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A GENERALIZED FERNIQUE THEOREM AND APPLICATIONS

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Abstract. We prove a generalisation of Fernique's theorem which applies to a class of (measurable) functionals on abstract Wiener spaces by using the isoperimetric inequality. Our motivation comes from rough path theory where one deals with iterated integrals of Gaussian processes (which are generically not Gaussian). Gaussian integrability with explicitly given constants for variation and Hölder norms of the (fractional) Brownian rough path, Gaussian rough paths and the Banach space valued Wiener process enhanced with its Lévy area [Ledoux, Lyons, Qian. "Lévy area of Wiener processes in Banach spaces", Ann. Probab., 30(2):546–578, 2002] then all follow from applying our main theorem.

1. A generalized Fernique theorem

Let $(E, |\cdot|)$ be a real, separable Banach space equipped with a Borel σ-algebra $\mathfrak{B}$ and a centered Gaussian measure μ. A famous result by X. Fernique states that $|\cdot|_* \mu$ has a Gauss tail; more precisely,

$$\int \exp\left(\eta |x|^2\right) d\mu(x) < \infty \text{ if } \eta < \frac{1}{2\sigma^2},$$

where

$$\sigma := \sup_{\xi \in E^*, |\xi|_{E^*}=1} \left(\int \langle \xi, x \rangle^2 \, d\mu(x) \right)^{1/2} < \infty, \tag{1}$$

and this condition on η is sharp (see [8, Thm. 4.1] for instance). We recall the notion of a *reproducing kernel Hilbert space* H, continuously embedded in E, $|h| \le \sigma |h|_H$ for all $h \in H$, so that (E, H, μ) is an abstract Wiener space in the sense of L. Gross. We can then cite Borell's inequality, e.g. [8, Theorem 4.3].

Theorem 1. *Let (E, H, μ) be an abstract Wiener space and $A \subset E$ a Borel set with $\mu(A) > 0$. Take $a \in (-\infty, \infty]$ such that*

$$\mu(A) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx =: \Phi(a).$$

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Then, if $\mathcal{K}$ denotes the unit ball in H and μ_* stands for the inner measure[1] *associated to μ,*

$$\mu_*(A + r\mathcal{K}) = \mu_*\{x + rh : x \in A,\, h \in \mathcal{K}\} \geq \Phi(a + r). \tag{2}$$

The reader should observe that the following theorem reduces to the usual Fernique result when applied to the Banach norm on E.

Theorem 2 (Generalized Fernique)**.** *Let (E, H, μ) be an abstract Wiener space. Assume $f : E \to \mathbb{R} \cup \{-\infty, \infty\}$ is a measurable map and $N \subset E$ a null-set and c some positive constant such that for all $x \notin N$,*

$$|f(x)| < \infty, \tag{3}$$

$$\forall h \in H\colon\ |f(x)| \leq c\{|(f(x-h))| + \sigma |h|_H\}. \tag{4}$$

Then, with the definition of σ given in (1)*,*

$$\int \exp\left(\eta |f(x)|^2\right) d\mu(x) < \infty \quad \text{if } \eta < \frac{1}{2c^2\sigma^2}.$$

Proof. We have for all $x \notin N$ and all $h \in r\mathcal{K}$, where $\mathcal{K}$ denotes the unit ball of H and $r > 0$,

$$\begin{aligned}\{x : |f(x)| \leq M\} &\supset \{x : c(|f(x-h)| + \sigma |h|_H) \leq M\}\\ &\supset \{x : c(|f(x-h)| + \sigma r) \leq M\}\\ &= \{x + h : |f(x)| \leq M/c - \sigma r\}.\end{aligned}$$

Since $h \in r\mathcal{K}$ was arbitrary,

$$\begin{aligned}\{x : |f(x)| \leq M\} &\supset \bigcup_{h \in r\mathcal{K}} \{x + h : |f(x)| \leq M/c - \sigma r\}\\ &= \{x : |f(x)| \leq M/c - \sigma r\} + r\mathcal{K}\end{aligned}$$

and we see that

$$\begin{aligned}\mu[|f(x)| \leq M] &= \mu_*[|f(x)| \leq M]\\ &\geq \mu_*(\{x : |f(x)| \leq M/c - \sigma r\} + r\mathcal{K}).\end{aligned}$$

