvature 1 in $\mathbb{H}^3(-1)$. In this case, the mean curvature equation (4.1) reads

$$\sinh(2t) = \frac{d}{dt}\,\frac{f_t(t)\sinh(t)\cosh^2(t)}{\left(1+\cosh^2(t)f_t^2(t)\right)^{1/2}}, \tag{4.16}$$

which yields

$$\frac{f_t(t)\sinh(t)\cosh^2(t)}{\left(1+\cosh^2(t)f_t^2(t)\right)^{1/2}} = \frac{1}{2}\cosh(2t) - d,$$

for some constant $d \in \mathbb{R}$. For a solution to exist, d needs to be positive so that we may assume that $2d = e^{-2a}$ for some $a \in \mathbb{R}$, and we obtain

$$f_t = \frac{e^a\big(\cosh(2t) - e^{-2a}\big)}{\sqrt{2}\cosh(t)\sqrt{\cosh(2t) - \cosh(2a)}}, \quad t \geq |a|. \tag{4.17}$$

► We now limit ourselves to the embedded case and assume that $a > 0$. ◄ Equation (4.17) yields embedded catenary cousins $\{C_{1,a}\}_{a>0}$, given by

$$\lambda_1(a,t) = \int_a^t \frac{e^a\big(\cosh(2\tau) - e^{-2a}\big)}{\sqrt{2}\cosh(\tau)\sqrt{\cosh(2\tau) - \cosh(2a)}}\, d\tau, \quad \text{for} \quad t \geq a, \tag{4.18}$$

where the lower index 1 refers to $H = 1$. Notice that the function λ_1 describes the upper halves of catenary-like curves. We compute the arc-length function

$$\begin{cases} S_1(a,t) = \frac{e^a}{\sqrt{2}}\sqrt{\cosh(2t) - \cosh(2a)}, \\ \cosh(2t) = 2e^{-2a}S_1^2(a,t) + \cosh(2a), \quad t \geq a. \end{cases} \tag{4.19}$$

As in Section 4.2, we obtain the following result.

Proposition 4.11. *For $a > 0$ and $s \in \mathbb{R}$, define the functions $y_1(a,s)$ and $\Lambda_1(a,s)$ by the formulas*

$$y_1(a,s) = a + \int_0^s \frac{2e^{-2a}t\,dt}{\sqrt{\big(2e^{-2a}t^2 + \cosh(2a)\big)^2 - 1}} \quad \textit{and} \tag{4.20}$$

$$\Lambda_1(a,s) = \int_0^s \frac{e^a\big(2t^2 + e^{2a}\sinh(2a)\big)\,dt}{2\big(t^2 + e^{2a}\cosh^2(a)\big)\sqrt{t^2 + e^{2a}\sinh^2(a)}}. \tag{4.21}$$

(1) *The function y_1 is smooth, even, and satisfies*
$$\cosh(2y_1(a,s)) = 2e^{-2a}s^2 + \cosh(2a).$$
(2) *The function Λ_1 is smooth, odd, and satisfies $\Lambda_1(a,s) = \lambda_1(a, y_1(a,s))$ for $s \geq 0$.*
(3) *For $a > 0$, the maps $\mathbb{R} \ni s \mapsto \big(y_1(a,s), \Lambda_1(a,s)\big) \in \mathbb{H}^2_{\{u,v\}}$ are arc-length parametrizations of the family of embedded catenary cousins $\{C_{1,a}\}$ which generate the family $\{\mathcal{C}_{1,a}\}_{a>0}$ of embedded catenoid cousins.*
(4) *The parametrization of the family $\{\mathcal{C}_{1,a}\}_{a>0}$ in $\mathbb{H}^3_{\{x_1,x_2,x_3\}}$ is given by*

$$Y_1(a,s) = \Big(e^{\Lambda_1(a,s)}\tanh(y_1(a,s))\,\omega_\theta\,,\ \frac{e^{\Lambda_1(a,s)}}{\cosh(y_1(a,s))}\Big). \tag{4.22}$$

4.3.2. *Jacobi fields on $\mathcal{C}_{1,a}$.* As in Section 4.2, we define the vertical and variation Jacobi fields on $\mathcal{C}_{1,a}$.

