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ŁOJASIEWICZ INEQUALITY FOR WEIGHTED HOMOGENEOUS POLYNOMIAL WITH ISOLATED SINGULARITY

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Professor Charles Fefferman on the occasion of his 60th birthday

ABSTRACT. Let ∇f be a gradient vector field of a weighted homogenous polynomial with isolated critical point at the origin. Let $(w_1, \ldots, w_n)$ be the weights of f. In this paper, we prove that the Łojasiewicz Exponent θ of f is precisely equal to $\max\limits_{0\le i\le n} w_i - 1$. This means that for some constant c, $|\nabla f(z)| \ge c|z|^\theta$ in a neighborhood of 0, which provides the optimal lower estimate of $|\nabla f(z)|$.

1. INTRODUCTION

In geometric analysis, it is important to get the best lower bound of the gradient of a function f in a certain norm. The famous Poincaré inequality states that for any compact C^∞ Riemannian manifold M of dimension n and for any C^1 function defined on M,

$$C_1 \cdot \inf_{a\in\mathbb{R}} \|f-a\|_2^2 \le \|\nabla f\|_2^2, \tag{1.1}$$

where $\|\cdot\|_2$ denotes the L^2-norm and C_1 is a real constant.

Another important analytic inequality is the Sobolev inequality

$$C_2 \cdot \inf_{a\in\mathbb{R}} \|f-a\|_{n/(n-1)}^n \le \|\nabla f\|_1^n \tag{1.2}$$

for all functions in $H_{1,1}$, the Sobolev space of functions which has L^1 derivatives. The best constant C_2 such that (1.2) holds is known as the Sobolev constant.

Let f be a real analytic function on $\mathbb{R}^n$ and let $V = \{x \in \mathbb{R}^n : f(x) = 0\}$. The celebrated Łojasiewicz inequality states that for any compact set K there are a positive constant C and α such that

$$\operatorname{dist}(x, V)^\alpha \le C\cdot \mid f(x) \mid \quad \text{for every } x \in K, \tag{1.3}$$

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where $\operatorname{dist}(x,V) := \inf_{z\in V}\|x-z\|$, and $\|\cdot\|$ denotes the Eulidean norm. As we shall see below, the Łojasiewicz inequality will give a lower estimate of $|\nabla f(z)|$ in the pointwise sense for all real analytic functions.

If $V \subseteq \mathbb{C}^n$ is defined by complex analytic equations $f_1 = \cdots = f_k = 0$, then $V = \{z \in \mathbb{C}^n : f(z) := |f_1(z)|^2 + \cdots + |f_k(z)|^2 = 0\}$. Thus the Łojasiewicz inequality applies to complex analytic or algebraic sets.

In [Br], Brownawell proved a deep theorem which says that one can estimate the exponent α in terms of the degree of f_i if the f_i's are polynomials.

Theorem 1.1 (Brownawell). *Let $f_1, \cdots, f_k \in \mathbb{C}[z_1, \cdots, z_n]$ and $D = \max \deg f_i$. Let $V = \{z \in \mathbb{C}^n : f_1(z) = \cdots = f_k(z) = 0\}$. Then there is a constant $C > 0$ such that*

$$\left(\frac{\min[\operatorname{dist}(z,V),1]}{1+\|z\|^2}\right)^{(n+1)^2 D^{\min(n,k)}} \le C \cdot \max_{1\le i\le k} |\ f_i(z)\ |,$$

where $\|z\|^2 = |z_1| + \cdots + |z_n|^2$.

In their remarkable paper [J-K-S], Ji, Kollár and Shiffman found the best possible exponent in terms of the degree of f_i. Given natural numbers $n \ge 2$ and $d_1 \ge \cdots \ge d_k$, let

$$B(n, d_1, \cdots, d_k) = \begin{cases} d_1 \cdots d_k & \text{if } k \le n, \\ d_1 \cdots d_{n-1} d_k & \text{if } k > n, \end{cases}$$

$$\bar{B}(n, d_1, \cdots, d_k) = (\frac{3}{2})^j B(n, d_1, \cdots, d_k) + \theta,$$

where $j = \sharp\{i < \min(k,n) - 1 : d_i = 2\}$ and

$$\theta = \begin{cases} 1 & \text{if } k > n \text{ and } d_{n-1} = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2 (Ji-Kollár-Shiffman). *Let $f_1, \cdots, f_k \in \mathbb{C}[x_1, \cdots, x_n]$ be polynomials and let $d_i = \deg(f_i)$. Assume that $n \ge 2$. Let $V = V(f_1, \cdots, f_k) \subseteq \mathbb{C}^n$ be the common zero set of these polynomials. Assume that V is nonempty. Then there is a positive integer $m \le \bar{B}(n, d_1, \cdots, d_k)$ and a constant $C > 0$ (both depending on the f_i) such that*

$$\operatorname{dist}(x, V)^m \le C \cdot \max_i\{|\ f_i(x)\ |\} \cdot (1+ |\ x\ |)^{\bar{B}(n,d_1,\cdots,d_k)}$$

holds for all $x \in \mathbb{C}^n$.

