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LINEAR SERIES ON RIBBONS

DAWEI CHEN

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ABSTRACT. A ribbon is a double structure on $\mathbb{P}^1$. The geometry of a ribbon is closely related to that of a smooth curve. In this paper we consider linear series on ribbons. Our main result is an explicit determinantal description for the locus W^r_{2n} of degree $2n$ line bundles with at least $(r+1)$-dimensional sections on a ribbon. We also discuss some results of Clifford and Brill-Noether type.

1. INTRODUCTION

In this section, we recall some basic theory of ribbons. In the literature a ribbon is also called a Fossum-Ferrand doubling structure. Here we will mainly follow Bayer-Eisenbud [BE] for the related terminology. Many results and much of the notation below come from their paper.

We work over an algebraically closed field k of characteristic 0. A ribbon on $\mathbb{P}^1$ is a scheme C equipped with an isomorphism $\mathbb{P}^1 \to C_{red}$, such that the ideal sheaf $\mathscr{L}$ of $\mathbb{P}^1$ in C satisfies

$$\mathscr{L}^2 = 0.$$

Because of this condition, $\mathscr{L}$ can be regarded as a line bundle on $\mathbb{P}^1$. It is called the *conormal bundle* of $\mathbb{P}^1$ in C. There is a short exact sequence called the *conormal sequence*:

$$0 \to \mathscr{L} \to \mathscr{O}_C \to \mathscr{O}_{\mathbb{P}^1} \to 0.$$

Define the arithmetic genus g of a ribbon C as

$$g = 1 - \chi(\mathscr{O}_C).$$

From the conormal sequence, we see that C has genus g if and only if the conormal bundle $\mathscr{L}$ on $\mathbb{P}^1$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-g-1)$.

There is another short exact sequence called the *restricted cotangent sequence*:

$$0 \to \mathscr{L} \to \Omega_C|_{\mathbb{P}^1} \to \Omega_{\mathbb{P}^1} \to 0.$$

This restricted cotangent sequence defines the *extension class* of C:

$$e_c \in \operatorname{Ext}^1_{\mathbb{P}^1}(\Omega_{\mathbb{P}^1}, \mathscr{L}).$$

We will say that two ribbons are *isomorphic* over $\mathbb{P}^1$ if there is an isomorphism between them that extends the identity on $\mathbb{P}^1$. A ribbon C is *split* if the inclusion $\mathbb{P}^1 \hookrightarrow C$ admits a section. Such a section is a scheme-theoretically degree two map from C to $\mathbb{P}^1$. We also call C *hyperelliptic* if it is split.

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Theorem 1.1. *Given any line bundle $\mathscr{L}$ on $\mathbb{P}^1$ and any class $e \in \mathrm{Ext}^1_{\mathbb{P}^1}(\Omega_{\mathbb{P}^1}, \mathscr{L})$, there is a unique ribbon C on $\mathbb{P}^1$ with $e_c = e$. If there is another class $e' \in \mathrm{Ext}^1_{\mathbb{P}^1}(\Omega_{\mathbb{P}^1}, \mathscr{L})$ corresponding to a ribbon C', then $C \cong C'$ if and only if $e = ae'$ for some $a \in k^*$. A hyperelliptic ribbon corresponds to the split extension. The set of nonhyperelliptic ribbons of genus g, up to isomorphism over $\mathbb{P}^1$, is the set*

$$\mathbb{P}^{g-3} = \mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}^1}(g-3))).$$

Proof. The above results are essentially from [BE, Thm. 1.2, 2.1]. □

There is an explicit way to write down the structure of a ribbon by a gluing method; cf. [BE, Sec. 3]. Define two open sets

$$u_1 = \mathrm{Spec}\ k[s],\ u_2 = \mathrm{Spec}\ k[t]$$

that cover $\mathbb{P}^1$ via the identification $s^{-1} = t$ on $u_1 \cap u_2$.