We can take $M = (1 + \varepsilon) c\sigma r$ and obtain

$$\mu[|f(x)| \leq (1 + \varepsilon) c\sigma r] \geq \mu_*(\{x : |f(x)| \leq \varepsilon\sigma r\} + r\mathcal{K}).$$

Keeping ε fixed, take $r \geq r_0$, where r_0 is chosen large enough such that $\mu[\{x : |f(x)| \leq \varepsilon\sigma r_0\}] > 0$. Letting Φ denote the distribution function of a standard Gaussian, it follows from Borell's inequality that

$$\mu[|f(x)| \leq (1 + \varepsilon) c\sigma r] \geq \Phi(a + r)$$

for some $a > -\infty$. Equivalently,

$$\mu[|f(x)| \geq x] \leq \bar{\Phi}\left(a + \frac{x}{(1 + \varepsilon) c\sigma}\right)$$

[1]Measurability of the so-called Minkowski sum $A + r\mathcal{K}$ is a delicate topic. Use of the inner measure bypasses this issue and is not restrictive in applications.

with $\bar{\Phi} \equiv 1 - \Phi$ and using $\bar{\Phi}(z) \lesssim \exp\left(-z^2/2\right)$, we see that this implies that

$$\int \exp\left(\eta \left|f(x)\right|^2\right) d\mu(x) < \infty$$

provided $\eta < \frac{1}{2}\left(\frac{1}{(1+\varepsilon)c\sigma}\right)^2$. Sending $\varepsilon \to 0$ finishes the proof. □

2. Applications

The examples below apply Theorem 2 with $f(.) = |||\cdot|||_{p\text{-var};[0,T]} \circ \phi$ or $f(.) = |||\cdot|||_{1/p\text{-Höl};[0,T]} \circ \phi$, where

$$\phi : C_o([0,T], B) \to C_o\left([0,T], \{1\} \oplus B \oplus B^{\otimes 2}\right)$$

is constructed (typically through an almost-sure convergence result) as a measurable map, in general not continuous, and, for $\mathbf{x} = \left(1, \mathbf{x}^1, \mathbf{x}^2\right) \in C_o\left([0,T], \{1\} \oplus B \oplus B^{\otimes 2}\right)$,

$$|||\mathbf{x}|||_{p\text{-var};[0,T]} = \max_{i=1,2}\left(\sup_{D=(t_i)} \sum_{i:t_i \in D} \left|\mathbf{x}^i_{t_i,t_{i+1}}\right|^{p/i}_{B^{\otimes i}}\right)^{1/p} \in [0,\infty],$$

$$|||\mathbf{x}|||_{1/p\text{-Höl};[0,T]} = \max_{i=1,2}\left(\sup_{s,t\in[0,T], s\neq t} \frac{\left|\mathbf{x}^i_{s,t}\right|^{1/i}_{B^{\otimes i}}}{|t-s|^{1/p}}\right) \in [0,\infty].$$

This setting is standard in rough path theory (we refer to [10],[11]). The first example to have in mind is $B = \mathbb{R}^d$ and $E = C_o\left([0,T], \mathbb{R}^d\right)$ equipped with the Wiener measure. We now discuss this in detail, followed by more complex examples.

2.1. **Brownian motion on $\mathbb{R}^d$.** Take the usual Wiener space (E, H, μ); i.e., E is the space of continuous paths on $[0,T]$ in $\mathbb{R}^d$ vanishing at 0, H is the standard Cameron-Martin space and μ is the Wiener measure. The coordinate process $B_t(x) = x(t)$ for $x \in E$ is a standard Brownian motion. In this case,

$$\phi : C_o\left([0,T], \mathbb{R}^d\right) \to C_o\left([0,T], G^2\left(\mathbb{R}^d\right)\right)$$

(we recall that $G^2\left(\mathbb{R}^d\right) \subset \{1\} \oplus \mathbb{R}^d \oplus \left(\mathbb{R}^d\right)^{\otimes 2}$ is the natural state space for geometric rough paths) is given by

$$\phi(x) = \left(1, x, \int_0^{\cdot} x \otimes \circ dx\right)$$

and almost surely well-defined (the final integral is a stochastic integral in the Stratonovich sense). We also write $\mathbf{B} = \left(1, \mathbf{B}^1, \mathbf{B}^2\right)$ for the coordinate process of the lift $\phi(.)$ and call $\mathbf{B}$ the enhanced Brownian motion. Using the Theorem 2 it is easy to see that $|||\mathbf{B}|||_{1/p\text{-Höl};[0,T]}$ has a Gauss tail.