In this paper, we are interested in the special case that $k = n$ and $(f_1, \cdots, f_n)$ is a gradient vector field.

Let $f = f(z_1, \cdots, z_n) \in \mathbb{C}\{z_1, \cdots, z_n\}$ be a convergent power series defining an isolated singularity at the origin 0 in $\mathbb{C}^n$, i.e. $f(0) = 0$, and the gradient of f,

$$\triangledown f := \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0),$$

has an isolated zero at 0 in $\mathbb{C}^n$. The Łojasiewicz Exponent $\mathcal{L}_0(f)$ of f is by definition the smallest $\theta > 0$ such that

$$|\triangledown f(z)| \ge c|z|^\theta \text{ for all } z \in U,$$

where U is a neighborhood of 0 in $\mathbb{C}^n$ and C is a positive constant. The computation or estimation of the Łojasiewicz Exponent is a quite interesting problem not only in geometric analysis but also in singularity theory. For example, B. Teissier proved that $\mathcal{L}_0(f)$ is equal to the maximal polar invariant $\eta_0(f)$ of the singularity of f ([T], Cor. 2). Let $\mathrm{Suff}_0(f)$ be the C^0-degree of sufficiency of f. $\mathrm{Suff}_0(f)$ is the smallest integer r such that f is topologically equivalent to $f+g$ for all g with $\mathrm{ord}(g) \geq r+1$; i.e., there exists a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = (f+g) \circ \varphi$. Then it is well known that $\mathrm{Suff}_0(f) = [\mathcal{L}_0(f)] + 1$ ([T], Theorem 8), where $[a]$ denotes the integral part of $a \in \mathbb{R}$. It is well known (see [T]) that the Łojasiewicz Exponent can be calculated by means of analytic paths $\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t)) \in \mathbb{C}\{t\}^n$, $\varphi(0) = 0$, $\varphi(t) \neq 0$ in $\mathbb{C}\{t\}^n$. If $\mathrm{ord}(\varphi) := \inf_{i=1}^n \mathrm{ord}(\varphi_i)$, then

$$\mathcal{L}_0(f) = \sup_{\varphi} \frac{\mathrm{ord}((\nabla f) o \varphi)}{\mathrm{ord}(\varphi)}$$

(by the Curve Selection Lemma; see also [L-T]). In the two-dimensional case there are many explicit formulas for $\mathcal{L}_0(f)$ in various forms (see [KL], [CK1], [CK2], [L], [K-O-P]). In this paper we investigate the problem of determining the Łojasiewicz Exponent for general weighted homogeneous isolated singularities. Let us recall that if $(w_1, \cdots, w_n)$ is a sequence of n rational numbers such that $w_i \geq 2$ for $i = 1, \cdots, n$, then a polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ is called *weighted homogeneous* of type $(w_1, \cdots, w_n)$ if f can be written as linear combinations of monomials $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with

$$\frac{\alpha_1}{w_1} + \cdots + \frac{\alpha_n}{w_n} = 1.$$

For an isolated weighted homogeneous surface singularity, Krasinski, Oleksik and Ploski [K-O-P] gave a formula for the Łojasiewicz Exponent of weighted homogeneous isolated sigularites in terms of the weights. More precisely, the Łojasiewicz Exponent is equal to the maximum of the weights minus one. Unfortunately, it is very difficult if not impossible to generalize the method used in [K-O-P] for weighted homogeneous polynomials in n variables for $n \geq 4$.

Estimates of the Łojasiewicz Exponent for quasi-homogeneous isolated singularities in the real and complex cases are in a recent preprint by Haraux and Pham [HP]. Estimations in the general case can be found in [Lt], [F], [P1], [A].

The main result of this paper is the following.

