

QUANTIZATION DIMENSION FOR SOME MORAN MEASURES

MRINAL KANTI ROYCHOWDHURY

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ABSTRACT. The quantization dimension function for some Moran measures has been determined, and a relationship between the quantization dimension function and the temperature function of the thermodynamic formalism arising in multifractal analysis is established.

1. INTRODUCTION

The term quantization in this paper refers to the idea of estimating a given probability on \mathbb{R}^d with a discrete probability, that is, a ‘quantized’ version of the probability supported on a finite set. The problems surrounding this idea have been of major interest in signal compression and communications for quite some time (cf. [7]). Following the work of Graf and Luschgy, we define the *quantization dimension* (or perhaps better, the *quantization dimension function*) as follows (cf. [8, 9]): Given a Borel probability measure μ on \mathbb{R}^d , a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the n th *quantization error* of order r for μ is defined by

$$e_{n,r} = \inf\left\{\left(\int d(x, \alpha)^r d\mu(x)\right)^{\frac{1}{r}} : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n\right\},$$

where $d(x, \alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on \mathbb{R}^d . We note that if $\int \|x\|^r d\mu(x) < \infty$, then there is some set α for which the infimum is achieved (cf. [8]). Graf and Luschgy also defined $e_{n,r}$ where $r = 0$ and $r = +\infty$, but in the paper we only deal with the case $0 < r < +\infty$. The *quantization dimension of order r* for μ is defined to be

$$D_r = D_r(\mu) = \lim_{n \rightarrow \infty} \frac{\log n}{-\log e_{n,r}}$$

if the limit exists. If the limit does not exist, then we define \overline{D}_r as the lim sup of the sequence and \underline{D}_r as the lim inf. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $e_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log e_{n,r} \sim \log\left(\frac{1}{n}\right)^{1/D_r}.$$

Let S_1, S_2, \dots, S_N be a set of contractive similarity mappings on \mathbb{R}^d with the similarity ratios s_1, s_2, \dots, s_N for $N \geq 2$. Then for a given probability vector

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(p_1, p_2, \dots, p_N) there exists a unique probability measure μ (cf. [12]) satisfying the condition

$$\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}.$$

Let the iterated function system $\{S_1, S_2, \dots, S_N\}$ satisfy the open set condition: there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{i=1}^N S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. Then, Graf and Luschgy showed that the quantization dimension function $D_r := D_r(\mu)$ for the probability measure μ exists and satisfies the following relation (cf. [8, 10]):

$$\sum_{i=1}^N (p_i s_i^r)^{\frac{D_r}{r+D_r}} = 1.$$

Note that from the above relation it is clear that the quantization dimension function for a self-similar probability measure has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis (cf. [5]). Later the result was extended by Lindsay and Mauldin to the F-conformal measures with finitely many conformal mappings (cf. [16]). In the paper [20], the author considered the ergodic measure with bounded distortion on the coding space, determined the quantization dimension function for the image measure via the coding map on a self-similar set, and showed its functional relationship with the temperature function of the thermodynamic formalism. In this paper, we give a generalization of the result of Graf and Luschgy and determine the quantization dimension function for a Moran measure, and its relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis is established.

2. BASIC DEFINITIONS AND PROPOSITIONS

Let us write

$$V_{n,r} = \inf \left\{ \int d(x, \alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

$$u_{n,r} = \inf \left\{ \int d(x, \alpha \cup U^c)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

where U is a set which comes from the open set condition and U^c denotes the complement of U . We see that

$$u_{n,r}^{1/r} \leq V_{n,r}^{1/r} = e_{n,r}.$$

We call sets $\alpha_n \subset \mathbb{R}^d$, for which the above infimums are achieved, n -optimal sets for $e_{n,r}, V_{n,r}$ or $u_{n,r}$ respectively. As stated above, Graf and Luschgy have shown that n -optimal sets exist when $\int \|x\|^r d\mu(x) < \infty$.

2.1. Moran set. Let $\{n_k\}_{k \geq 1}$ be a sequence of positive integers, and $\{\Phi_k\}_{k \geq 1}$ be a sequence of positive real vectors with

$$\Phi_k = (c_{k1}, c_{k2}, \dots, c_{kn_k}), \quad \sum_{j=1}^{n_k} c_{kj} \leq 1, \quad k \in \mathbb{N}.$$

Let D_0 be the empty set. For $k \geq 1$ write

$$D_{m,k} = \{(i_m, i_{m+1}, \dots, i_k) : 1 \leq i_j \leq n_j, m \leq j \leq k\},$$

and $D_k = D_{1,k}$, $D_\infty = \lim_{k \rightarrow \infty} D_k$. Define $D := \bigcup_{k \geq 0} D_k$. Elements of D are called words. For any $\sigma \in D$ if $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in D_n$ we write $|\sigma| = n$ to denote the length of σ , and $\sigma|_k := (\sigma_1, \sigma_2, \dots, \sigma_k)$, $k \leq n$, to denote the truncation of σ to the length k . If $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in D_k$ and $\tau = (\tau_1, \tau_2, \dots, \tau_m) \in D_{k+1,m}$, then we write $\sigma\tau = \sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m)$ to denote the juxtaposition of $\sigma, \tau \in D$. A word of length zero is called the empty word and is denoted by \emptyset . For $\sigma \in D$ and $\tau \in D \cup D_\infty$ we say that τ is an extension of σ , written as $\sigma \prec \tau$, if $\tau|_{|\sigma|} = \sigma$. Let $J \subset \mathbb{R}^d$ be a compact set such that $J = \text{cl}(\text{int}J)$, and $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ be a collection of subsets of \mathbb{R}^d . We say that the collection \mathcal{F} fulfills the Moran structure provided it satisfies the following Moran structure conditions (MSC):

(M1) $J_\emptyset = J$.

