PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 11, November 2010, Pages 4045–4057
S 0002-9939(2010)10406-9
Article electronically published on May 17, 2010

QUANTIZATION DIMENSION FOR SOME MORAN MEASURES

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(Communicated by Bryna Kra)

ABSTRACT. The quantization dimension function for some Moran measures has been determined, and a relationship between the quantization dimension function and the temperature function of the thermodynamic formalism arising in multifractal analysis is established.

1. INTRODUCTION

The term quantization in this paper refers to the idea of estimating a given probability on $\mathbb{R}^d$ with a discrete probability, that is, a 'quantized' version of the probability supported on a finite set. The problems surrounding this idea have been of major interest in signal compression and communications for quite some time (cf. [7]). Following the work of Graf and Luschgy, we define the *quantization dimension* (or perhaps better, the *quantization dimension function*) as follows (cf. [8, 9]): Given a Borel probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error* of order r for μ is defined by

$$e_{n,r} = \inf\{\big(\int d(x,\alpha)^r d\mu(x)\big)^{\frac{1}{r}} : \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \leq n\},$$

where $d(x, \alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r d\mu(x) < \infty$, then there is some set α for which the infimum is achieved (cf. [8]). Graf and Luschgy also defined $e_{n,r}$ where $r = 0$ and $r = +\infty$, but in the paper we only deal with the case $0 < r < +\infty$. The *quantization dimension of order* r for μ is defined to be

$$D_r = D_r(\mu) = \lim_{n\to\infty} \frac{\log n}{-\log e_{n,r}}$$

if the limit exists. If the limit does not exist, then we define $\overline{D}_r$ as the lim sup of the sequence and $\underline{D}_r$ as the lim inf. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $e_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log e_{n,r} \sim \log(\frac{1}{n})^{1/D_r}.$$

Let $S_1, S_2, \cdots, S_N$ be a set of contractive similarity mappings on $\mathbb{R}^d$ with the similarity ratios $s_1, s_2, \cdots, s_N$ for $N \geq 2$. Then for a given probability vector

Received by the editors November 12, 2009 and, in revised form, January 22, 2010.
2010 *Mathematics Subject Classification.* Primary 37A50; Secondary 28A80, 94A34.
Key words and phrases. Moran measure, quantization dimension, temperature function.

$(p_1, p_2, \cdots, p_N)$ there exists a unique probability measure μ (cf. [12]) satisfying the condition

$$\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}.$$

Let the iterated function system $\{S_1, S_2, \cdots, S_N\}$ satisfy the open set condition: there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{i=1}^N S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. Then, Graf and Luschgy showed that the quantization dimension function $D_r := D_r(\mu)$ for the probability measure μ exists and satisfies the following relation (cf. [8, 10]):

$$\sum_{i=1}^N (p_i s_i^r)^{\frac{D_r}{r+D_r}} = 1.$$

Note that from the above relation it is clear that the quantization dimension function for a self-similar probability measure has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis (cf. [5]). Later the result was extended by Lindsay and Mauldin to the F-conformal measures with finitely many conformal mappings (cf. [16]). In the paper [20], the author considered the ergodic measure with bounded distortion on the coding space, determined the quantization dimension function for the image measure via the coding map on a self-similar set, and showed its functional relationship with the temperature function of the thermodynamic formalism. In this paper, we give a generalization of the result of Graf and Luschgy and determine the quantization dimension function for a Moran measure, and its relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis is established.

2. Basic definitions and propositions

Let us write

$$V_{n,r} = \inf\{\int d(x,\alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d,\ \text{card}(\alpha) \leq n\},$$
$$u_{n,r} = \inf\{\int d(x,\alpha \cup U^c)^r d\mu(x) : \alpha \subset \mathbb{R}^d,\ \text{card}(\alpha) \leq n\},$$

where U is a set which comes from the open set condition and U^c denotes the complement of U. We see that

$$u_{n,r}^{1/r} \leq V_{n,r}^{1/r} = e_{n,r}.$$

We call sets $\alpha_n \subset \mathbb{R}^d$, for which the above infimums are achieved, n-optimal sets for $e_{n,r}, V_{n,r}$ or $u_{n,r}$ respectively. As stated above, Graf and Luschgy have shown that n-optimal sets exist when $\int \|x\|^r d\mu(x) < \infty$.

2.1. **Moran set.** Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers, and $\{\Phi_k\}_{k\geq 1}$ be a sequence of positive real vectors with

$$\Phi_k = (c_{k1}, c_{k2}, \cdots, c_{kn_k}),\ \sum_{j=1}^{n_k} c_{kj} \leq 1,\ k \in \mathbb{N}.$$

Let D_0 be the empty set. For $k \geq 1$ write

$$D_{m,k} = \{(i_m, i_{m+1}, \cdots, i_k) : 1 \leq i_j \leq n_j,\ m \leq j \leq k\},$$

and $D_k = D_{1,k}$, $D_\infty = \lim_{k\to\infty} D_k$. Define $D := \bigcup_{k\geq 0} D_k$. Elements of D are called words. For any $\sigma \in D$ if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in D_n$ we write $|\sigma| = n$ to denote the length of σ, and $\sigma|_k := (\sigma_1, \sigma_2, \cdots, \sigma_k)$, $k \leq n$, to denote the truncation of σ to the length k. If $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k$ and $\tau = (\tau_1, \tau_2, \cdots, \tau_m) \in D_{k+1,m}$, then we write $\sigma\tau = \sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m)$ to denote the juxtaposition of $\sigma, \tau \in D$. A word of length zero is called the empty word and is denoted by $\emptyset$. For $\sigma \in D$ and $\tau \in D \cup D_\infty$ we say that τ is an extension of σ, written as $\sigma \prec \tau$, if $\tau|_{|\sigma|} = \sigma$. Let $J \subset \mathbb{R}^d$ be a compact set such that $J = \mathrm{cl}(\mathrm{int}J)$, and $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ be a collection of subsets of $\mathbb{R}^d$. We say that the collection $\mathcal{F}$ fulfills the Moran structure provided it satisfies the following Moran structure conditions (MSC):

$(M1)$ $J_\emptyset = J$.