Theorem 1.3. *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial of type* $(w_0, \cdots, w_n)$ *defining an isolated singularity at the origin* $0 \in \mathbb{C}^n$*. Then*

$$\mathcal{L}_0(f) = \max_{0 \leq i \leq n} w_i - 1.$$

In particular, if f is a homogeneous polynomial of degree $d > 1$ with an isolated singularity at the origin, then $\mathcal{L}_0(f) = d - 1$.

Thus, under the hypothesis of Theorem 1.3, our result is better than Ji-Kollár-Shiffman in Theorem 1.2. This provides an optimal lower estimate of $|\nabla f(z)|$.

Corollary 1.1. *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with weights* $(w_0, \cdots, w_n)$ *defining an isolated singularity at the origin* $0 \in \mathbb{C}^n$*. Then* $\mathrm{Suff}_0(f) = [\max_{1 \leq i \leq n} w_i]$.

Let $\deg(f)$ be the degree of the polynomial f. Then we have a lower estimate of $\mathrm{suff}_o(f)$ in terms of $\deg(f)$.

Corollary 1.2. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with weights $(w_0, \cdots, w_n)$ defining an isolated singularity at the origin $0 \in \mathbb{C}^n$. Then* $\deg f \leq \mathrm{Suff}_0(f)$.

The above inequality may be strict.

Actually, we can prove the following more general theorem using the same methods.

Theorem 1.4. *Let $f = (f_0, \cdots, f_n) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ be a weighted homogeneous map, i.e. $f_i, 0 \leq i \leq n$ are weighted homogeneous polynomials with the same weight type $(w_0, \cdots, w_n)$, which has an isolated common zero at $0 \in \mathbb{C}^{n+1}$. Then*

$$l_0(f) = \max_{0 \leq i \leq n} w_i - 1.$$

2. Preliminaries

Definition 2.1. Let $f = (f_1, \cdots, f_p) \in \mathbb{C}\{z_1, \cdots, z_n\}^p$ define a germ of the holomorphic mapping $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with an isolated zero at $0 \in \mathbb{C}^n$. The Lojasiewicz exponent $l_0(f)$ of f is by definition the smallest $\theta > 0$ such that there exists a neighborhood U of $0 \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|f(z)| \geq c|z|^\theta, \text{ for all } z \in U.$$

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with an isolated singularity at the origin $0 \in \mathbb{C}^{n+1}$. Clearly it follows from the above definition that $\mathscr{L}_0(f) = l_0(\nabla f)$.

Definition 2.2. Let f be a nonzero holomorphic function defined in an open neighborhood of the origin $0 \in \mathbb{C}^n$. We denote by $\mathrm{ord}(f)$ the multiplicity of f at the origin; i.e., if $f = \sum_{i \geq m} f_i$, $f_m \neq 0$, is the expansion of f in a series of homogeneous polynomials, then $\mathrm{ord}(f) = m$. By definition, we put $\mathrm{ord}(0) = +\infty$. For any holomorphic mapping $f = (f_1, \cdots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, we define $\mathrm{ord}(f) = \min\limits_{1 \leq i \leq p}(\mathrm{ord}(f_i))$.

Lemma 2.1 (see [L-T]). *Let $\phi(t) = (\phi_0(t), \cdots, \phi_n(t)) \in \mathbb{C}\{t\}^{n+1}$, $\phi(0) = 0$, and $\phi(t) \neq 0$ in $\mathbb{C}\{t\}^{n+1}$. If $\mathrm{ord}(\phi) := \inf\limits_{0 \leq i \leq n}(\mathrm{ord}\phi_i)$, then*

$$\mathscr{L}_0(f) = \sup_\phi \frac{\mathrm{ord}((\nabla f) \circ \phi)}{\mathrm{ord}\phi}.$$

Lemma 2.2. *Let f_i, $0 \leq i \leq n$, be a polynomial whose support* $\mathrm{supp}(f_i)$ *lies in the hyperplane $q_0\alpha_0 + q_1\alpha_1 + \cdots + q_n\alpha_n = d_i$, where $q_0, \cdots, q_n, d_i > 0$ are integers. Suppose that $f = (f_0, \cdots, f_n)$ has an isolated zero at $0 \in \mathbb{C}^{n+1}$. Then*

$$l_0(f) \leq \frac{\max\limits_{0 \leq i \leq n}(d_i)}{\min\limits_{0 \leq i \leq n}(q_i)}.$$

Proof. See [P2], Proposition 2.2. □

Lemma 2.3 ([S]). *Suppose that $f \in \mathbb{C}[[z_1, z_2, \cdots, z_n]]$ has an isolated singularity at the origin and that no monomial of the form $z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k}$ occurs in f. Then for at least k monomials of the form $x_j x_1^{m_{j,1}} \cdots x_k^{m_{j,k}}$, $k+1 \leq j \leq n$ occur in f.*

Proof. See [S], Lemma 1.5. □

Definition 2.3. Suppose that Y is a topological space and that $\varphi : Y \to \mathbb{Z}$ is an integer-valued function. We say that φ is upper semicontinuous if for every $y \in Y$ there exists a neighborhood U of y such that $\varphi(y') \leq \varphi(y)$ for all y' in U.