If C is a genus g ribbon on $\mathbb{P}^1$, we can write

$$U_1 := C|_{u_1} \cong \mathrm{Spec}\ k[s,\epsilon]/\epsilon^2,$$
$$U_2 := C|_{u_2} \cong \mathrm{Spec}\ k[t,\eta]/\eta^2.$$

C may be specified by a gluing isomorphism between U_1 and U_2 over $u_1 \cap u_2$. The ideal sheaf $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^1}(-g-1)$ of $\mathbb{P}^1$ in C is generated by ϵ on u_1 and by η on u_2. So we can further write

$$\epsilon = t^{-g-1}\eta,$$
$$s^{-1} = t + F(t)\eta$$

on $u_1 \cap u_2$, with $F(t) \in k[t, t^{-1}]$. Conversely, any such gluing data can determine a ribbon of genus g on $\mathbb{P}^1$.

If we change the coordinates with

$$s' = s + p(s)\epsilon,\ t = t' + q(t)\eta$$

on U_1 and U_2 with polynomials $p(s), q(t)$, then we get

$$\begin{aligned} s'^{-1} &= s^{-1} - s^{-2}p(s)\epsilon \\ &= t' + \big(F(t) - t^{-g+1}p(t^{-1} - F(t)\eta) + q(t)\big)\eta \\ &= t + F(t)\eta - (t + F(t)\eta)^2 p(t^{-1}) t^{-g-1}\eta \\ &= t + F(t)\eta - t^{1-g}p(t^{-1})\eta \\ &= t' + \big(F(t) + q(t) - t^{1-g}p(t^{-1})\big)\eta \\ &= t' + \big(F(t') + q(t') - t'^{1-g}p(t'^{-1})\big)\eta. \end{aligned}$$

The fact that $t\eta = t'\eta$ is used in the last step. If we multiply s or t by a scalar, F will also be multiplied by the same scalar. Therefore, F can be determined as an element of the projective space of lines in the quotient

$$k[t, t^{-1}]/(k[t] + t^{-g+1}k[t^{-1}]).$$

From now on, we shall write F as

$$F = \sum_{i=1}^{g-2} F_i t^{-i}. \tag{1}$$

$F = 0$ corresponds to a hyperelliptic ribbon. This explicit expression recovers the fact in Theorem 1.1 that nonhyperelliptic ribbons of genus g, up to isomorphism over $\mathbb{P}^1$, are parameterized by the set

$$\mathbb{P}^{g-3} = \mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}^1}(g-3))).$$

Now we consider line bundles. Let L be a line bundle on a ribbon C. The degree of L is defined as

$$\deg L := \chi(L) - \chi(\mathscr{O}_C).$$

Proposition 1.2. *If $L|_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(n)$, then $\deg L = 2n$. The Picard group of C is*

$$\text{Pic } C = H^1(\mathscr{O}_{\mathbb{P}^1}(-g-1)) \times \mathbb{Z} \cong k^g \times \mathbb{Z},$$

where the projection to $\mathbb{Z}$ is given by the degree of the restriction $L|_{\mathbb{P}^1}$.

Proof. See [BE, Props. 4.1, 4.2]. □

Bayer and Eisenbud remarked that one must switch to torsion-free sheaves in order to obtain the analogue of line bundles of odd degree. For simplicity, here we only consider line bundles of even degree $2n$.

A line bundle L can also be constructed by gluing. Suppose $L|_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(n)$. Using the above notation, we have

$$L|_{U_1} = k[s,\epsilon]e_1,$$
$$L|_{U_2} = k[t,\eta]e_2,$$

and

$$e_1 = (t+F\eta)^n(1+G\eta)e_2$$

on $U_1 \cap U_2$ for some $G \in k[t,t^{-1}]$. Conversely, any such G can be used to construct a line bundle.