(M2) For any $\sigma \in D$, J_σ is geometrically similar to J ; i.e., there exists a similarity mapping $S_\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $J_\sigma = S_\sigma(J)$.

(M3) For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \dots, J_{\sigma*n_{k+1}}$ are subsets of J_σ , and $\text{Int}(J_{\sigma*i}) \cap \text{Int}(J_{\sigma*j}) = \emptyset$ for $1 \leq i \neq j \leq n_{k+1}$, where $\text{Int}(A)$ denotes the interior of A .

(M4) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $1 \leq j \leq n_k$, $\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_{kj}$.

Since given $\sigma = (\sigma_i)_{i=1}^\infty \in D_\infty$ the diameters of the compact sets $J_{\sigma|_k}$, $k \geq 1$ converge to zero and since they form a descending family, the set

$$\bigcap_{k=0}^\infty J_{\sigma|_k}$$

is a singleton; and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : D_\infty \rightarrow J$. The main object of our interest is the limit set

$$E = \pi(D_\infty) = \bigcup_{\sigma \in D_\infty} \bigcap_{k=1}^\infty J_{\sigma|_k}.$$

The set $E := E(\mathcal{F})$ is called the Moran set associated with the collection \mathcal{F} . Let $\mathcal{F}_k = \{J_\sigma : \sigma \in D_k\}$ and $\mathcal{F} = \bigcup_{k \geq 0} \mathcal{F}_k$. The elements of \mathcal{F}_k are called the basic elements of order k , and the elements of \mathcal{F} are called the basic elements of the Moran set E . Note that the set E satisfies the invariance equality

$$(1) \quad E = \bigcup_{j_k=1}^{n_k} S_{k j_k}(E),$$

where $\{S_{k1}, S_{k2}, \dots, S_{kn_k}\}$ ($k \geq 1$) are the similarity mappings that arise in the Moran structure condition (M2). Let us assume that the collection $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ satisfies the *open set condition* (OSC): there exists a bounded nonempty open set $U \subset J$ such that for $\sigma \in D_k$ ($k \geq 0$), $\bigcup_{i=1}^{n_{k+1}} S_{\sigma i}(U) = \bigcup_{i=1}^{n_{k+1}} S_i(S_\sigma(U)) \subset U$ and $S_{\sigma i}(U) \cap S_{\sigma j}(U) = \emptyset$ for each pair $i, j \in \{1, 2, \dots, n_{k+1}\}$ with $i \neq j$, where the similarity mappings S_σ are the similarity mappings as in (M2) arising in the Moran construction. Furthermore, the collection satisfies the *strong open set condition* (SOSC) if U can be chosen such that $U \cap E \neq \emptyset$, where $E := E(\mathcal{F})$ is the Moran set.

Suppose that the set J and the sequences $\{n_k\}$ and $\{\Phi_k\}$ are given. We denote by $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ the class of Moran sets satisfying the MSC (M1)–(M4). We call $\mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ a Moran class associated with the triplet $(J, \{n_k\}, \{\Phi_k\})$. For more details about the Moran sets one is referred to [13].

Let $A := \{a_1, a_2, \dots, a_m\}$ be a finite set of distinct positive integers and define

$$A^{\mathbb{N}} := \{(t_j)_{j=1}^{\infty} \in D_{\infty} : t_j \in A\}.$$

For $\omega = (s_1, s_2, \dots) \in A^{\mathbb{N}}$ let $\omega|_k = (s_1, s_2, \dots, s_k)$, and let $\|\omega|_k\|_{a_i} := \#\{s_j = a_i : s_j \text{ appears in } \omega|_k\}$ denote the times that the element a_i appears in the word $\omega|_k$. For $\omega \in A^{\mathbb{N}}$ let $\lim_{k \rightarrow \infty} \frac{\|\omega|_k\|_{a_i}}{k} = \eta_i > 0$ for every $a_i \in A$. Then we say that ω has the frequency vector $\eta = (\eta_1, \eta_2, \dots, \eta_m)$. Note that $\sum_{i=1}^m \|\omega|_k\|_{a_i} = k$ and $\sum_{i=1}^m \eta_i = 1$. For $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ define

$$A^{\mathbb{N}}_{\eta} := \{t = (t_j)_{j \geq 1} : t_j \in A, \lim_{k \rightarrow \infty} \frac{\|t|_k\|_{a_i}}{k} = \eta_i, 1 \leq i \leq m\}.$$

Suppose $m_i \in \mathbb{N}$, $1 \leq i \leq m$ and let $\Phi_i = (c_{i1}, c_{i2}, \dots, c_{im_i})$ be a positive real vector with $\sum_{j=1}^{m_i} c_{ij} \leq 1$. For $\omega \in A^{\mathbb{N}}_{\eta}$ and $E \in \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$, if $s_k = a_i$, $k \geq 1$, we take $n_k = m_i$, $\Phi_k := \Phi_i = (c_{i1}, c_{i2}, \dots, c_{im_i})$. Then we obtain a class of Moran sets associated with $\omega \in A^{\mathbb{N}}_{\eta}$. Let us denote such a Moran set by $E(\omega)$.