Proposition 2.1. *A function $\varphi : Y \to \mathbb{Z}$ is upper semicontinuous if and only if for each $n \in \mathbb{Z}$, the set $\{y \in Y \mid \varphi(y) \geq n\}$ is a closed subset of Y.*

Proof. See [Ha], Remark 12.7.1. □

Notation. Let $\mathbb{H}(w)$ denote the $\mathbb{C}$-vector space of weighted homogeneous polynomials with the same weight $w = (w_1, w_2, \cdots, w_n)$.

Proposition 2.2 (see [Di], Prop. 7.15)**.** *The set $A \subset \mathbb{H}(w)$ of polynomial mapping germs which do not have an isolated hypersurface singularity at the origin is a closed algebraic subset.*

Proof. Let $\mathbb{P}(w)$ be the weighted projective space $\mathbb{P}(w) = (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*$, where the quotient is taken with respect to the $\mathbb{C}^*$ action defined below:

$$t \cdot z = t \cdot (z_0, z_1, \cdots, z_n) = (t^{w_0} z_0, t^{w_1} z_1, \cdots, t^{w_n} z_n).$$

It is known that $\mathbb{P}(w)$ is a complete algebraic variety, in general with singularities; see for instance [Do]. The set

$$B = \{(x, f) \in \mathbb{P}(w) \times \mathbb{H}(w) : f(x) = \frac{\partial f}{\partial x_0}(x) = \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\}$$

is clearly a closed algebraic set. From the completeness of the weighted projective space, it follows that $q(B)$ is a closed algebraic set in $\mathbb{H}(w)$, where $q : \mathbb{P}(w) \times \mathbb{H}(w) \to \mathbb{H}(w)$ denotes the projection. It is easy to see that $q(B) = A$. □

Let $\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t)) \in \mathbb{C}\{t\}^n$, $\varphi(0) = 0$, $\varphi(t) \neq 0$ in $\mathbb{C}\{t\}^n$. By Definition 2.2, we have a function $\operatorname{ord}(\bigtriangledown(\cdot) \circ \varphi) : \mathbb{H}(w) - A \to \mathbb{Z}$ defined by $\operatorname{ord}(\bigtriangledown(\cdot) \circ \varphi)(f) = \operatorname{ord}(\bigtriangledown(f) \circ \varphi)$, where $f \in H(w) - A$.

Proposition 2.3. *The function $\operatorname{ord}(\bigtriangledown(\cdot) \circ \varphi) : \mathbb{H}(w) - A \to \mathbb{Z}$ defined by $\operatorname{ord}(\bigtriangledown(\cdot) \circ \varphi)(f) = \operatorname{ord}(\bigtriangledown(f) \circ \varphi)$, where $f \in \mathbb{H}(w) - A$, is upper semicontinuous.*

Proof. It follows from Proposition 2.1 that we only need to show that $\forall N \in \mathbb{Z}$, $\{f \in \mathbb{H}(w) - A \mid \operatorname{ord}(\bigtriangledown(f) \circ \varphi) \geq N\}$ is a closed subset of $\mathbb{H}(w) - A$. It is trivially true if $N \leq 0$. So we only need to consider $N \geq 1$. Since $\mathbb{H}(w) - A$ is an open dense set in $\mathbb{H}(w)$, we can choose a basis $\{g_1, g_2, \cdots, g_m\}$ of $\mathbb{H}(w)$, such that $\{g_1, g_2, \cdots, g_m\} \subseteq \mathbb{H}(w) - A$, where m is the dimension of $\mathbb{H}(w)$. Then there exist $t_1, t_2, \cdots, t_m \in \mathbb{C}$ such that $f = \sum_{i=1}^{m} t_i g_i$ and $\bigtriangledown(f) \circ \varphi = (\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_0} \circ \varphi, \sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_1} \circ \varphi, \cdots, \sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_n} \circ \varphi)$. By definition, $\operatorname{ord}(\bigtriangledown(f) \circ \varphi) = \min_{0 \leq j \leq n} \operatorname{ord}(\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi)$. Thus

$$\{f \in \mathbb{H}(w) - A \mid \operatorname{ord}(\bigtriangledown(f) \circ \varphi) \geq N\} = \{f \in \mathbb{H}(w) - A \mid \min_{0 \leq j \leq n} \operatorname{ord}(\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi) \geq N\}.$$