If we change the coordinates by

$$e_1 = (1+m(s)\epsilon)^{-1}e_1', \ e_2 = (1+n(t)\eta)e_2'$$

on U_1 and U_2 with polynomials $m(s), n(t)$, then we get

$$e_1' = (t+F\eta)^n\big(1+(G-m(t^{-1})t^{-g-1}+n(t))\eta\big)e_2'.$$

In order to classify L, it suffices to specify G as an element of

$$k[t,t^{-1}]/(k[t]+t^{-g-1}k[t^{-1}]) = H^1(\mathscr{O}_{\mathbb{P}^1}(-g-1)).$$

This also recovers the fact in Proposition 1.2 that $H^1(\mathscr{O}_{\mathbb{P}^1}(-g-1))$ parameterizes line bundles of fixed degree. We will also write G as

$$G = \sum_{j=1}^{g} G_j t^{-j}. \tag{2}$$

Let L be a line bundle on C of degree $2n$ given by the above gluing data. We would like to find out the space of global sections of L. Suppose $p = p(t)$ is a polynomial of degree $\leq n$. Then $pe_2|_{\mathbb{P}^1}$ determines an element

$$\sigma \in H^0(L|_{\mathbb{P}^1}) = H^0(\mathscr{O}_{\mathbb{P}^1}(n)).$$

Define

$$\delta_L(p) = -(p'F+pG) \in k[t,t^{-1}]/(k[t]+t^{n-g-1}k[t^{-1}]), \tag{3}$$

where $p' = \frac{\partial p}{\partial t}$.

Theorem 1.3. *The space of sections of L restricted to $U_2 = \operatorname{Spec} k[t,\eta]$ can be identified as the direct sum of the space of elements $q(t)\eta$ and the space of expressions $p(t)+p_1(t)\eta$, where q is a polynomial of degree $\leq n-g-1$, p is a polynomial of degree $\leq n$ satisfying $\delta_L(p)=0$ in (3), and $p_1 \in k[t]$ is the polynomial part of $p'F+pG$, i.e. $p_1(t) \equiv p'(t)F(t)+p(t)G(t) \bmod t^{-1}k[t^{-1}]$.*

Proof. This is exactly [BE, Thm. 4.3]. □

At first glance, the above way to identify $H^0(L)$ seems quite messy. Nevertheless, a further observation will imply an important conclusion immediately.

Corollary 1.4. *Let L be a line bundle of degree $2n$ on a ribbon C. If $n \geq g$, then $h^0(L) = 1-g+2n$.*

Proof. Let a section of L restricted to U_2 correspond to the data $(q(t)\eta, p(t)+p_1(t))$ in the above setup. When $n \geq g$, we have $k[t,t^{-1}]/(k[t]+t^{n-g-1}k[t^{-1}]) \equiv 0$. Then by its definition, δ_L always takes the value of 0. So $\delta_L(p)=0$ does not impose any condition on $p(t)$. The only constraint on $p(t)$ and $q(t)$ is the upper bound of their degree. Here $q(t)$ has degree $\leq n-g-1$ and $p(t)$ has degree $\leq n$. In total, they yield a $(1-g+2n)$-dimensional space for sections of L. □

Remark 1.5. Notice that if $n \geq g$, then the degree d of L satisfies $d = 2n > 2g-2$. In case C is a smooth curve of genus g, $h^0(L) = 1-g+d$ holds for any line bundle L on C with degree $d > 2g-2$. So the above corollary can be viewed as a similarity between ribbons and smooth curves.

2. The locus W^r_{2n}

For smooth curves, the theory of special linear series can be best characterized by the Brill-Noether theory. We refer readers to [ACGH, Chap. V] for bibliographical notes on this topic. Let C be a curve of genus g. We introduce the variety $W^r_d(C)$ as

$$W^r_d(C) = \{L \in \operatorname{Pic}^d(C) \mid h^0(C,L) \geq r+1\}.$$

We also define the Brill-Noether number ρ as

$$\rho = g-(r+1)(g-d+r).$$

The basic results of the Brill-Noether theory can be summarized as follows.

Theorem 2.1. *Let C be a smooth curve of genus g. Let d, r be integers such that $d \geq 1, r \geq 0$.*

(Existence) If $\rho \geq 0$, $W^r_d(C)$ is non-empty. Furthermore, every component of $W^r_d(C)$ has dimension at least equal to ρ provided $r \geq d-g$.

(Connectedness) Assume that $\rho \geq 1$. Then $W^r_d(C)$ is connected.

(Dimension) Assume that C is a general curve. If $\rho < 0$, then $W^r_d(C)$ is empty. If $\rho \geq 0$, then $W^r_d(C)$ is reduced and of dimension ρ.