Lemma 4.12. *The vertical Jacobi field v_1 is a smooth odd function of s. It is given by $v_1(a,s) = \cosh(y_1(a,s))y_{1,s}(a,s) = e^{-a}s\big(s^2 + e^{2a}\sinh^2(a)\big)^{-1/2}$ and satisfies $v_1(a,0) = 0$, $v_1(a,\infty) = e^{-a}$.*

The *variation Jacobi field* $e_1(a,s)$ on $\mathcal{C}_{1,a}$ is given by

$$e_1(a,s) = \cosh\big(y_1(a,s)\big)\big(\Lambda_{1,a}y_{1,s} - \Lambda_{1,s}y_{1,a}\big)(a,s),$$

which we can write as

$$e_1(a,s) = v_1(a,s)\Lambda_{1,a}(a,s) - \cosh(y_1(a,s))y_{1,a}(a,s)\Lambda_{1,s}(a,s).$$

Using Proposition 4.11 and Lemma 4.12, we find the formula

$$\cosh(y_1)\,\Lambda_{1,s}\,y_{1,a} = \frac{\sinh^2(2a) - 4e^{-4a}s^4}{4\big(e^{-2a}s^2 + \cosh^2(a)\big)\big(e^{-2a}s^2 + \sinh^2(a)\big)}. \tag{4.23}$$

By (4.21), we can write $\Lambda_1(a,s)$ as $\int_0^s A(a,t)\,dt$, where the integrand $A(a,t)$ is given by

$$\begin{cases} A(a,t) &= \dfrac{2e^a t^2 + e^{3a}\sinh(2a)}{2\big(t^2 + e^{2a}\cosh^2(a)\big)\big(t^2 + e^{2a}\sinh^2(a)\big)^{1/2}} \\ &= \dfrac{A_1(a,t)}{2A_2(a,t)A_3^{1/2}(a,t)}, \end{cases} \tag{4.24}$$

where the second equality defines the functions A_i. One can now compute the derivative of $A(a,t)$ with respect to the variable a:

$$A_a(a,t) = \frac{A_{1,a}(a,t)}{2A_2(a,t)A_3^{1/2}(a,t)} - \frac{A_1(a,t)B_2(a)}{2A_2^2(a,t)A_3^{1/2}(a,t)} - \frac{A_1(a,t)B_3(a)}{4A_2(a,t)A_3^{3/2}(a,t)},$$

where $B_2(a) = \partial_a\big(e^{2a}\cosh^2(a)\big)$, $B_3(a) = \partial_a\big(e^{2a}\sinh^2(a)\big)$. It follows that

$$\begin{aligned} A_a(a,t) = \ & \frac{2e^a t^2 + e^{3a}(3\sinh(2a) + 2\cosh(2a))}{2A_2(a,t)A_3^{1/2}(a,t)} - \frac{A_1(a,t)B_2(a)}{2A_2^2(a,t)A_3^{1/2}(a,t)} \quad \cdots \\ & - \frac{A_1(a,t)B_3(a)}{4A_2(a,t)A_3^{3/2}(a,t)}, \end{aligned}$$

i.e.

$$\begin{cases} A_a(a,t) = B(a,t) - C(a,t), \quad \text{where} \\ B(a,t), C(a,t) > 0, \quad \text{for} \quad a > 0, t \in \mathbb{R}, \\ B(a,t) \sim \frac{e^a}{|t|}, \quad \text{at infinity}, \\ C(a,t) = O(\frac{1}{|t|^3}), \quad \text{at infinity}. \end{cases} \tag{4.25}$$

Finally, with the above notation, we can write the variation Jacobi field as

$$\begin{aligned} e_1(a,s) = \ & -\frac{e^{4a}\sinh^2(a)\cosh^2(a) - s^4}{\big(s^2 + e^{2a}\cosh^2(a)\big)\big(s^2 + e^{2a}\sinh^2(a)\big)} \\ & - v_1(a,s)\int_0^s C(a,t)\,dt + v_1(a,t)\int_0^s B(a,t)\,dt. \end{aligned}$$

We have proved

Lemma 4.13. *The variation Jacobi field e_1 is a smooth, even function of s which can be written as*

$$e_1(a,s) = -f_1(a,s) + v_1(a,s)\int_0^s B(a,t)\,dt, \tag{4.26}$$

where the function f_1 is a smooth, even function of s, such that $f_1(a,0) = 1$ and $f_1(a,\infty)$ is finite. Furthermore,

$$\lim_{s\to\infty} v_1(a,s)\int_0^s B(a,t)\,dt = +\infty.$$

4.3.3. *Stable domains on the embedded catenoid cousins.* We can now investigate the stability properties of the embedded catenoids cousins $\{\mathcal{C}_{1,a}\}_{a>0}$ in $\mathbb{H}^3(-1)$.