Suppose $\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi \neq 0$. Since $\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi \in \mathbb{C}[t_1, t_2, \cdots, t_m]\{t\}$, so it can be expanded as a series

$$\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi = \sum_{k=1}^{\infty} a_{j,d}[t_1, t_2, \cdots, t_m] t^d,$$

where $a_{j,d}[t_1, t_2, \cdots, t_m]$ is a polynomial depending on the variables $t_1, t_2, \cdots, t_m$. It follows that $\mathrm{ord}(\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi) \geq N$ is equivalent to $a_{j,d}(t_1, t_2, \cdots, t_m) = 0$ for all $d \leq N-1$. If $\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi = 0$, then $\mathrm{ord}(\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi) = \mathrm{ord}(0) = \infty \geq N$. So $\{f \in \mathbb{H}(w) - A \mid \min_{j=0}^{n} \mathrm{ord}(\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi) \geq N\}$ is equivalent to $a_{j,d}(t_1, t_2, \cdots, t_m) = 0$ for all $d \leq N-1$ and $0 \leq j \leq n$. Therefore, $\forall N \in \mathbb{Z}$, $\{f \in \mathbb{H}(w) - A \mid \min_{0 \leq j \leq n} \mathrm{ord}(\sum_{i=1}^{m} t_i \frac{\partial g_i}{\partial z_j} \circ \varphi) \geq N\}$ is a closed subset of $\mathbb{H}(w) - A$. □

Given natural numbers n and $d_1 \geq \cdots \geq d_k$, let

$$N(d_1, \cdots, d_k, n) = \begin{cases} \prod_{i=1}^{k} d_i & \text{if } n \geq k \geq 1, \\ (\prod_{i=1}^{n-1} d_i) d_k & \text{if } k > n > 1, \\ d_k & \text{if } n = 1. \end{cases}$$

Theorem 2.5 ([Je])**.** *Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^m$ be an affine n-dimensional variety of degree D. Let $f_1, \cdots, f_k \in \mathbb{K}[X]$ be nonzero polynomials. Assume that $\deg(f_i) = d_i$, where $d_1 \geq \cdots \geq d_k$. If $I = (f_1, \cdots, f_k)$, then*

$$\sqrt{I}^{DN(d_1, \cdots, d_k, n)} \subset I.$$

Proof. See Theorem 1.3 in [Je]. □

3. Proof of main theorem

Let $f \in \mathbb{C}\{z_0, z_1, \cdots, z_n\}$ be a complex analytic function with an isolated singularity at the origin, and let $l = \sum_{i=0}^{n} a_i z_i$ be a nonzero linear form. A (local) *polar curve* of f relative to l is the germ $\Gamma_l(f)$ of the analytic set defined by the equations

$$\frac{\partial(f, l)}{\partial(z_i, z_j)} := \begin{vmatrix} \frac{\partial f}{\partial z_i} & \frac{\partial f}{\partial z_j} \\ \frac{\partial l}{\partial z_i} & \frac{\partial l}{\partial z_j} \end{vmatrix} = 0, 1 \leq i < j \leq n$$

near the origin.

It is easy to check that $\dim \Gamma_l(f) = 1$. In particular $\Gamma_{z_k}(f)$ is defined by the equations

$$\frac{\partial f}{\partial z_0} = \cdots = \frac{\partial f}{\partial z_{k-1}} = \frac{\partial f}{\partial z_{k+1}} = \cdots = \frac{\partial f}{\partial z_n} = 0.$$

By Definition 2.1, we have a function $\mathscr{L}_0 : H(w) - A \to \mathbb{R}$.