We would like to investigate linear series for ribbons. The importance of such a study is three-fold. In the first place, $W^r_d(C)$ essentially carries a determinantal structure for a smooth curve C. In case C is a ribbon, the determinantal characterization can even be made explicit. Secondly, the Brill-Noether theory for a special member in a family of curves usually reveals information for a general one. Ribbons do arise as the degeneration of smooth curves; cf. [F]. Finally, Lazarsfeld [L] proved that a general curve contained in certain K3 surfaces satisfies the above dimension theorem. Correspondingly, ribbons lie on the so-called K3 carpet, i.e. a double structure on a rational normal scroll, which has the same numerical invariants as a smooth K3 surface; cf. [BE, Sec. 8]. Hence, it would be interesting to figure out some results of Brill-Noether type for ribbons.

Let C be a ribbon determined by $[F_1, \dots, F_{g-2}]$, the coefficients of F in (1), up to a scalar. Let L be a line bundle of degree $2n$ on C determined by $(G_1, \dots, G_g)$, the coefficients of G in (2). If $n \geq g$, there is no special linear system because of Corollary 1.4. Actually we only need to consider $2n \leq g-1$, since the Riemann-Roch formula also holds for ribbons; cf. [BE, Sec. 5]. From now on, assume that $2n \leq g-1$. Define a $(g-n) \times (n+1)$ matrix $\mathcal{A}_F(G)$ with entries $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}$, namely,

$$\mathcal{A}_F(G) = \begin{pmatrix} G_1 & G_2 + F_1 & \cdots & G_{n+1} + nF_n \\ G_2 & G_3 + F_2 & \cdots & G_{n+2} + nF_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{g-n} & G_{g-n+1} + F_{g-n} & \cdots & G_g \end{pmatrix}.$$

Now we can state our main result.

Theorem 2.2. *In the above setting, the locus $W^r_{2n}(C)$ is isomorphic to the following affine algebraic set:*

$$W^r_{2n}(C) = \{(G_1, \dots, G_g) \in \mathbb{A}^g \mid \operatorname{rank} \mathcal{A}_F(G) \leq n - r\}.$$

Proof. By Theorem 1.3, the space $H^0(L)$ of global sections of L can be identified as the direct sum of two spaces:

$$\langle q(t)\eta \rangle \oplus \langle p(t) + p_1(t)\eta \rangle.$$

Since $q(t)$ is a polynomial of degree $\leq n-g-1$ and $n \leq g-2$, the first summand is the null space. For the second, $p(t)$ is a polynomial of degree $\leq n$ satisfying $\delta_L(p) = 0$ as in (3). Then $p_1(t)$ is determined by $p(t)$, the polynomial part of $p'F + pG$. Let $p(t) = \sum_{i=0}^n a_i t^i$. The condition $\delta_L(p) = 0$ means

$$p'F + pG \in k[t] + t^{n-g-2} k[t^{-1}],$$

which is equivalent to the following:

$$\mathcal{A}_F(G) \cdot \vec{a} = 0,$$

where $\vec{a}$ is the vector $(a_0, \dots, a_n)^t$ determined by the coefficients of $p(t)$. Note that $g - n \geq n+1$ by the assumption on n. Hence, $W^r_{2n}(C)$ can be identified as the desired determinantal locus. ☐

The following Clifford theorem for ribbons is a direct consequence of Theorem 2.2.

Theorem 2.3. *Let C be a ribbon and let L be a line bundle of degree $2n$ on C, $1 \leq n \leq g-2$. Then $h^0(C, L) \leq n+1$. The equality holds if and only if C is a hyperelliptic ribbon and L is the pullback of $\mathscr{O}_{\mathbb{P}^1}(n)$ from $C_{red} \cong \mathbb{P}^1$ to C.*

Proof. By the above determinantal description for $W^r_{2n}(C)$, we know that $h^0(C, L) \leq n+1$. If the equality holds, then $r = n$. We have $G_i = 0$ and $F_j = 0$ for all i, j. Thus C is hyperelliptic and L is isomorphic to the pullback of $\mathscr{O}_{\mathbb{P}^1}(n)$ from $C_{red} \cong \mathbb{P}^1$. ☐

When C is a hyperelliptic ribbon, i.e. $F_i = 0$ for all i, $\mathcal{A}_F(G)$ has entries $s_{ij} = G_{i+j-1}$:

$$\begin{pmatrix} G_1 & G_2 & \cdots & G_{n+1} \\ G_2 & G_3 & \cdots & G_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ G_{g-n} & G_{g-n+1} & \cdots & G_g \end{pmatrix}.$$

Such a matrix is called the catalecticant matrix. We cite the result [E, Prop. 4.3] as follows.