Note 2.2. Let $\omega = (s_1, s_2, \dots) \in A^{\mathbb{N}}$. For the rest of the paper we will keep our Moran set $E(\omega)$ fixed and assume its structure satisfies the strong open set condition as defined earlier.

2.3. Moran measure. Let J_{σ} be a basic element of order k of the Moran set $E(\omega)$. Note that by construction, if $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in D_k$, $k \geq 1$, and $s_n = a_i \in A$ ($1 \leq i \leq m$), then $\sigma_n \in \{1, 2, \dots, m_i\}$ ($1 \leq n \leq k$). For such a $\sigma \in D_k$ ($k \geq 1$) let us define $\sigma(a_1)\sigma(a_2) \dots \sigma(a_m)$ as follows: Fix i ($1 \leq i \leq m$) and calculate $\|\omega|_k\|_{a_i}$. If $\|\omega|_k\|_{a_i} = 0$ assume $\sigma(a_i) = \emptyset$. If $\|\omega|_k\|_{a_i} \neq 0$, let $i_1, i_2, \dots, i_{\|\omega|_k\|_{a_i}}$ be the positions where the symbol a_i appears in the word $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$. Note that $\sigma_{i_j} \in \{1, 2, \dots, m_i\}$ for $1 \leq j \leq \|\omega|_k\|_{a_i}$. Write $\sigma(a_i) = (\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_{\|\omega|_k\|_{a_i}}})$. Then $\sigma(a_1)\sigma(a_2) \dots \sigma(a_m)$ is another arrangement of $(\sigma_1, \sigma_2, \dots, \sigma_k)$. For every $a_i \in A$, let $P_{a_i} = (p_{i_1}, p_{i_2}, \dots, p_{i_{m_i}})$ ($1 \leq i \leq m$) be probability vectors, i.e., $p_{i_j} > 0$ and $\sum_{j=1}^{m_i} p_{i_j} = 1$ ($1 \leq i \leq m$). Define $p_{\sigma(a_i)} = p_{i\sigma_{i_1}} p_{i\sigma_{i_2}} \dots p_{i\sigma_{i_{\|\omega|_k\|_{a_i}}}}$, $1 \leq i \leq m$, where $p_{i\sigma_{i_j}} \in \{p_{i_1}, p_{i_2}, \dots, p_{i_{m_i}}\}$ for $1 \leq j \leq \|\omega|_k\|_{a_i}$. If $\sigma(a_i) = \emptyset$ assume that $p_{\sigma(a_i)} = 1$. We define

$$p_{\sigma} := p_{\sigma(a_1)} p_{\sigma(a_2)} \dots p_{\sigma(a_m)}.$$

It is easy to see that $\sum_{\sigma \in D_k} p_{\sigma} = \sum_{\sigma \in D_k} \prod_{i=1}^m p_{\sigma(a_i)} = 1$ for any $k \geq 1$.

Suppose μ is a mass distribution on $E(\omega)$, denoted by μ_{ω} , such that for any J_{σ} ($\sigma \in D_k$), $\mu(J_{\sigma}) = p_{\sigma(a_1)} p_{\sigma(a_2)} \dots p_{\sigma(a_m)}$ and $\mu(\sum_{\sigma \in D_k} J_{\sigma}) = 1$. Measure μ_{ω} is a Moran measure and $\text{supp} \mu_{\omega} = E(\omega)$, which is an extension of a self-similar measure in [12]. Note that for $E(\omega)$, there exists $D_{E(\omega)} \subset D_{\infty}$ such that $\pi(D_{E(\omega)}) = E(\omega)$. Define the left shift map $T : D_{E(\omega)} \rightarrow D_{E(\omega)}$ by $T(\sigma) = T(\sigma_1, \sigma_2, \dots) = (\sigma_2, \sigma_3, \dots)$ for $\sigma = (\sigma_1, \sigma_2, \dots) \in D_{E(\omega)}$. For each $i \in \{1, 2, \dots, m\}$, $i_j \in \{1, 2, \dots, m_i\}$, we define a right shift map $\tau_{i_j} : D_{E(\omega)} \rightarrow D_{E(\omega)}$ as follows: $\tau_{i_j}(\sigma) = i_j \sigma$, for all $\sigma = \sigma_1 \sigma_2 \dots \in D_{E(\omega)}$. For the probability vectors $P_{a_1}, P_{a_2}, \dots, P_{a_m}$ we denote by ν_{ω} the corresponding product measure on $D_{E(\omega)}$; then $\mu_{\omega} = \nu_{\omega} \circ \pi^{-1}$. Moreover, we have

$$(2) \quad \nu_{\omega} = \sum_{j=1}^{m_i} p_{i_j} \nu_{\omega} \circ \tau_{i_j}^{-1} \text{ and } \mu_{\omega} = \sum_{j=1}^{m_i} p_{i_j} \mu_{\omega} \circ S_{i_j}^{-1},$$

where S_{ij} is the similarity mapping corresponding to the similarity ratio c_{ij} for the Moran set $E(\omega)$ ($1 \leq i \leq m$ and $1 \leq j \leq m_i$). For this measure μ_ω we will determine the quantization dimension and its relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis.