Proposition 3.4. *Let $\mathbb{H}(w)$ denote the $\mathbb{C}$-vector space of weighted homogeneous polynomial mapping germs with weight w and let A be defined as above in Proposition* 2.2. *Consider the function $\mathscr{L}_0 : \mathbb{H}(w) - A \to \mathbb{R}$. If $f_{t_0} \in \mathbb{H}(w) - A$, then there exists an open neighborhood $U \subset \mathbb{H}(w) - A$ of f_{t_0} such that for any $f_t \in U \subset \mathbb{H}(w) - A$, $\mathscr{L}_0(f_t) \le \mathscr{L}_0(f_{t_0})$.*

Proof. Without loss of generality, we suppose that $\mathbb{H}(w) - A$ is nonempty. Let $f_{t_0} \in \mathbb{H}(w) - A$. It follows from Lemma 2.1 that the Łojasiewicz Exponent can be calculated by means of analytic paths:

$$\mathcal{L}_0(f_{t_0}) = \sup_{\varphi} \frac{\operatorname{ord}((\nabla f_{t_0}) o \varphi)}{\operatorname{ord} \varphi},$$

where $\varphi(t) = (\varphi_0(t), \cdots, \varphi_n(t)) \in \mathbb{C}\{t\}^{n+1}$, $\varphi(0) = 0$, $\varphi(t) \neq 0$ in $\mathbb{C}\{t\}^{n+1}$. Consider the function $\operatorname{ord}(\bigtriangledown(\cdot) \circ \varphi) : \mathbb{H}(w) - A \to \mathbb{Z}$ defined by $\operatorname{ord}(\bigtriangledown(\cdot) \circ \varphi)(f) = \operatorname{ord}(\bigtriangledown(f) \circ \varphi)$, where $f \in \mathbb{H}(w) - A$. By Proposition 2.3, it is an upper semicontinuous function. Then there exists an open subset U of f_{t_0} such that for any $f_t \in U$, $\operatorname{ord}(\bigtriangledown(\cdot)\circ\varphi)(f_t) \le \operatorname{ord}(\bigtriangledown(\cdot)\circ\varphi)(f_{t_0})$, i.e. $\operatorname{ord}(\bigtriangledown(f_t)\circ\varphi) \le \operatorname{ord}(\bigtriangledown(f_{t_0})\circ\varphi)$. Therefore

$$\sup_{\varphi} \frac{\operatorname{ord}((\nabla f_t) o \varphi)}{\operatorname{ord} \varphi} \le \sup_{\varphi} \frac{\operatorname{ord}((\nabla f_{t_0}) o \varphi)}{\operatorname{ord} \varphi}$$

and $\mathscr{L}_0(f_t) \le \mathscr{L}_0(f_{t_0})$. □

If I is an ideal in a ring, we use $V(I)$ to denote the zero locus defined by $\{z : f(z) = 0, \forall f \in I\}$.

Proposition 3.5. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin and weight $w = (w_0, \cdots, w_n)$. Without loss of generality, we assume $w_0 = \max_{0\le j\le n} w_j$. Let A, $\mathbb{H}(w)$ be defined as before in Proposition* 2.2. *Then there exists a proper closed set $B \subset \mathbb{H}(w)$ such that for any $f_t \in \mathbb{H}(w) - A - B$, $\Gamma_{z_0}(f_t) \nsubseteq V(z_0)$ and $\mathbb{H}(w) - A - B$ is a dense open set in $\mathbb{H}(w)$.*

Proof. Since $f \in \mathbb{H}(w) - A$, so $\mathbb{H}(w) - A$ is not empty. Note that $\mathbb{H}(w)$ is a finite-dimensional vector space. We choose a standard basis $z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ of $\mathbb{H}(w)$, where $\frac{\alpha_0}{w_0} + \frac{\alpha_1}{w_1} + \cdots + \frac{\alpha_n}{w_n} = 1$. For any $f_t \in \mathbb{H}(w)$, we have

$$f_t = \sum_{\frac{\alpha_0}{w_0} + \frac{\alpha_1}{w_1} + \cdots + \frac{\alpha_n}{w_n} = 1} t_{\alpha_0, \alpha_1, \cdots, \alpha_n} z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

where $t_{\alpha_0,\alpha_1,\cdots,\alpha_n}$ is a complex number. Recall that

$$\begin{aligned} &\Gamma_{z_0}(f_t(z_0, z_1, \cdots, z_n)) \\ &\quad = V\Big(\frac{\partial f_t(z_0, z_1, \cdots, z_n)}{\partial z_1}, \frac{\partial f_t(z_0, z_1, \cdots, z_n)}{\partial z_2}, \cdots, \frac{\partial f_t(z_0, z_1, \cdots, z_n)}{\partial z_n}\Big). \end{aligned}$$