Proposition 3. *If $p > 2$,*

$$\mathbb{E}\left[\exp\left(\frac{\eta}{T^{1-2/p}} |||\mathbf{B}|||^2_{1/p\text{-}Höl;[0,T]}\right)\right] < \infty$$

for all $\eta < \eta_0$. Moreover, one can take $\eta_0 = \frac{1}{\left(1+\sqrt{2}\right)}$.

Proof. We recall that $|||\mathbf{B}|||_{p\text{-var};[0,T]} < \infty$ a.s. (see [10]) and the stronger statement $|||\mathbf{B}|||_{1/p\text{-Höl};[0,T]} < \infty$ a.s. is found in [3]. Since

$$|||\mathbf{B}|||_{1/p\text{-Höl};[0,T]} \sim T^{1/2-1/p} |||\mathbf{B}|||_{1/p\text{-Höl};[0,1]}$$

we can assume w.l.o.g. that $T = 1$ and one checks (4) by simple Riemann–Stieltjes estimates. Note that by construction of $\mathbf{B}$ (as an a.s. limit) and continuity properties of the integrals[2] it follows that the set

$$\left\{ x \in E : \mathbf{B}(x+h) = (x+h) + \left(\mathbf{B}^2(x) + \int x \otimes dh + \int h \otimes dx + \int h \otimes dh \right) \right.$$
$$\left. \text{for all } h \in H \right\}$$

has full measure (see [1]). Hence, there exists a nullset N s.t. for $x \notin N$,

$$\left| \mathbf{B}^1(x+h)_{s,t} \right| = |x_{s,t} + h_{s,t}| \leq |||\mathbf{B}(x)|||_{1/p\text{-Höl};[0,1]} (t-s)^{1/p} + |h_{s,t}|,$$
$$\sqrt{\left| \mathbf{B}^2(x+h)_{s,t} \right|} = \sqrt{\left| \mathbf{B}^2_{s,t} + \int_s^t x_{s,r} \otimes dh_r + \int_s^t h_{s,r} \otimes dx_r + \int_s^t h_{s,r} \otimes dh_r \right|}$$

for all $h \in H$. Now,

$$\sqrt{\left| \int_s^t x_{s,r} \otimes dh_r + \int_s^t h_{s,r} \otimes dx_r \right|} \leq \sqrt{|h|_{1-\text{var};[s,t]} |x_{s,t}|} \leq \frac{1}{\sqrt{2}} \left(|h|_{1\text{-var};[s,t]} + |x_{s,t}| \right)$$

and by Cauchy–Schwarz, $|h|_{1-var;[s,t]} \leq |t-s|^{1/2} |h|_H$, which implies that

$$\sqrt{\left| \int_s^t h_{s,r} \otimes dh_r \right|} \leq |t-s|^{1/2} |h|_H .$$

Combining these estimates leads to

$$\begin{aligned} \frac{\sqrt{\left| \mathbf{B}^2(x+h)_{s,t} \right|}}{(t-s)^{1/p}} &\leq \left(1 + 1/\sqrt{2}\right) \left(|||\mathbf{B}|||_{1/p\text{-Höl};[0,1]} + (t-s)^{1/2-1/p} |h|_H \right) \\ &\leq \left(1 + 1/\sqrt{2}\right) \left(|||\mathbf{B}|||_{1/p\text{-Höl};[0,1]} + |h|_H \right) \end{aligned}$$

and (4) holds for $|f(.)| = |||\mathbf{B}(.)|||_{1/p\text{-Höl};[0,1]}$ with $c = \left(1 + 1/\sqrt{2}\right)$ and $\sigma = \sqrt{E[B_1^2]} = 1$. □

Remark 4. This implies an (exponential) integrability of Lévy area which cannot be obtained by integrability properties of the second Wiener-Itô chaos due to the non-linearity of area increments, i.e. $A_{s,t} \neq A_{0,t} - A_{0,s}$.