2.4. Topological pressure. For $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in D_k$ let us write

$$c_\sigma = \begin{cases} c_{1\sigma_1} c_{2\sigma_2} \cdots c_{k\sigma_k} & \text{if } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases}$$

For $q, t \in \mathbb{R}$, let us now define the function

$$Z_n(q, t) = \sum_{\sigma \in D_n} p_\sigma^q c_\sigma^t = \prod_{i=1}^m \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right)^{\|\omega\|_n a_i}.$$

For $n \in \mathbb{N}$ let us write $c_n(q, t) = \frac{\log Z_n(q, t)}{n}$. Then for $n, p \in \mathbb{N}$ with $n > p \geq 1$ we have

$$\begin{aligned} |c_n(q, t) - c_p(q, t)| &= \left| \sum_{i=1}^m \frac{1}{n} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right)^{\|\omega\|_n a_i} - \sum_{i=1}^m \frac{1}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right)^{\|\omega\|_p a_i} \right| \\ &= \left| \sum_{i=1}^m \frac{\|\omega\|_n a_i}{n} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right) - \sum_{i=1}^m \frac{\|\omega\|_p a_i}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right) \right| \\ &\leq \left| \sum_{i=1}^m \frac{\|\omega\|_n a_i}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right) - \sum_{i=1}^m \frac{\|\omega\|_p a_i}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right) \right| \\ &= \left| \sum_{i=1}^m \frac{\|s_{p+1} s_{p+2} \cdots s_n\| a_i}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t \right) \right|, \end{aligned}$$

which tends to zero as p tends to infinity. Hence the sequence $\{c_n(q, t)\}$ is a Cauchy sequence and so by the ‘Cauchy convergence criteria’ $\{c_n(q, t)\}_{n=1}^\infty$ is a convergent sequence. Let us now define the function $P(q, t)$ as follows:

$$(3) \quad P(q, t) = \lim_{n \rightarrow \infty} c_n(q, t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\sigma \in D_n} p_\sigma^q c_\sigma^t = \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{i_j}^q c_{i_j}^t.$$

The function $P(q, t)$ is called the *topological pressure* corresponding to the given Moran construction.

The following proposition states the well-known properties of the function $P(q, t)$ (cf. [6, 18]).

Proposition 2.5. (i) $P(q, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(ii) $P(q, t)$ is strictly decreasing in each variable separately.

(iii) For fixed q we have

$$\lim_{t \rightarrow +\infty} P(q, t) = -\infty \text{ and } \lim_{t \rightarrow -\infty} P(q, t) = +\infty.$$

(iv) $P(q, t)$ is convex: if $q_1, q_2, t_1, t_2 \in \mathbb{R}$, $a_1, a_2 \geq 0$, $a_1 + a_2 = 1$, then

$$P(a_1 q_1 + a_2 q_2, a_1 t_1 + a_2 t_2) \leq a_1 P(q_1, t_1) + a_2 P(q_2, t_2).$$

Now for fixed q , the function $P(q, t)$ is a continuous function of t . Its value ranges from $-\infty$ (when $t \rightarrow +\infty$) to $+\infty$ (when $t \rightarrow -\infty$). Therefore, by the intermediate value theorem there is a real number β such that $P(q, \beta) = 0$. The solution β is unique, since $P(q, \cdot)$ is strictly decreasing. This defines β implicitly as a function of q : for each q there is a unique $\beta = \beta(q)$ such that $P(q, \beta(q)) = 0$.

The following proposition gives the well-known properties of the function $\beta(q)$ (cf. [6, 18]).

Proposition 2.6. *Let $\beta = \beta(q)$ be defined by $P(q, \beta(q)) = 0$. Then*

- (i) β is a continuous function of the real variable q .
- (ii) β is strictly decreasing: if $q_1 < q_2$, then $\beta(q_1) > \beta(q_2)$.
- (iii) $\lim_{q \rightarrow -\infty} \beta(q) = +\infty$ and $\lim_{q \rightarrow +\infty} \beta(q) = -\infty$.
- (iv) β is a convex function: if $q_1, q_2, a_1, a_2 \in \mathbb{R}$ with $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$, then

$$\beta(a_1 q_1 + a_2 q_2) \leq a_1 \beta(q_1) + a_2 \beta(q_2).$$

The function $\beta(q)$ is sometimes denoted by $T(q)$ and called the *temperature function*. A more general discussion of this function can be found in [11], where our $\beta(q)$ function would correspond to $-\tau(q)$ in their notation.

Remark 2.7. Note that

$$\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} c_{ij}^{\beta(0)} = \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{ij}^0 c_{ij}^{\beta(0)} = P(0, \beta(0)) = 0.$$

Hence, $\beta(0)$ gives the Hausdorff dimension $\text{Dim}_H(E(\omega))$ of the Moran set $E(\omega)$ (cf. [13]). Note that

$$P(1, 0) = \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{ij} = \sum_{i=1}^m \eta_i \log 1 = 0,$$

and hence $\beta(1) = 0$ (see Figure 1).

