

## ON AN EXAMPLE OF $\delta$ -KOSZUL ALGEBRAS

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ABSTRACT. The main purpose of this paper is to study a concrete example of  $\delta$ -Koszul algebras, which is related to three questions raised by Green and Marcos.

### 1. INTRODUCTION

It is well known that whether the Yoneda algebra of a graded algebra is finitely generated or not is too complicated to be answered. As an attempt to discuss this thesis, Green and Marcos introduced the notion of  $\delta$ -Koszul algebras in [3] in 2005. In particular, they ended the paper with three questions:

- For which functions  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  is there a  $\delta$ -resolution-determined algebra?
- For which functions  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  is there a  $\delta$ -Koszul algebra?
- Is there a bound  $N_0 \in \mathbb{N}$ , such that if  $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$  is a  $\delta$ -Koszul algebra, then the Yoneda algebra  $E(A) = \bigoplus_{n \geq 0} \text{Ext}_A^n(A_0, A_0)$  is generated by  $\text{Ext}_A^0(A_0, A_0)$ ,  $\text{Ext}_A^1(A_0, A_0)$ ,  $\cdots$ ,  $\text{Ext}_A^{N_0}(A_0, A_0)$ ?

In this paper, we give a sufficient condition for the resolution map  $\delta$  such that there do exist  $\delta$ -resolution determined algebras and  $\delta$ -Koszul algebras. Furthermore, we give an explicit procedure to construct concrete examples of  $\delta$ -resolution determined algebras and  $\delta$ -Koszul algebras satisfying this condition. It should be noted that such examples are a special class of *almost Koszul algebras* introduced by Brenner, Butler and King with the aim of finding periodic resolutions for the trivial extension algebras of path algebras of Dynkin quivers in bipartite orientation (see [1] and [2] for the further details) and giving an answer to the third question introduced above.

Now let us introduce some notations and recall some definitions.

Throughout the paper,  $\mathbb{k}$  denotes a fixed field and  $\mathbb{N}$  denotes the set of natural numbers. All the positively graded  $\mathbb{k}$ -algebra  $A = \bigoplus_{i \geq 0} A_i$  are assumed with the following conditions hold:

- $A_0 = \mathbb{k} \times \cdots \times \mathbb{k}$ , a finite product of  $\mathbb{k}$ ;
- $A_i \cdot A_j = A_{i+j}$  for all  $0 \leq i, j < \infty$ ;
- $\dim_{\mathbb{k}} A_i < \infty$  for all  $i \geq 0$ .

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**Definition 1.1** ([3]). Let  $A$  be a positively graded algebra.  $A$  is called  $\delta$ -Koszul provided the following two conditions hold:

- (1) The trivial  $A$ -module  $A_0$  admits a minimal graded projective resolution

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A_0 \rightarrow 0,$$

such that each  $P_n$  is generated in a single degree, say  $\delta(n)$  for all  $n \geq 0$ , where  $\delta$  is a strictly increasing set function;

- (2) The Yoneda-Ext algebra,  $E(A) = \bigoplus_{n \geq 0} \text{Ext}_A^n(A_0, A_0)$ , is finitely generated as a graded algebra.

If  $A$  only satisfies condition (1), we call  $A$  a  $\delta$ -resolution determined algebra.

## 2. MAIN RESULTS

We begin with

**Definition 2.1.** A set map  $f : \mathbb{N} \rightarrow \mathbb{N}$  is called *good* if and only if there exists  $N_0 \in \mathbb{N} - \{0\}$ , such that

- (1)  $f(i) = i$  for all  $0 \leq i < N_0$ ;  
 (2)  $f(i) = f(i - N_0) + f(N_0)$  for all  $i \geq N_0$ . In particular, if  $N_0 \geq 3$ , then  $f(N_0) = N_0 + 1$ .

**Lemma 2.2.** Let  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  be a good set map. Then  $A$  is a  $\delta$ -resolution determined algebra if and only if  $A$  is a  $\delta$ -Koszul algebra.

*Proof.* It is immediate from Theorem 3.6 in [3] and Definition 1.1.  $\square$

**Lemma 2.3.** Let  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  be a good set map. Then there exist  $\delta$ -resolution determined algebras.

*Proof.* By hypothesis,  $\delta$  satisfies  $\delta(i) = \delta(i - N_0) + \delta(N_0)$  for all  $i \geq N_0$  and  $\delta(i) = i$  for  $i = 0, 1, \dots, N_0 - 1$ , where  $N_0 \in \mathbb{N} - \{0\}$ . We divide the proof into three cases.

(i) If  $N_0 = 1$ , Koszul algebras are the desired  $\delta$ -resolution determined algebras with  $\delta(i) = i$  for all  $i \geq 0$  and there are many Koszul algebras.