Remark 5. It is well known that there exists $C = C(d)$ such that

$$C^{-1} \|\mathbf{x}\| \leq |||\mathbf{x}||| \leq C \|\mathbf{x}\|,$$

[2]The integral $\int_s^t h_{s,r} \otimes dx_r$ is defined as the Riemann-Stieltjes integral $h_{s,t} \otimes x_t - \int_s^t dh_r \otimes x_r$. The other integrals make immediate sense as Riemann-Stieltjes integrals due to $|h|_{1-\text{var}} < \infty$.

where $\mathbf{x} = \left(1, \mathbf{x}^1, \mathbf{x}^2\right)$, $|||\mathbf{x}||| = \max\left(\left|\mathbf{x}^1\right|, \left|\mathbf{x}^2\right|\right)$ and $\|.\|$ denotes the Carnot-Carathéodory norm on the step-2 free nilpotent group with d generators $G^2\left(\mathbb{R}^d\right)$. As a consequence,

$$C^{-1}\left\|\mathbf{x}\right\|_{p\text{-var};[0,T]} \leq |||\mathbf{x}|||_{p\text{-var};[0,T]} \leq C\left\|\mathbf{x}\right\|_{p\text{-var};[0,T]},$$

where $\left\|\mathbf{x}\right\|_{p\text{-var};[0,T]} = \left(\sup_{D=(t_i)}\sum_{i:t_i\in D}\left\|\mathbf{x}_{t_i,t_{i+1}}\right\|^p\right)^{1/p}$ and a similar estimate holds for $\left\|\mathbf{x}\right\|_{1/p\text{-Höl};[0,T]} = \sup_{s,t\in[0,T],s\neq t}\frac{\|\mathbf{x}_{s,t}\|}{|t-s|^{1/p}}$. Hence, the Gauss tail of $|||\mathbf{B}|||_{1/p\text{-Höl};[0,T]}$ is consistent with the known Gauss tail of $\left\|\mathbf{B}\right\|_{1/p\text{-Höl};[0,T]}$ as obtained in [5] by a precise tracking of constants in the Garsia-Rodemich-Rumsey estimate.

Remark 6. Optimal variation of Brownian motion is not measured in a $p-$variation norm but in a $\psi-$variation norm with $\psi(x) = x^2/\max(\log\log 1/x, 1)$. More precisely, in [12] it is established that

$$\sup_{D=(t_i)}\sum_{i:t_i\in D}\psi\left(\left|B_{t_{i+1}} - B_{t_i}\right|\right) < \infty \text{ a.s.}$$

This gives rise to a $\psi-$variation norm $|B|_{\psi\text{-var}}$ and likewise, one can show (see [2]) that the rough path $\mathbf{B}$ has finite ψ-variation $\left\|\mathbf{B}\right\|_{\psi-\text{var}}$, which is the optimal variational regularity enjoyed by $\mathbf{B}$. This is important as it allows us to solve rough differential equations driven by $\mathbf{B}$ under minimal regularity assumptions on vector fields. It is then interesting to know that the generalized Fernique estimate can be used to see that the random variable $\left\|\mathbf{B}\right\|_{\psi-\text{var}}$ also has a Gauss tail, a fact which would be difficult to obtain from tracking constants in the Garsia-Rodemich-Rumsey estimate. (If one aims for the Lévy modulus, the optimal modulus regularity enjoyed by $\mathbf{B}$, it is to the contrary possible to obtain the Gauss tail by tracking GRR constants, [4].)

2.2. **Gaussian processes on $\mathbb{R}^d$.** Let us generalize from Brownian to a d-dimensional, continuous, centered Gaussian process $X = \left(X^1, \ldots, X^d\right)$ with independent components, assuming the covariance of X to be of finite ρ-variation in the 2D-sense, $|R|_{\rho\text{-var};[0,T]} < \infty$ for some $\rho \in [1,2)$ (as introduced in [6]; this setting covers for instance fractional Brownian motion where $\rho = \frac{1}{2H}$ for a Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right]$, Ornstein-Uhlenbeck process, etc.). The setup is as in the Brownian case; we just replace Wiener measure by a more general Gaussian measure μ and an appropriate Cameron-Martin space H and make the assumption of complementary Young regularity, i.e. $\exists q : 1/q + 1/(2\rho) > 1$ s.t. $H \hookrightarrow C^{q\text{-var}}$ (this assumption is always satisfied when $\rho \in [1, 3/2)$ and hence includes fBM with $H \in \left(\frac{1}{3}, \frac{1}{2}\right]$).