Remark 2.8. If $p_{ij} = p_j$, $c_{ij} = c_j$ and $m_i = m$ for $1 \leq j \leq m_i$, $1 \leq i \leq m$, then the Moran set $E(\omega)$ reduces to the self-similar set E , and the Moran measure μ_ω reduces to the self-similar measure μ generated by the contractive similarity mappings S_1, S_2, \dots, S_m with the similarity ratios c_1, c_2, \dots, c_m , for which Graf and Luschgy determined the quantization dimension (cf. [8, 10]).

3. MAIN RESULT

The relationship between the quantization dimension function D_r and the temperature function $\beta(q)$ for the probability measure μ_ω is given by the following theorem. For a graphical description see Figure 1.

Theorem 3.1. *Let μ_ω be the Moran measure on the Moran set $E(\omega)$. Let $\beta = \beta(q)$ be the temperature function of the thermodynamic formalism. For each $r \in (0, +\infty)$ choose q_r such that $\beta(q_r) = r q_r$. Then the quantization dimension for the probability measure μ_ω is given by*

$$D_r = \frac{\beta(q_r)}{1 - q_r}.$$

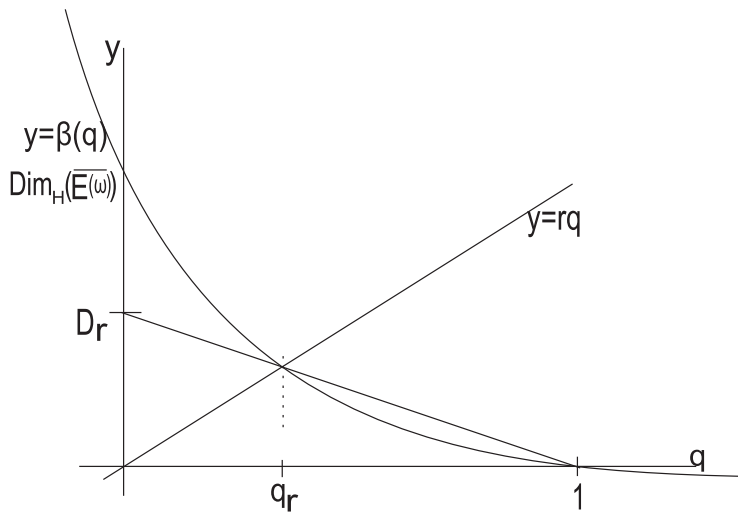


FIGURE 1. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y -intercept of the line through this point and the point $(1, 0)$.

Lemma 3.2. *Let $0 < r < +\infty$ be fixed. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that*

$$\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} = 0.$$

Proof. By Equation (3), we have

$$P(t, rt) = \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^t.$$

Proposition 2.5 says that $P(t, rt)$ is continuous, convex and strictly decreasing, and hence there exists a unique $t \in \mathbb{R}$ such that $P(t, rt) = 0$.

It $t = 0$, then $P(0, 0) = \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} 1 = \sum_{i=1}^m \eta_i \log m_i > 0$; if $t = 1$, then

$$\begin{aligned} P(1, r1) &= \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r) \\ &\leq \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{i_j} c_{\max}^r \\ &= \sum_{i=1}^m \eta_i (\log \sum_{j=1}^{m_i} p_{i_j} + \log c_{\max}^r) \\ &= r \log c_{\max} < 0, \end{aligned}$$

where $c_{\max} = \max\{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq m_i\}$. Therefore by the intermediate value theorem, the unique $t \in \mathbb{R}$ for which $P(t, rt) = 0$ must lie between 0 and 1. Then $\kappa_r = \frac{rt}{1-t}$ satisfies the conclusion of the lemma. \square

We call $\Gamma \subset D$ a *finite maximal antichain* if Γ is a finite set of words in D , such that every sequence in D_∞ is an extension of some word in Γ , but no word of Γ is an extension of another word in Γ . Of course, this requires that the index set $\{1, 2, \dots, m_i : 1 \leq i \leq m\}$ is finite. We will make this assumption in the remainder of this paper. By $|\Gamma|$ we denote the cardinality of Γ . Note that from the definition of Γ it follows that a finite maximal antichain does not contain the empty word \emptyset as all words are extensions of \emptyset .

Remark 3.3. If Γ is a finite maximal antichain and $n = \max\{|\sigma| : \sigma \in \Gamma\}$, then every word $\sigma \in D$ with $|\sigma| \geq n$ is an extension of some word in Γ .

Lemma 3.4. *Let Γ be a finite maximal antichain. Then $\mu_\omega = \sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}$.*

Proof. For $\sigma \in D_k$ let us write $E_\sigma(\omega) = S_\sigma(E(\omega))$, which is called a cylinder set in $E(\omega)$ of length $k \geq 0$, where S_σ are the similarity mappings that arise in the construction of $E(\omega)$. The Borel σ -algebra on $E(\omega)$ is generated by all sets of the form $E_\sigma(\omega)$ for $\sigma \in D$.