(ii) If  $N_0 = 2$ ,  $d$ -Koszul algebras are the desired  $\delta$ -resolution determined algebras, where the set function  $\delta$  is defined as

$$\delta(i) = \begin{cases} \frac{id}{2}, & i \equiv 0 \pmod{2}, \\ \frac{(i-1)d}{2} + 1, & i \equiv 1 \pmod{2}. \end{cases}$$

(iii) If  $N_0 \geq 3$ , let  $\Gamma$  be the quiver:

$$\bullet^1 \xrightleftharpoons[\beta_1]{\alpha_1} \bullet^2 \xrightleftharpoons[\beta_2]{\alpha_2} \bullet^3 \xrightleftharpoons[\beta_3]{\alpha_3} \dots \xrightleftharpoons[\beta_{N_0-1}]{\alpha_{N_0-1}} \bullet^{N_0}.$$

Now let

$$A = \frac{\mathbb{k}\Gamma}{\langle \alpha_i \beta_i - \beta_{i+1} \alpha_{i+1}, \alpha_{i+1} \alpha_i, \beta_i \beta_{i+1} : i = 1, 2, \dots, N_0 - 2 \rangle}.$$

Now we will compute the minimal graded projective resolution of the trivial  $A$ -module  $A_0$  as follows.

Let  $P_i$  denote the simple  $A$ -module related to the vertex  $i$ .

If  $N_0 = 3$ , then  $\mathbb{k}^{\otimes 3}$  has the following minimal graded projective resolution:

$$\begin{aligned} \cdots \rightarrow (A \oplus P_2)[6] \rightarrow (A \oplus P_2)[5] \rightarrow A[4] \rightarrow (A \oplus P_2)[2] \\ \rightarrow (A \oplus P_2)[1] \rightarrow A \rightarrow \mathbb{k}^{\otimes 3} \rightarrow 0. \end{aligned}$$

If  $N_0 = 4$ , then  $\mathbb{k}^{\otimes 4}$  has the following minimal graded projective resolution:

$$\begin{aligned} \cdots \rightarrow (A \oplus P_2 \oplus P_3)[7] &\rightarrow (A \oplus P_2 \oplus P_3)[6] \rightarrow A[5] \rightarrow (A \oplus P_2 \oplus P_3)[3] \\ &\rightarrow (A \oplus P_2 \oplus P_3)[2] \rightarrow (A \oplus P_2 \oplus P_3)[1] \rightarrow A \rightarrow \mathbb{k}^{\otimes 4} \rightarrow 0. \end{aligned}$$

By an induction, we get that the minimal graded projective resolution of the trivial  $A$ -module  $\mathbb{k}^{\otimes N_0}$  has the following general form:

$$\begin{aligned} \cdots \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[N_0 + 2] &\rightarrow A[N_0 + 1] \rightarrow A[N_0 - 1] \rightarrow \cdots \\ \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0-1}{2}} \oplus P_{\frac{N_0+1}{2}} \oplus P_{\frac{N_0+3}{2}})[\frac{N_0+1}{2}] \\ \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0+1}{2}})[\frac{N_0-1}{2}] &\rightarrow \cdots \\ \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}))[2] &\rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[1] \\ \rightarrow A \rightarrow \mathbb{k}^{\otimes N_0} &\rightarrow 0 \end{aligned}$$

for  $N_0$  being odd;

$$\begin{aligned} \cdots \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[N_0 + 2] &\rightarrow A[N_0 + 1] \rightarrow A[N_0 - 1] \rightarrow \cdots \\ \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0}{2}} \oplus P_{\frac{N_0+1}{2}})[\frac{N_0}{2}] \\ \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0}{2}} \oplus P_{\frac{N_0+1}{2}})[\frac{N_0}{2} - 1] &\rightarrow \cdots \\ \rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}))[2] &\rightarrow (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[1] \\ \rightarrow A \rightarrow \mathbb{k}^{\otimes N_0} &\rightarrow 0 \end{aligned}$$

for  $N_0$  being even.

It is obvious that most terms of the above resolutions are made of many brackets. In order to avoid some misunderstandings, we stipulate the following: Given a concrete  $N \in \mathbb{N}$ , whether the bracket appears or not is completely determined by the subscripts of the first object and the last object in the bracket. If the subscript of the first object is smaller than that of the last object, then a bracket appears. Otherwise, the bracket does not appear.