Proposition 7. *Let X be a centered, continuous Gaussian process in $\mathbb{R}^d$ on $[0,T]$ with independent components and with covariance of finite $2D$ ρ-variation $|R|_{\rho\text{-var};[0,T]}$ for $\rho < 2$. Assume furthermore complementary Young regularity. Then there exists a lift to a Gaussian rough path $\mathbf{X}(.) \in C_0\left([0,T], G^2\left(\mathbb{R}^d\right)\right)$ of finite homogeneous p-variation, $p > 2\rho$ and $|||\mathbf{X}|||_{p\text{-var};[0,T]}$ has a Gauss tail. More precisely,*

$$\mathbb{E}\left[\exp\left(\eta\frac{1}{|R|_{\rho\text{-var};[0,T]^2}}|||\mathbf{X}|||^2_{p\text{-var};[0,T]}\right)\right] < \infty$$

for every $\eta < \eta_0$. *Moreover, one can take* $\eta_0 = \left(\sqrt{2}3^{(1/2-1/p)}\left(\sqrt{c_{\rho,\rho}}^p + \sqrt{c_{\rho,p}}^p/\sqrt{2}\right)^{1/p}\right)^{-2}$ *where* $c_{u,v} = 2.4^{1/u+1/v}\zeta\left(\frac{1}{u}+\frac{1}{v}\right)$ *(with the Riemann-Zeta function* $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ *).*

Proof. The lift constructed in [6] gives a rough path of finite p-variation and we verify the translation estimate (4). By construction of the rough path lift (as an a.s. limit) and the assumption of complementary Young regularity one has that

$$\left\{x : \mathbf{X}(x+h) = (x+h) + \left(\mathbf{X}^2(x) + \int h \otimes dx + \int x \otimes dh + \int h \otimes dh\right) \quad \text{for all } h \in H\right\}$$

has full measure. The estimate $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ together with $|h|_{\rho\text{-var};[s,t]} \leq |h|_H \sqrt{|R|_{\rho\text{-var};[s,t]^2}}$ (see [6]) implies that

$$\left|\mathbf{X}^1(x+h)_{s,t}\right|^p \leq 2^{p-1}\left(|x_{s,t}|^p + |h|_H^p\left(|R|^{\rho}_{\rho\text{-var};[s,t]^2}\right)^{p/(2\rho)}\right),$$

and since by assumption $p/(2\rho) > 1$ and $|R|^{\rho}_{\rho\text{-var};[0,T]^2}$ is a 2D-control, summing up yields

$$\sum_i \left|\mathbf{X}^1(x+h)_{t_i,t_{i+1}}\right|_B^p \leq 2^{(p-1)/p}\left(|||\mathbf{X}(x)|||^p_{p\text{-var};[0,T]} + |h|_H^p\left(|R|^{\rho}_{\rho\text{-var};[0,T]^2}\right)^{p/(2\rho)}\right).$$

Similarly for the second level,

$$\left|\mathbf{X}^2(x+h)_{s,t}\right|^{p/2}_{B^{\otimes 2}} \leq 3^{p/2-1}\left(\left|\mathbf{X}^2_{s,t}(x)\right|^{p/2}_{B\otimes B} + \left|\int_s^t h_{s,r} \otimes dx_r + \int_s^t x_{s,r} \otimes dh_r\right|^{p/2}_{B\otimes B} + \left|\int_s^t h_{s,r} \otimes dh_r\right|^{p/2}_{B\otimes B}\right).$$

By Young's inequality (and using an i.b.p.) there exists a constant $c_{\rho,p}$ such that

$$\begin{aligned}\left|\int_s^t x_{s,r} \otimes dh_r + \int_s^t h_{s,r} \otimes dx_r\right|^{p/2} &= \left|\int_s^t dh_r \otimes x_{s,t}\right|^{p/2} \\ &\leq \left(c_{\rho,p}|h|_{\rho\text{-var};[s,t]}|x|_{p\text{-var};[s,t]}\right)^{p/2} \\ &\leq \frac{\sqrt{c_{\rho,p}}^p}{\sqrt{2}}\left(|h|_H^p\left(|R|^{\rho}_{\rho\text{-var};[0,T]^2}\right)^{p/(2\rho)} + |x|^p_{p\text{-var};[s,t]}\right).\end{aligned}$$