Let $n \in \mathbb{N}$ and $n \geq \max\{|\sigma| : \sigma \in \Gamma\}$. As the Moran set $E(\omega)$ satisfies the invariance equality (1), it is enough to prove that for any cylinder set $E_\tau(\omega)$ of length n ,

$$\mu_\omega(E_\tau(\omega)) = \sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}(E_\tau(\omega)).$$

Since Γ is a finite maximal antichain, for $\tau \in D_n$ there exists $x \in \Gamma$ such that $\tau = xy$ for some $y \in D$. Then $E_\tau(\omega) = E_{xy}(\omega) = S_{xy}(E(\omega)) = S_x(S_y(E(\omega))) = S_x(E_y(\omega))$, and

$$\begin{aligned} & \sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}(E_\sigma(\omega)) \\ &= p_x \mu_\omega \circ S_x^{-1}(S_x(E_y(\omega))) = p_x \mu_\omega(E_y(\omega)) = p_x p_y = p_\tau = \mu_\omega(E_\tau(\omega)), \end{aligned}$$

and thus the proof is complete. \square

Lemma 3.5. *Let $0 < r < +\infty$ and κ_r be as in Lemma 3.2 and let Γ be a finite maximal antichain. Then $\sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq 1$.*

Proof. For any fixed positive integer n by Lemma 3.2, we have

$$(4) \quad 0 = \lim_{p \rightarrow \infty} \frac{1}{np} \log \sum_{\sigma \in D_{np}} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} = \lim_{p \rightarrow \infty} \frac{1}{np} \sum_{i=1}^m \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right)^{\|\sigma\|_{np} \|a_i\}}.$$

Note that $\|\sigma\|_{np} \|a_i\| = \|\sigma\|_n \|a_i\| + \|T^p(\sigma)\|_n \|a_i\| + \dots + \|T^{(n-1)p}(\sigma)\|_n \|a_i\|$. Let

$$L := L(n, p) = \min \left\{ 1, \frac{\|T^p(\sigma)\|_n \|a_i\|}{\|\sigma\|_n \|a_i\|}, \dots, \frac{\|T^{(n-1)p}(\sigma)\|_n \|a_i\|}{\|\sigma\|_n \|a_i\|} \right\},$$

and then $\|\sigma|_{np}\|_{a_i} \geq pL\|\sigma|_n\|_{a_i}$. Hence from (4) we have

$$\begin{aligned} 0 &= \lim_{p \rightarrow \infty} \frac{\|\sigma|_{np}\|_{a_i}}{np} \sum_{i=1}^m \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right) \\ &\geq \lim_{p \rightarrow \infty} \frac{pL\|\sigma|_n\|_{a_i}}{np} \sum_{i=1}^m \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right) \\ &= L \sum_{i=1}^m \frac{\|\sigma|_n\|_{a_i}}{n} \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right) = L \sum_{i=1}^m \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right)^{\frac{\|\sigma|_n\|_{a_i}}{n}} \\ &= L \log \prod_{i=1}^m \left(\sum_{j=1}^{m_i} (p_{i_j} c_{i_j}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right)^{\frac{\|\sigma|_n\|_{a_i}}{n}} = L \log \sum_{\sigma \in D_n} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}}, \end{aligned}$$

and so,

$$(5) \quad \sum_{\sigma \in D_n} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq 1.$$

Now let $\ell = \min\{|\sigma| : \sigma \in \Gamma\}$. Since Γ does not contain the empty word, we have $\ell \geq 1$. Then for each $\sigma \in \Gamma$ there exists $\tau(\sigma) \in D$ with $|\tau(\sigma)| = \ell$ and $\tau(\sigma) \prec \sigma$; i.e., there exists $x(\sigma) \in D$ such that $\sigma = \tau(\sigma)x(\sigma)$. Since $p_\sigma \leq p_{\tau(\sigma)}$ and $c_\sigma \leq c_{\tau(\sigma)}$ for $\sigma \in \Gamma$, using (5) we have

$$\sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq \sum_{\sigma \in \Gamma} (p_{\tau(\sigma)} c_{\tau(\sigma)}^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq \sum_{\tau \in D_\ell} (p_\tau c_\tau^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq 1. \quad \square$$

Lemma 3.6. *Let $\Gamma \subset D$ be a finite maximal antichain, $n \in \mathbb{N}$ with $n \geq |\Gamma|$, and $0 < r < +\infty$. Then $V_{n,r}(\mu_\omega) \leq \inf \left\{ \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r V_{n_\sigma,r}(\mu_\omega) : 1 \leq n_\sigma, \sum_{\sigma \in \Gamma} n_\sigma \leq n \right\}$.*

Proof. Suppose $n_\sigma \geq 1$ for each $\sigma \in \Gamma$, and $\sum_{\sigma \in \Gamma} n_\sigma \leq n$. For each $\sigma \in \Gamma$ let α_σ be an n_σ -optimal set for $V_{n_\sigma,r}(\mu_\omega)$. Since $|\bigcup_{\sigma \in \Gamma} S_\sigma(\alpha_\sigma)| \leq n$, $\mu_\omega = \sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}$, we have

$$\begin{aligned} V_{n,r}(\mu_\omega) &\leq \int d(x, \bigcup_{\sigma \in \Gamma} S_\sigma(\alpha_\sigma))^r d\mu_\omega(x) \\ &= \sum_{\sigma \in \Gamma} p_\sigma \int d(x, \bigcup_{\sigma \in \Gamma} S_\sigma(\alpha_\sigma))^r d(\mu_\omega \circ S_\sigma^{-1})(x) \\ &\leq \sum_{\sigma \in \Gamma} p_\sigma \int d(S_\sigma(x), S_\sigma(\alpha_\sigma))^r d\mu_\omega(x) \\ &= \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r \int d(x, \alpha_\sigma)^r d\mu_\omega(x) \\ &= \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r V_{n_\sigma,r}(\mu_\omega), \end{aligned}$$

which implies the lemma. □

Proposition 3.7. *Let $0 < r < +\infty$ and let κ_r be as in Lemma 3.2. Then $\limsup_{n \rightarrow \infty} n e_{n,r}^{\kappa_r} < +\infty$.*