Assume that $\Gamma_{z_0}(f_t) \subset V(z_0)$. It follows from the Local Nullstellensatz that

$$z_0 \in \sqrt{\Big(\frac{\partial f_t}{\partial z_1}, \frac{\partial f_t}{\partial z_2}, \cdots, \frac{\partial f_t}{\partial z_n}\Big)}.$$

By Theorem 2.4 there exists a $k \in \mathbb{Z}_+$ such that $z_0^k = \sum_{i=1}^{n} g_i(t, z_0, z_1, \cdots, z_n) \frac{\partial f_t}{\partial z_i}$, where $g_i(t, z_0, z_1, \cdots, z_n) \in \mathbb{C}\{t, z_0, z_1, \cdots, z_n\}$. Let $\mathbb{P}(w)$, where $w = (w_0, w_1, \cdots, w_n)$, be the weighted projective space which was defined as before. The set

$$C = \{(z, t) \in \mathbb{P}(w) \times \mathbb{H}(w) : z_0^k = \sum_{i=1}^{n} g_i(t, z_0, z_1, \cdots, z_n) \frac{\partial f_t}{\partial z_i}\}$$

is clearly a closed algebraic set. From the completeness of the weighted projective space, it follows that $B := q(C)$ is a closed algebraic set in $\mathbb{H}(w)$, where $q : \mathbb{P}(w) \times \mathbb{H}(w) \to \mathbb{H}(w)$ denotes the projection. It follows from the construction of B that for any $f_t \in \mathbb{H}(w) - A - B$, $\Gamma_{z_0}(f_t) \nsubseteq V(z_0)$. One thing left is to prove that B is a proper closed set. By abusing the notation, in $\mathbb{H}(w)$, we shall identify t_α with $t_\alpha z^\alpha$. It follows from Lemma 2.3 that for any $0 \le i \le n$, either $z_i^{m_i}$ or $z_i^{m_i} z_j$ is in $\mathbb{H}(w)$, where $j \ne i$ and m_i is an integer. If $z_i^{m_i} \in \mathbb{H}(w)$, where $1 \le i \le n$ or $z_i^{m_i} z_j \in \mathbb{H}(w)$,where $1 \le i \le n, m_i \ge 2$ and $j \ne i$ or $z_i z_j \in \mathbb{H}(w)$, where $1 \le i, j \le n, j \ne i$, then it is easy to prove that they are not in B. Otherwise, $z_i z_0 \in \mathbb{H}(w)$, where $1 \le i \le n$. Then $w_0 = w_1 = \cdots = w_n = 2$. Thus $z_1 z_2$ is also contained in $\mathbb{H}(w)$, so B is proper closed set. It follows that $\mathbb{H}(w) - A - B$ is a dense open set in $\mathbb{H}(w)$. □

Proposition 3.6. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin in $\mathbb{C}^{n+1}$ and weight $w = (w_0, \cdots, w_n)$. Then*

$$\mathscr{L}_0(f) \ge \max_{0 \le i \le n} (w_i - 1).$$

Proof. Let $q_0, \cdots, q_n$ and d be positive integers such that $q_i w_i = d$, for $i = 0, \cdots, n$. Without loss of generality, suppose $w_0 = \max_{0 \le i \le n} w_i$, i.e. $q_0 = \min_{0 \le i \le n} q_i$. Since f is a weighted homogeneous isolated singularity, it follows from Proposition 3.4 and Proposition 3.5 that there exists a proper closed algebraic set $A \subset \mathbb{H}(w)$ such that $f \in \mathbb{H}(w) - A$, and a proper subset $B \subset \mathbb{H}(w)$, such that $\mathbb{H}(w) - A - B$ is dense in $\mathbb{H}(w) - A$. There are two cases to be considered.

Case 1. If $f \in \mathbb{H}(w) - A - B$, then $\Gamma_{z_0}(f) \nsubseteq V(z_0)$. We choose an open neighborhood U of $0 \in \mathbb{C}^{n+1}$, such that if $\bigtriangledown f(z) = 0$, and $z \in U$, then $z = 0$. Since f has an isolated singularity, we have $\frac{\partial f}{\partial z_i} \ne 0$ for $i = 0, \cdots, n$. From the assumption that $\Gamma_{z_0}(f) \nsubseteq V(z_0)$, it follows that the system of equations $\frac{\partial f}{\partial z_1} = \frac{\partial f}{\partial z_2} = \cdots = \frac{\partial f}{\partial z_n} = 0$ has a solution $b = (b_0, b_1, \cdots, b_n)$ in U such that $b_0 \ne 0$. Take an analytic path $\varphi(t) = (b_0 t^{q_0}, b_1 t^{q_1}, \cdots, b_n t^{q_n})$. Since $\mathrm{Supp}(\frac{\partial f}{\partial z_i})$ lies on the the hyperplane $q_0 \alpha_0 + q_1 \alpha_1 + \cdots + q_n \alpha_n = d - q_i$ we get