Further,

$$\left|\int_s^t h_{s,r} \otimes dh_r\right|^{p/2}_{B\otimes B} \leq \sqrt{c_{\rho,\rho}}^p |h|^p_{\rho\text{-var};[s,t]} \leq \sqrt{c_{\rho,\rho}}^p |h|_H^p\left(|R|^{\rho}_{\rho\text{-var};[0,T]^2}\right)^{p/2\rho}.$$

Combining these estimates yields

$$\begin{aligned}\sum \left|\mathbf{X}^2(x+h)_{t_i,t_{i+1}}\right|_B^{p/2} &\leq 3^{(p/2-1)}\left(\sqrt{c_{\rho,\rho}}^p+\sqrt{c_{\rho,p}}^p/\sqrt{2}\right)\\ &\quad\times\left(|||\mathbf{X}(x)|||^p_{p\text{-var};[0,T]}+|h|_H^p\left(|R|^\rho_{\rho\text{-var};[0,T]^2}\right)^{p/(2\rho)}\right)\\ &\leq 3^{(p/2-1)}\left(\sqrt{c_{\rho,\rho}}^p+\sqrt{c_{\rho,p}}^p/\sqrt{2}\right)\left(\frac{\sqrt{|R|_{\rho\text{-var};[0,T]^2}}}{\sigma}\right)^p\\ &\quad\times\left(|||\mathbf{X}(x)|||^p_{p\text{-var};[0,T]}+|h|_H^p\,\sigma^p\right).\end{aligned}$$

Noting that $\sqrt{|R|_{\rho\text{-var};[0,T]^2}}/\sigma \geq 1$ (which follows directly from the definition of $|R|_{\rho\text{-var}}$ and σ), the above estimates imply that (4) holds with $c = 3^{(1/2-1/p)}(\sqrt{c_{\rho,\rho}}^p + \sqrt{c_{\rho,p}}^p/\sqrt{2})^{1/p}\sqrt{|R|_{\rho\text{-var};[0,T]^2}}/\sigma$, where $c_{u,v} = 2.4^{1/u+1/v}\zeta\left(\frac{1}{u}+\frac{1}{v}\right)$ and σ is as in (1). □

Remark 8. One can replace the variation norm by a stronger Hölder norm (if finite) and follow the proof of Proposition 3. A sufficient condition for Hölder regularity is that the covariance of X satisfies an appropriate Hölder condition (see [6]).

Remark 9. Explicit estimates for $|R|_{\rho\text{-var};[0,T]}$ are sometimes known. For example, for Brownian motion, $|R|_{1\text{-var};[0,T]} = T$ (compare this to Proposition 10, where one has the same scaling in T but a better constant η_0 due to the use of Riemann-Stieltjes instead of Young integrals) and for fBM with Hurst parameter $H \in (0, 1/2]$, $|R|_{1/(2H)\text{-var};[0,T]} \leq C(H)\,T^{\,1/(2H)}$ for a known constant $C(H)$ (see [6]).

2.3. **Banach space valued Brownian motion.** Let (B, G, ν) be an abstract Wiener space. Then there exists a stochastic process $X = (X_t)$ with continuous paths, taking values in B and such that $X_0 = 0$ a.s., the increments $X_t - X_s$ are independent and $\xi(X_t - X_s) \sim N\left(0, (t-s)\,|\xi|_G^2\right)$ for $\xi \in B^*$, $s < t$. This process is called a Wiener process based on (the abstract Wiener space) (B, G, ν). The associated Lévy area (and lift to a rough path $\mathbf{X}$) was constructed in [9] (the choice of a norm $|.|_{B^{\otimes 2}}$ on $B^{\otimes 2}$ is a subtle issue and reflected in the condition of exactness; the injective tensor norm is always exact; see [9] for the definition and further examples).