Proof. Let $q_r = \frac{\kappa_r}{r+\kappa_r}$. Then $\beta(q_r) = r q_r$. Choose ϵ_0 so that $0 < \epsilon_0 < 1$. Fix $m \in \mathbb{N}$. Choose any $n \in \mathbb{N}$ so that $\frac{m}{n} < \epsilon_0$, and set $\epsilon = \epsilon_0 \frac{m}{n}$, and then $0 < \epsilon < 1$. Let $\Gamma = \Gamma(\epsilon) = \{\sigma \in D : (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} < \epsilon \leq (p_{\sigma^-} c_{\sigma^-}^r)^{\frac{\kappa_r}{r+\kappa_r}}\}$, where σ^- is the word obtained by deleting the last letter of σ ; i.e., if $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$, then $\sigma^- = (\sigma_1, \sigma_2, \dots, \sigma_{k-1})$ for $k \geq 2$, and $\sigma^- = \emptyset$ if $k = 1$. Since the index set $\{1, 2, \dots, m_i : 1 \leq i \leq m\}$ is finite, Γ is a finite maximal antichain.

Hence by the previous lemma we have

$$\begin{aligned} V_{n,r}(\mu_\omega) &\leq \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r V_{m,r}(\mu_\omega) \\ &= \sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} (p_\sigma c_\sigma^r)^{\frac{r}{r+\kappa_r}} V_{m,r}(\mu_\omega) \\ &< \sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu_\omega) \\ &\leq \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu_\omega) \quad (\text{by Lemma 3.5}) \\ &= \epsilon_0^{\frac{r}{\kappa_r}} \left(\frac{m}{n}\right)^{\frac{r}{\kappa_r}} V_{m,r}(\mu_\omega) \\ &\implies n V_{n,r}^{\frac{\kappa_r}{r}}(\mu_\omega) \leq \epsilon_0 m V_{m,r}^{\frac{\kappa_r}{r}}(\mu_\omega). \end{aligned}$$

Since the inequality holds for all but a finite number of n we have

$$\limsup_{n \rightarrow \infty} n e_{n,r}^{\kappa_r} \leq \epsilon_0 m e_{m,r}^{\kappa_r} < +\infty. \quad \square$$

Lemma 3.8. *Let $\Gamma \subset D$ be a finite maximal antichain. Then there exists $n_0 = n_0(\Gamma)$ such that for every $n \geq n_0$ there exists a set of positive integers $\{n_\sigma := n_\sigma(n)\}_{\sigma \in \Gamma}$ such that $\sum_{\sigma \in \Gamma} n_\sigma \leq n$ and*

$$u_{n,r} \geq \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r u_{n_\sigma,r}.$$

Proof. Let U be the open set from the strong open set condition. Then there exists $\tau \in D$ such that $S_\tau(J) \subset U$. Note that here the similarity mappings S_σ for $\sigma \in D$ are the similarity mappings as defined in (M2) of the Moran set construction. Let $\epsilon = d(S_\tau(J), U^c)$ and $\lambda = \min\{c_\sigma : \sigma \in \Gamma\}$. Then for $\sigma \in \Gamma$ we have $d(S_\sigma S_\tau(J), S_\sigma(U^c)) \geq \lambda d(S_\tau(J), U^c) = \lambda \epsilon$, which implies $d(x, U^c) \geq d(x, S_\sigma(U^c)) \geq \lambda \epsilon$ for any $x \in S_\sigma(S_\tau(J))$. For each n , let α_n be an n -optimal set for $u_{n,r}$ and let $\delta_n = \max\{d(x, \alpha_n \cup U^c) : x \in E(\omega)\}$. Since $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ we can choose n_0 such that $\delta_n < \lambda \epsilon$ for all $n \geq n_0$. Suppose $n \geq n_0$ and $x \in S_\sigma(S_\tau(E(\omega)))$. Then there exists $a \in \alpha_n \cup U^c$ such that $d(x, \alpha_n \cup U^c) = d(x, a) \leq \delta_n < \lambda \epsilon$, and so $a \in S_\sigma(U)$. Therefore, letting $\alpha_{n_\sigma} = \alpha_n \cap S_\sigma(U)$, we have $n_\sigma := |\alpha_{n_\sigma}| \geq 1$

and $\sum_{\sigma \in \Gamma} n_\sigma \leq n$. It can be proved that for any $x \in E(\omega)$, $d(S_\sigma(x), \alpha_n \cup U^c) \geq d(S_\sigma(x), \alpha_n \cup S_\sigma(U^c))$ and $d(S_\sigma(x), \alpha_n \cup S_\sigma(U^c)) = d(S_\sigma(x), \alpha_{n_\sigma} \cup S_\sigma(U^c))$. Hence,