Proposition 2.4. *The space of rank $\leq m$ catalecticant matrices is isomorphic to a cone over S_m, where S_m is the union of m-secant $(m-1)$-planes to a rational normal curve of degree $g-1$.*

This exactly describes $W^r_{2n}(C)$ for a hyperelliptic ribbon.

Theorem 2.5. *Let C be a hyperelliptic ribbon. Then $W^r_{2n}(C)$ is isomorphic to a cone over S_{n-r} for $r < n$. In particular, $W^r_{2n}(C)$ has dimension equal to $2n-2r$.*

Proof. $W^r_{2n}(C)$ can be identified as the space of rank $\leq n-r$ catalecticant matrices, which is isomorphic to a cone over S_{n-r} by Proposition 2.4. Also, S_{n-r} has dimension $2n-2r-1$, so $W^r_{2n}(C)$ has dimension $2n-2r$. □

We have seen that for a nonhyperelliptic ribbon, its structure can be determined by the data $[F_1, \ldots, F_{g-2}]$ in (1) as a point of $\mathbb{P}^{g-3}$. Note that the expected dimension of $W^r_{2n}(C)$ would still be $g-(r+1)(g-2n+r)$, which equals the Brill-Noether number ρ. We would like to study the actual dimension of $W^r_{2n}(C)$. First, let us focus on a natural compactification of $W^r_{2n}(C)$ as follows.

Define another $(g-n)\times(n+1)$ matrix $\overline{\mathcal{A}}_F(G)$ with entries $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}G_0$:

$$\overline{\mathcal{A}}_F(G) = \begin{pmatrix} G_1 & G_2 + F_1G_0 & \cdots & G_{n+1} + nF_nG_0 \\ G_2 & G_3 + F_2G_0 & \cdots & G_{n+2} + nF_{n+1}G_0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{g-n} & G_{g-n+1} + F_{g-n}G_0 & \cdots & G_g \end{pmatrix}.$$

Let

$$\overline{W}^r_{2n}(C) = \{[G_0, G_1, \ldots, G_g] \in \mathbb{P}^g \mid \text{rank } \overline{\mathcal{A}}_F(G) \leq n-r\}.$$

There is an inclusion $W^r_{2n}(C) \subset \overline{W}^r_{2n}(C)$ given by

$$(G_1, \ldots, G_g) \to [1, G_1, \ldots, G_g].$$

The complement of $W^r_{2n}(C)$ in $\overline{W}^r_{2n}(C)$ is just the hyperplane section $\{G_0 = 0\} \cap \overline{W}^r_{2n}(C)$.

We also need to introduce generic determinantal varieties. Let M be the space of $(g-n)\times(n+1)$ matrices. Denote by M_l the locus of rank $\leq l$ matrices. Here M_l is called the l-generic determinantal variety, $l \leq n+1$. Denote by $\mathbf{M}$ and $\mathbf{M}_l$ the projectivization of M and M_l respectively.

Proposition 2.6. *$\mathbf{M}_l$ is an irreducible subvariety of codimension $(g-n-l)(n+1-l)$ in $\mathbf{M}$.*

One can refer to [ACGH, Chap. II] for a general discussion on determinantal varieties. Our next result is about the global geometry of $\overline{W}^r_{2n}(C)$.