$$\frac{\partial f}{\partial z_i} \circ \varphi(t) = t^{d - q_i} \frac{\partial f}{\partial z_i}(b) = 0, \text{ for } i > 0,$$
$$\frac{\partial f}{\partial z_0} \circ \varphi(t) = t^{d - q_0} \frac{\partial f}{\partial z_0}(b) \ne 0.$$

Therefore we get

$$\mathscr{L}_0(f) \geq \frac{\mathrm{ord}((\nabla f)\circ\varphi(t))}{\mathrm{ord}\,\varphi(t)} = \frac{\mathrm{ord}(\frac{\partial f}{\partial z_0}(\varphi(t)))}{\mathrm{ord}\,\varphi(t)} = \frac{d-q_0}{q_0} = w_0 - 1;$$

i.e. $\mathscr{L}_0(f) \geq \max_{0\leq i\leq n} w_i - 1$.

Case 2. Let $f \in B \cap (\mathbb{H}(w) - A)$. Suppose $f = f_{t_o}$. It follows from Proposition 3.4 that there exists an open subset $U \subset \mathbb{H}(w) - A$ of f_{t_0} such that for any $f_t \in U$, $\mathscr{L}_0(f_t) \leq \mathscr{L}_0(f_{t_0})$. Since $U \cap (\mathbb{H}(w) - A - B)$ is not empty, we can take an $f_t \in \mathbb{H}(w) - A - B$. It follows from Case 1 that $\mathscr{L}_0(f_t) \geq \max_{0\leq i\leq n} w_i - 1$. Therefore $\mathscr{L}_0(f) = \mathscr{L}_0(f_{t_0}) \geq \mathscr{L}_0(f_t) \geq \max_{0\leq i\leq n} w_i - 1$. □

In the following, let us recall a result in [K-O-P].

Proposition 3.7. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous isolated singularity of type $(w_0, \cdots, w_n)$ at $0 \in \mathbb{C}^n$. Then*

$$\mathcal{L}_0(f) \leq \max_{0\leq i\leq n} (w_i - 1).$$

Proof. Let $q_0, \cdots, q_n$ and d be positive integers such that $q_i w_i = d$ for $i = 0, \cdots, n$. Since f is an isolated singularity, we have $\frac{\partial f}{\partial z_i} \neq 0$ for $i = 0, \cdots, n$. Obviously, $\mathrm{supp}(\frac{\partial f}{\partial z_i})$ lies on the hyperplane $q_0\alpha_0 + \cdots + q_n\alpha_n = d - q_i$. Using Lemma 2.2, we get

$$\mathscr{L}_0(f) = l_0(\nabla f) \leq \frac{\max_{0\leq i\leq n}(d - q_i)}{\min_{0\leq i\leq n}(q_i)} = \max_{0\leq i\leq n}(\frac{d}{q_i} - 1) = \max_{0\leq i\leq n}(w_i - 1). \qquad \square$$

Proof of Theorem 1.3. It follows from Proposition 3.6 and Proposition 3.7 above. □

Proof of Corollary 1.1. This is an immediate consequence of Theorem 1.3 and the result proven by B. Teissier that $\mathrm{suff}_0(f) = [\mathscr{L}_0(f)] + 1$ ([T], Theorem 8). □

Proof of Corollary 1.2. Suppose $f = \sum_{\alpha\in I} c_\alpha z^\alpha \in \mathbb{C}[z_0, z_1, \cdots, z_n]$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index in I. Since f is a weighted homogeneous polynomial, we have $\frac{\alpha_0}{w_0} + \cdots + \frac{\alpha_n}{w_n} = 1$, which implies $\frac{\alpha_0+\alpha_1+\cdots+\alpha_n}{\max_{0\leq i\leq n} w_i} \leq 1$. Thus $\deg(f) \leq \max_{0\leq i\leq n} w_i$. Since $\deg(f)$ is an integer, we have $\deg(f) \leq [\max_{0\leq i\leq n} w_i]$. By Corollary 1.1, $\deg(f) \leq \mathrm{suff}_0(f)$. □

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