Proposition 10. *Let X be a Wiener process based on the abstract Wiener space (B, G, ν) and let $\sigma_B^2 = \sup_{|\varphi|_{B^*}\leq 1}\int_B \varphi(x)^2\,\mu(dx)$. If we complete the algebraic tensor product $B\otimes B$ with a norm $|.|_{B\otimes B}$ such that the pair $\left(|.|_{B\otimes B}, \nu\right)$ is exact, then there exists a lift to a geometric rough path $\mathbf{X} = \left(1, \mathbf{X}^1, \mathbf{X}^2\right) \in C^{p\text{-var}}([0,T], 1\oplus B \oplus B^{\otimes 2})$, $p > 2$ and $|||\mathbf{X}|||_{p\text{-var}}$ has a Gauss tail. More precisely,*

$$\mathbb{E}\left[\exp\left(\eta\frac{1}{\sigma_B^2 T}|||\mathbf{X}|||^2_{p\text{-var};[0,T]}\right)\right] < \infty$$

for every $\eta < \eta_0$. Moreover, one can take $\eta_0 = \left(\sqrt{2}3^{(1/2-1/p)}\left(1+1/\sqrt{2}\right)^{1/p}\right)^{-2}$.

Let $E = C_0([0,T], B)$ and set $|x|_E = \sup_{t\in[0,T]} |x_t|$. Denote the associated Borel $\sigma-$algebra by $\mathcal{E}$ and define the RKHS $H \subset E$ as $H = i^*\left(L^2(E,\mathcal{E},\mu)\right)$, where $i^* : L^2(E,\mathcal{E},\mu;\mathbb{R}) \to E$ maps φ to $\int_E x\varphi(x)\,\mu(dx)$ (μ is the measure given by construction of the Wiener process (X_t)). (E,H,μ) is then an abstract Wiener space (following the setup of [8]). We first estimate the regularity of elements in the Hilbert space H.

Proposition 11. *For all $h \in H$, $|h|_{1\text{-var};[s,t]} \le |h|_H\, \sigma_B\, |t-s|^{1/2}$.*

Proof. Let (t_j) be a dissection of $[s,t]$. By definition $h = \int_E x\varphi(x)\,\mu(dx)$ for some $\psi \in L^2(\mu)$ and we also write $h_t = \mathbb{E}(\psi(X)X_t)$. Then

$$\begin{aligned}\sum_j \left|h_{t_j,t_{j+1}}\right|_B &= \sup_{\xi_j\in B^*:|\xi_j|_{B^*}\le 1} \sum_j \xi_j h_{t_j,t_{j+1}} = \sup \mathbb{E}\left[\psi(x)\sum_j \xi_j X_{t_j,t_{j+1}}\right] \\ &\le |\psi|_{L^2(\mu)} \sup_{\substack{\xi_j\in B^*:|\xi_j|_{B^*}\le 1\\ \xi_k\in B^*:|\xi_k|_{B^*}\le 1}} \sqrt{\sum_{j,k}\mathbb{E}\left[\xi_j X_{t_j,t_{j+1}}\xi_k X_{t_k,t_{k+1}}\right]} \\ &= |h|_H^2 \sup_{\substack{\xi_j\in B^*:|\xi_j|_{B^*}\le 1\\ \xi_k\in B^*:|\xi_k|_{B^*}\le 1}} \sqrt{\sum_{j,k}\mathbb{E}\left[\xi_j X_{t_j,t_{j+1}}\xi_k X_{t_k,t_{k+1}}\right]},\end{aligned}$$

where we used Cauchy-Schwarz and the isometry $\left\langle i^*\left(\tilde{\psi}\right), i^*(\psi)\right\rangle_H = \left\langle \tilde{\psi},\psi\right\rangle_{L^2(\mu)}$. The characteristic property of the tensor product yields an isomorphism between the space of (algebraic) linear functionals on $B\otimes B$, $L(B\otimes B,\mathbb{R})$, and bilinear functionals on $B\times B$, $BL(B\times B,\mathbb{R})$. Since $\xi_j(.)\,\xi_k(.) \in BL(B\times B,\mathbb{R})$,

$$\xi_j(.)\,\xi_k(.) = (\xi_j\otimes\xi_k)(.\otimes .) \text{ for some } \xi_j\otimes\xi_k \in L(B\otimes B,\mathbb{R})$$

and $|\xi_j\otimes\xi_k| \le 1$ since $|\xi_j|\le 1, |\xi_j|\le 1$. So we have the estimate

$$\sum_j \left|h_{t_j,t_{j+1}}\right| \le |h|_H^2 \sqrt{\left(\sum_{j,k}\left|\mathbb{E}\left(X_{t_j,t_{j+1}}\otimes X_{t_k,t_{k+1}}\right)\right|_{B\otimes B}\right)}.$$