Lemma 4.14. *The upper and lower halves $\mathcal{D}_{1,a,\pm} = Y_1(a, \mathbb{R}^\bullet_\pm, [0, 2\pi])$ of the embedded catenoid cousins are stable. As a consequence, a Jacobi field $w(a, s)$, which only depends on the radial variable s, on $\mathcal{C}_{1,a}$ can have at most one zero on $\mathbb{R}^\bullet_+$ and on $\mathbb{R}^\bullet_-$.*

Proof. Use Property 2.1 (3) and the fact that $v_1(a, s)$ is a Jacobi field which only vanishes at $s = 0$. □

Lemma 4.15. *The embedded catenoid cousins $\mathcal{C}_{1,a}$ have at most index* 1.

Proof. Same proof as for Lemma 4.6. □

We can now state the main theorem of this section. Recall that the Jacobi fields $v_1(a, s)$ and $e_1(a, s)$ are given respectively by Lemmas 4.12 and 4.13.

Theorem 4.16. *Let $\{\mathcal{C}_{1,a}, a > 0\}$ be the family of embedded catenoid cousins in $\mathbb{H}^3$ given by the parametrization Y_1, equation* (4.22).

(1) *The Jacobi field $e_1(a, s)$ has exactly one positive zero $z_1(a)$, and the domains $\mathcal{D}_{1,a,z_1(a)} = Y_1(a,]-z_1(a), z_1(a)[, [0, 2\pi])$ are stable-unstable.*
(2) *For any $\alpha > 0$, there exists a $\beta(\alpha) > 0$ such that the domains*

$$\mathcal{D}_{1,a,-\alpha,\beta(\alpha)} = Y_1(a,]-\alpha, \beta(\alpha)[, [0, 2\pi])$$

are stable-unstable.
(3) *In particular, the embedded catenoid cousins $\{\mathcal{C}_{1,a}\}_{a>0}$ satisfy Lindelöf's property: the upper and lower halves of the embedded catenoid cousins $\mathcal{D}_{1,a,\pm}$ are maximal rotationally symmetric domains.*
(4) *The index of the catenoid $\mathcal{C}_{1,a}$ is equal to* 1.

Proof. Assertion 1. As we have seen in Lemma 4.14, the function $e_1(a, s)$ can have at most one zero on $]0, \infty[$ and at most one zero on $]-\infty, 0[$. By Lemma 4.13, the function $e_1(a, s)$ is even, $e_1(a, 0) = -1$ and $e_1(a, \infty) = \infty$. It follows that $e_1(a, s)$ has exactly two symmetric zeroes in $\mathbb{R}$.

Assertion 2. Given any $\alpha > 0$, we introduce the Jacobi field $e_1(a, \alpha, s)$,

$$e_1(a, \alpha, s) = v_1(a, \alpha)e_1(a, s) + e_1(a, \alpha)v_1(a, s). \tag{4.27}$$

This Jacobi field vanishes at $s = -\alpha < 0$ so that it cannot vanish elsewhere in $]-\infty, 0[$ and can at most vanish once in $]0, \infty[$. Using Lemma 4.13, we can write

$$e_1(a, \alpha, s) = -v_1(a, \alpha)f_1(a, s) + v_1(a, s)\big(e_1(a, \alpha) + v_1(a, \alpha)\int_0^s B(a, t)\,dt\big).$$

It follows that $e_1(a, \alpha, -\alpha) = 0$, $e_1(a, \alpha, 0) < 0$ and $\lim_{s\to\infty} e_1(a, \alpha, s) = +\infty$, and hence that $e_1(a, \alpha, \cdot)$ must vanish at least once.

Assertion 3. This is a consequence of Assertion 2.

Assertion 4. This assertion follows from Assertion 1 and from Lemma 4.15. This has also been proved, using different methods, by Lima and Rossman [7]. □

4.4. **Further results.** One can also study rotation surfaces with constant mean curvature H, $0 \leq H < 1$ in $\mathbb{H}^3(-1)$. This is similar to the case of minimal surfaces. More precisely, H-rotation surfaces in $\mathbb{H}^3(-1)$, with $0 \leq H < 1$, come in a one-parameter family $\mathcal{C}_{H,a}$. For some values of a the surfaces are stable, for other values of a they have index 1. Furthermore, they do not satisfy Lindelöf's property. The computations are much more complicated but similar to the minimal case. The functions involved depend continuously on the parameter H, for $0 \leq H < 1$.

The method described in the previous sections can be applied to study the stable domains on higher-dimensional catenoids (minimal rotation hypersurfaces or constant mean curvature 1 rotation hypersurfaces) in $\mathbb{H}^{n+1}$ or other homogeneous spaces.

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