Theorem 2.7. *Let C be a ribbon of genus g. For $r < n$, $\overline{W}^r_{2n}(C)$ is always non-empty and has dimension equal to $2n-2r-1$ or $2n-2r$. Each irreducible component of $\overline{W}^r_{2n}(C)$ has dimension at least equal to ρ provided $\rho \geq 0$. Furthermore, $\overline{W}^r_{2n}(C)$ is connected provided $\rho > 0$.*

Proof. $\overline{W}^r_{2n}(C)$ is the intersection of $\mathbf{M}_{n-r}$ and a g-dimensional linear subspace of $\mathbf{M}$ determined by $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}G_0$. Therefore, each irreducible component of $\overline{W}^r_{2n}(C)$ has dimension $\geq \dim \mathbf{M}_{n-r} + g - \dim \mathbf{M} = \rho$.

When $G_0 = 0$, the matrix $\overline{\mathcal{A}}_F(G)$ reduces to a catalecticant matrix with entries $s_{ij} = G_{i+j-1}$. The space of rank $\leq n-r$ catalecticant matrices has dimension $2n-2r$ by Proposition 2.4. It implies that the hyperplane section $\{G_0 = 0\} \cap \overline{W}^r_{2n}(C)$ has dimension $2n-2r-1$. If the top dimensional component of $\overline{W}^r_{2n}(C)$ is contained in $\{G_0 = 0\}$, then $\overline{W}^r_{2n}(C)$ has dimension $2n-2r-1$. Otherwise it has dimension $2n-2r$.

$\overline{W}^r_{2n}(C)$ can also be regarded as the intersection of the $(n-r)$-generic determinantal variety $\mathbf{M}_{n-r}$ and a g-dimensional linear subspace of $\mathbf{M}$ defined by $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}G_0$ for a fixed lifting $(F_1, \ldots, F_{g-2})$. If $\rho > 0$, the sum of the dimensions of these two spaces is greater than the dimension of $\mathbf{M}$. The connectedness of their intersection $\overline{W}^r_{2n}(C)$ follows as a consequence of [L, Ex. 3.3.7]. □

Corollary 2.8. *Assume that $\rho \geq 0$. If $W^r_{2n}(C)$ is non-empty for a ribbon C, then $W^r_{2n}(C)$ has dimension at least equal to ρ.*

Proof. $W^r_{2n}(C)$ is the complement of the hyperplane section $\{G_0 = 0\} \cap \overline{W}^r_{2n}(C)$ in $\overline{W}^r_{2n}(C)$. By Theorem 2.7 we know that each component of $\overline{W}^r_{2n}(C)$ has dimension $\geq \rho$. So does $W^r_{2n}(C)$, assuming it is non-empty. □

We can also let $(F_1, \ldots, F_{g-2})$ vary as a point of $\mathbb{A}^{g-2}$ and define the global Brill-Noether locus $\mathcal{W}^r_{2n}$ as follows:

$$\mathcal{W}^r_{2n} = \{(G_1, \ldots, G_g; F_1, \ldots, F_{g-2}) \in \mathbb{A}^g \times \mathbb{A}^{g-2} \mid \text{rank } \mathcal{A}_F(G) \leq n-r\}.$$

$\mathcal{W}^r_{2n}$ is the intersection of the $(n-r)$-generic determinantal variety M_{n-r} and a $(2g-2)$-dimensional linear subspace S of M, where S is determined by relations $2s_{ij} = s_{i-1\ j+1} + s_{i+1\ j-1}$. Note that the expected dimension of $\mathcal{W}^r_{2n}$ would be $2g-2-(g-2n+r)(r+1) = g-2+\rho$, which implies the following conclusion right away.

Corollary 2.9. *If $\mathcal{W}^r_{2n}$ has dimension equal to $g-2+\rho$, then for $(F_1, \ldots, F_{g-2})$ corresponding to a general ribbon C, $W^r_{2n}(C)$ has dimension at most equal to ρ.*

In order to calculate the actual dimension of $\mathcal{W}^r_{2n}$, we introduce the concept of l-generic spaces developed by Eisenbud [E, Prop.-Def. 1.1].

Definition 2.10. Let L be a linear subspace of the space M of $(g-n) \times (n+1)$ matrices. Here L can be regarded as an associated $(g-n) \times (n+1)$ matrix of linear forms. We say that L is *m-generic* for some integer $1 \leq m \leq n+1$ if after arbitrary invertible row and column operations, any m of the linear forms L_{ij} in L are linearly independent.