Again for $\varphi\in L(B\otimes B,\mathbb{R})$, $\varphi\left(\mathbb{E}\left(X_{t_j,t_{j+1}}\otimes X_{t_k,t_{k+1}}\right)\right) = \mathbb{E}\left(\tilde{\varphi}\left(X_{t_j,t_{j+1}}, X_{t_k,t_{k+1}}\right)\right)$ for some $\tilde{\varphi}\in BL(B\times B,\mathbb{R})$ by linearity of φ. Writing $\tilde{\varphi}(x,y) = \sum_i f_i(x)\,g_i(y)$, $f_i, g_i \in L(B,\mathbb{R})$, and using the independence of increments in combination with Gaussianity and $|.|_{B\otimes B} = \sup\{\varphi(.), \varphi\in L(B\otimes B,\mathbb{R}), |\varphi|\le 1\}$ gives $\sum_{j\ne k}\left|\mathbb{E}\left(X_{t_j,t_{j+1}}\otimes X_{t_k,t_{k+1}}\right)\right|_{B\otimes B} = 0$.

By the compatibility of the tensor norm (and again using the characteristic property of tensor products)

$$\left|\mathbb{E}\left(X_{t_j,t_{j+1}}\otimes X_{t_j,t_{j+1}}\right)\right|_{B\otimes B} \le \mathbb{E}\left(\left|X_{t_j,t_{j+1}}\right|_B^2\right).$$

Now $X_{t_j,t_{j+1}}$ has the same distribution as $(t_{j+1}-t_j)\,X_1$ and since X_1 has distribution μ,

$$\mathbb{E}\left(\left|X_{t_j,t_{j+1}}\right|_B^2\right) = (t_{j+1}-t_j)\int_B |x|_B^2\,\mu(dx).$$

This gives $\sum_j \left|h_{t_j,t_{j+1}}\right| \le |h|_H\,\sigma_B\sqrt{t-s}$. □

Proof of Proposition 10. Since $|||\mathbf{X}|||_{p\text{-var};[0,T]} < \infty$ a.s. is shown in [9] we only have to check (4). We recall the scaling $|||\mathbf{X}|||_{p\text{-var};[0,T]} \sim \sqrt{T}|||\mathbf{X}|||_{p\text{-var};[0,1]}$ and that by construction of the Lyons-Ledoux-Qian rough path lift (as an a.s. limit) and continuity properties of Riemann integrals the set

$$\left\{x : \mathbf{X}(x+h) = (x+h) + \left(\mathbf{X}^2(x) + \int h \otimes dx + \int x \otimes dh + \int h \otimes dh\right) \quad \text{for all } h \in H\right\}$$

has full measure. The proof now works as in the finite dimensional case. □

Remark 12. The Gauss tail of $|||\mathbf{X}|||_{p\text{-var};[0,T]}$ was obtained in [7] by a careful tracking of the original estimates in [9] though no explicit constant η was given.

2.4. **Integrability of higher iterated integrals.** The above applications imply Gaussian integrability of norms of twice iterated integrals. To obtain integrability properties of N-times iterated integrals, $N \geq 3$, we recall a basic theorem of rough path theory (see [11]): a continuous $G^{[p]}\left(\mathbb{R}^d\right)$-valued path $\mathbf{x}$ of finite p-variation lifts for every $N \geq [p]$ uniquely to a $G^N\left(\mathbb{R}^d\right)$-valued path, say $S_N(\mathbf{x})$, of finite p-variation and there exists a constant $C(N,p)$ such that

$$\|S_N(\mathbf{x})\|_{p\text{-var}} \leq C(N,p)\|\mathbf{x}\|_{p\text{-var}}.$$

Applied to, say, d-dimensional Brownian motion, this yields in combination with Proposition 3 that Brownian motion and all its iterated Stratonovich integrals up to order N, written as $S_N(\mathbf{B})$ and viewed as a diffusion in the step-N nilpotent group with d generators, have Gaussian integrability in the sense that $\|S_N(\mathbf{B})\|_{p\text{-var}}$ has a Gauss tail.

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