$$\begin{aligned} u_{n,r} &= \int d(x, \alpha_n \cup U^c)^r d\mu_\omega(x) \\ &= \sum_{\sigma \in \Gamma} p_\sigma \int d(S_\sigma(x), \alpha_n \cup U^c)^r d\mu_\omega(x) \\ &\geq \sum_{\sigma \in \Gamma} p_\sigma \int d(S_\sigma x, \alpha_n \cup S_\sigma(U^c))^r d\mu_\omega(x) \\ &= \sum_{\sigma \in \Gamma} p_\sigma \int d(S_\sigma x, \alpha_{n_\sigma} \cup S_\sigma(U^c))^r d\mu_\omega(x) \\ &= \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r \int d(x, S_\sigma^{-1}(\alpha_{n_\sigma}) \cup U^c)^r d\mu_\omega(x) \\ &\geq \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r u_{n_\sigma,r}. \end{aligned} \quad \square$$

Proposition 3.9. *Let the Moran set construction satisfy the strong open set condition and let $0 < r < +\infty$. Moreover, let κ_r be as in Lemma 3.2 and let $0 < \ell < \kappa_r$. Then $\liminf_{n \rightarrow \infty} n c_{n,r}^\ell > 0$.*

Proof. Since $0 < \ell < \kappa_r$ and κ_r is unique for which

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\sigma \in D_n} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r + \kappa_r}} = 0,$$

we have

$$\sum_{\sigma \in D_n} (p_\sigma c_\sigma^r)^{\frac{\ell}{r + \ell}} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Choose $t \in \mathbb{N}$ so that $\sum_{\sigma \in D_t} (p_\sigma c_\sigma^r)^{\frac{\ell}{r + \ell}} \geq 1$. Let $\Gamma = \{\sigma \in D : |\sigma| = t\}$. Then Γ is a finite maximal antichain. The previous lemma yields an n_0 , and for $n \geq n_0$ the numbers $\{n_\sigma := n_\sigma(n)\}_{\sigma \in \Gamma}$ which satisfy the conclusion of that lemma. Set $c = \min\{n^{r/\ell} u_{n,r} : n \leq n_0\}$. Clearly each $u_{n,r} > 0$ and hence $c > 0$. Suppose $n \geq n_0$ and $k^{r/\ell} u_{k,r} \geq c$ for all $k < n$. Using the previous lemma we have

$$\begin{aligned} n^{r/\ell} u_{n,r} &\geq n^{r/\ell} \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r u_{n_\sigma,r} \\ &= n^{r/\ell} \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r (n_\sigma(n))^{-r/\ell} (n_\sigma(n))^{r/\ell} u_{n_\sigma,r} \\ &\geq c \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r \left(\frac{n_\sigma(n)}{n}\right)^{-r/\ell}. \end{aligned}$$

Using Hölder’s inequality (with exponents less than 1) we have

$$n^{r/\ell} u_{n,r} \geq c \left(\sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{\ell/(r+\ell)} \right)^{(1+r/\ell)} \left(\sum_{\sigma \in \Gamma} \left(\frac{n_\sigma(n)}{n}\right)^{(-r/\ell)(-\ell/r)} \right)^{-r/\ell}.$$

By our choice of Γ , which depends only on ℓ and not on n , and the fact that $\sum_{\sigma \in \Gamma} n_{\sigma}(n) \leq n$, we see that $n^{r/\ell} u_{n,r} \geq c$. Hence by induction,

$$\liminf_{n \rightarrow \infty} n u_{n,r}^{\ell/r} \geq c^{\ell/r} > 0, \text{ i.e., } \liminf_{n \rightarrow \infty} n e_{n,r}^{\ell} > 0. \quad \square$$

Proof of Theorem 3.1. From Proposition 11.3 of [8] we know:

(a) If $0 \leq t < \underline{D}_r < s$, then

$$\lim_{n \rightarrow \infty} n e_{n,r}^t = +\infty \text{ and } \liminf_{n \rightarrow \infty} n e_{n,r}^s = 0.$$

(b) If $0 \leq t < \overline{D}_r < s$, then

$$\limsup_{n \rightarrow \infty} n e_{n,r}^t = +\infty \text{ and } \lim_{n \rightarrow \infty} n e_{n,r}^s = 0.$$

From (a) and Proposition 3.9, we have $\ell \leq \underline{D}_r$ whenever $\ell < \kappa_r$. Hence, $\kappa_r \leq \underline{D}_r$. From (b) and Proposition 3.7, we have $\overline{D}_r \leq \kappa_r$. Hence, $\kappa_r \leq \underline{D}_r \leq \overline{D}_r \leq \kappa_r$; i.e., the quantization dimension D_r exists and $D_r = \kappa_r$. Note that for $q_r = \frac{\kappa_r}{r + \kappa_r}$ and $\beta(q_r) = r q_r$ we have $D_r = \frac{\beta(q_r)}{1 - q_r}$. Hence the proof of the theorem.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS-PAN AMERICAN, 1201 WEST UNIVERSITY DRIVE, EDINBURG, TEXAS 78539-2999

E-mail address: roychowdhurymk@utpa.edu