We also say that L meets M_l *properly* if their intersection has codimension equal to $(g-n-l)(n+1-l)$ in L.

Theorem 2.11. *Let $L \subset M$ be an m-generic space; then L meets M_{n+1-m} properly.*

Proof. This is part of [E, Thm. 2.1]. □

Note that the space of catalecticant matrices is 1-generic. One can also prove the 2-genericity for the space S of matrices of type $\mathcal{A}_F(G)$.

Proposition 2.12. *Consider $G_1, \ldots, G_g; F_1, \ldots, F_{g-2}$ as independent linear forms. The $(2g-2)$-dimensional vector space S of all matrices determined by $\mathcal{A}_F(G)$ is 2-generic.*

Proof. $\mathcal{A}_F(G)$ is the matrix with entries $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}$. Suppose there were two invertible matrices $A = (a_{ij})$ and $B = (b_{ij})$ corresponding to invertible row and column operations such that two entries of the new matrix $A \cdot \mathcal{A}_F(G) \cdot B = (s'_{ij})$ became equal to each other. We can always assume that these two entries are s'_{11} and s'_{22}. The case that they are in the same row or column would be even easier. Then the condition $s'_{11} = s'_{22}$ is equivalent to

$$\sum_{i+j=k+1} (a_{1i}b_{j1} - a_{2i}b_{j2})\big(G_k + (j-1)F_{k-1}\big) = 0$$

for any k. Namely,

$$\sum_{i+j=k+1} (a_{1i}b_{j1} - a_{2i}b_{j2}) = 0 \text{ and } \sum_{i+j=k+1} (a_{1i}b_{j1} - a_{2i}b_{j2})(j-1) = 0$$

since G_i and F_j can vary independently.

Define four polynomials as follows:

$$A_k(x) = \sum_i a_{ki}x^i \text{ and } B_k(x) = \sum_j b_{kj}x^j \text{ for } k = 1, 2.$$

We can deduce from the above two equalities that

$$A_1(x)B_1(x) = A_2(x)B_2(x) \text{ and } A_1(x)B'_1(x) = A_2(x)B'_2(x).$$

Since the matrices A and B are invertible, by these two relations we can get

$$B_1(x)B'_2(x) = B_2(x)B'_1(x),$$

which would imply that $(B_1(x)/B_2(x))' = 0$. Then $B_1(x)/B_2(x)$ would be a constant, which contradicts the assumption that the matrix B is invertible. □

Corollary 2.13. *For $r = 1$, $\mathcal{W}^1_{2n}$ has dimension $4n-4$ and $W^1_{2n}(C)$ has dimension at most equal to $\rho = 4n - g - 2$ for a general ribbon C, provided $\rho \geq 0$.*

Proof. Since the space of matrices $\overline{\mathcal{A}}_F(G)$ is 2-generic, it intersects M_{n-1} properly. So the intersection $\mathcal{W}^1_{2n}$ has dimension equal to $g-2+\rho = 4n-4$ by Theorem 2.11. Then by Corollary 2.9, $W^1_{2n}(C)$ has dimension at most equal to $\rho = 4n-2-g$. □

Based on Corollaries 2.8 and 2.13, we obtain the following conclusion.

Corollary 2.14. *For $r = 1$, if $W^1_{2n}(C)$ is non-empty for a general ribbon C, then $W^1_{2n}(C)$ has dimension equal to $\rho = 4n - 2 - g$ provided $\rho \geq 0$.*

It would be interesting to pin down the following question in general.

Question 2.15. For $\rho \geq 0$, is the dimension of $\mathcal{W}^r_{2n}$ equal to the expected dimension $g - 2 + \rho$? For a general ribbon C, is the locus $W^r_{2n}(C)$ non-empty and does it have dimension equal to ρ provided $\rho \geq 0$?

By the determinantal descriptions for $\mathcal{W}^r_{2n}$ and $W^r_{2n}(C)$, using Macaulay one can check that the above question does have a positive answer when the genus of C is small.

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Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60607

E-mail address: dwchen@math.uic.edu