Now it is easy to see that the algebra constructed above is the desired  $\delta$ -resolution determined algebra, where  $\delta$  is defined as follows:

$$\delta(i) = \begin{cases} \frac{i(N_0+1)}{N_0}, & i \equiv 0(\text{mod}N_0), \\ \frac{(i-1)(N_0+1)}{N_0} + 1, & i \equiv 1(\text{mod}N_0), \\ \dots & \dots \\ \frac{(i-N_0+1)(N_0+1)}{N_0} + N_0 - 1, & i \equiv N_0 - 1(\text{mod}N_0). \end{cases}$$

Therefore, Lemma 2.3 is proved. □

Now we will point out that the algebra constructed in the proof of Lemma 2.3 (iii) gives an answer to the third question.

**Lemma 2.4.** *Let  $A$  be the algebra constructed in the proof of Lemma 2.3 (iii) and let  $E(A) = \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$  be the Yoneda algebra of  $A$ . Then  $E(A)$  is minimally generated by  $\text{Ext}_A^0(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ ,  $\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$  and  $\text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ .*

*Proof.* We first prove that  $E(A)$  can be generated in degrees 0, 1, and  $N_0$ . By hypothesis, the resolution map  $\delta$  of  $A$  is defined as

$$\delta(i) = \begin{cases} \frac{i(N_0+1)}{N_0}, & i \equiv 0(\text{mod } N_0), \\ \frac{(i-1)(N_0+1)}{N_0} + 1, & i \equiv 1(\text{mod } N_0), \\ \dots & \dots \\ \frac{(i-N_0+1)(N_0+1)}{N_0} + N_0 - 1, & i \equiv N_0 - 1(\text{mod } N_0). \end{cases}$$

It is easy to see that  $\delta(i) = \delta(i - N_0) + \delta(N_0)$  for all  $i \geq N_0$  and  $\delta(i) = i$  for all  $0 \leq i \leq N_0 - 1$ . By Proposition 3.6 in [4] we have

$$\text{Ext}_A^i(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}) = (\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}))^i$$

for  $0 \leq i \leq N_0 - 1$  and

$$\text{Ext}_A^i(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}) = (\text{Ext}_A^i(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}))^k \cdot \text{Ext}_A^j(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$$

for  $i = kN_0 + j$ , where  $0 \leq j \leq N_0 - 1$  and  $k \in \mathbb{N} - \{0\}$ . Thus,  $E(A)$  can be generated by  $\text{Ext}_A^0(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ ,  $\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$  and  $\text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ .

Now we claim that  $\text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$  cannot be generated in lower degrees; i.e.,  $E(A)$  is minimally generated by  $\text{Ext}_A^0(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ ,  $\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ , and  $\text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ . In fact, it suffices to prove that  $\text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$  cannot be generated by  $\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})$ . Note that

$$(\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}))^{N_0} = (\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})_{-1})^{N_0} \subseteq \text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})_{-N_0}.$$

But recall that  $\delta(N_0) = N_0 + 1$ , which implies that  $\text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}) = \text{Ext}_A^{N_0}(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0})_{-N_0-1}$ . Thus,  $(\text{Ext}_A^1(\mathbb{k}^{\otimes N_0}, \mathbb{k}^{\otimes N_0}))^{N_0} = 0$ .

Therefore, Lemma 2.4 is proved. □

**Corollary 2.5.** *There does not exist a uniform bound of the generation degree for the Yoneda algebras of  $\delta$ -Koszul algebras.*

*Proof.* Suppose that we have a uniform bound of the generation degree for the Yoneda algebras of  $\delta$ -Koszul algebras, say,  $N \in \mathbb{N}$ . Now let  $N_0 = N + 1$  in the algebra constructed in the proof of Lemma 2.3 (iii). Then by Lemma 2.4, we have that  $E(A)$  is minimally generated by  $\text{Ext}_A^0(\mathbb{k}^{\otimes N+1}, \mathbb{k}^{\otimes N+1})$ ,  $\text{Ext}_A^1(\mathbb{k}^{\otimes N+1}, \mathbb{k}^{\otimes N+1})$  and  $\text{Ext}_A^{N+1}(\mathbb{k}^{\otimes N+1}, \mathbb{k}^{\otimes N+1})$ , which is a contradiction. □

Now putting Lemmas 2.2, 2.3, 2.4 and Corollary 2.5 together, we have the following result, which is the main result of this paper.

**Theorem 2.6.** *We have the following statements:*

- (1) *Let  $\delta : \mathbb{N} \rightarrow \mathbb{N}$  be a good set map. Then*
  - (a) *there exists a  $\delta$ -resolution determined algebra,*
  - (b) *there exists a  $\delta$ -Koszul algebra.*
- (2) *There does not exist any bound  $N \in \mathbb{N}$ , such that the Yoneda algebras of all the  $\delta$ -Koszul algebras can be generated in degrees of 0, 1,  $\dots$ , and  $N$ .*

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