$(M2)$ For any $\sigma \in D$, J_σ is geometrically similar to J; i.e., there exists a similarity mapping $S_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that $J_\sigma = S_\sigma(J)$.

$(M3)$ For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \cdots, J_{\sigma*n_{k+1}}$ are subsets of J_σ, and $\mathrm{Int}(J_{\sigma*i}) \cap \mathrm{Int}(J_{\sigma*j}) = \emptyset$ for $1 \leq i \neq j \leq n_{k+1}$, where $\mathrm{Int}(A)$ denotes the interior of A.

$(M4)$ For any $k \geq 1$ and $\sigma \in D_{k-1}$, $1 \leq j \leq n_k$, $\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_{kj}$.

Since given $\sigma = (\sigma_i)_{i=1}^\infty \in D_\infty$ the diameters of the compact sets $J_{\sigma|_k}$, $k \geq 1$ converge to zero and since they form a descending family, the set

$$\bigcap_{k=0}^{\infty} J_{\sigma|_k}$$

is a singleton; and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : D_\infty \to J$. The main object of our interest is the limit set

$$E = \pi(D_\infty) = \bigcup_{\sigma\in D_\infty} \bigcap_{k=1}^{\infty} J_{\sigma|_k}.$$

The set $E := E(\mathcal{F})$ is called the Moran set associated with the collection $\mathcal{F}$. Let $\mathcal{F}_k = \{J_\sigma : \sigma \in D_k\}$ and $\mathcal{F} = \bigcup_{k\geq 0} \mathcal{F}_k$. The elements of $\mathcal{F}_k$ are called the basic elements of order k, and the elements of $\mathcal{F}$ are called the basic elements of the Moran set E. Note that the set E satisfies the invariance equality

$$E = \bigcup_{j_k=1}^{n_k} S_{kj_k}(E), \tag{1}$$

where $\{S_{k1}, S_{k2}, \cdots, S_{kn_k}\}$ $(k \geq 1)$ are the similarity mappings that arise in the Moran structure condition $(M2)$. Let us assume that the collection $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ satisfies the *open set condition* (OSC): there exists a bounded nonempty open set $U \subset J$ such that for $\sigma \in D_k\,(k \geq 0)$, $\bigcup_{i=1}^{n_{k+1}} S_{\sigma i}(U) = \bigcup_{i=1}^{n_{k+1}} S_i(S_\sigma(U)) \subset U$ and $S_{\sigma i}(U) \cap S_{\sigma j}(U) = \emptyset$ for each pair $i, j \in \{1, 2, \cdots, n_{k+1}\}$ with $i \neq j$, where the similarity mappings S_σ are the similarity mappings as in $(M2)$ arising in the Moran construction. Furthermore, the collection satisfies the *strong open set condition* (SOSC) if U can be chosen such that $U \cap E \neq \emptyset$, where $E := E(\mathcal{F})$ is the Moran set.

Suppose that the set J and the sequences $\{n_k\}$ and $\{\Phi_k\}$ are given. We denote by $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ the class of Moran sets satisfying the MSC $(M1)-(M4)$. We call $\mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ a Moran class associated with the triplet $(J, \{n_k\}, \{\Phi_k\})$. For more details about the Moran sets one is referred to [13].

Let $A := \{a_1, a_2, \cdots, a_m\}$ be a finite set of distinct positive integers and define

$$A^{\mathbb{N}} := \{(t_j)_{j=1}^{\infty} \in D_\infty : t_j \in A\}.$$

For $\omega = (s_1, s_2, \cdots) \in A^{\mathbb{N}}$ let $\omega|_k = (s_1, s_2, \cdots, s_k)$, and let $\|\omega|_k\|_{a_i} := \#\{s_j = a_i : s_j \text{ appears in } \omega|_k\}$ denote the times that the element a_i appears in the word $\omega|_k$. For $\omega \in A^{\mathbb{N}}$ let $\lim_{k\to\infty} \frac{\|\omega|_k\|_{a_i}}{k} = \eta_i > 0$ for every $a_i \in A$. Then we say that ω has the frequency vector $\eta = (\eta_1, \eta_2, \cdots, \eta_m)$. Note that $\sum_{i=1}^m \|\omega|_k\|_{a_i} = k$ and $\sum_{i=1}^m \eta_i = 1$. For $\eta = (\eta_1, \eta_2, \cdots, \eta_m)$ define

$$A_\eta^{\mathbb{N}} := \{t = (t_j)_{j\geq 1} : t_j \in A, \lim_{k\to\infty} \frac{\|t|_k\|_{a_i}}{k} = \eta_i, 1 \leq i \leq m\}.$$

Suppose $m_i \in \mathbb{N}$, $1 \leq i \leq m$ and let $\Phi_i = (c_{i1}, c_{i2}, \cdots, c_{im_i})$ be a positive real vector with $\sum_{j=1}^{m_i} c_{ij} \leq 1$. For $\omega \in A_\eta^{\mathbb{N}}$ and $E \in \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$, if $s_k = a_i$, $k \geq 1$, we take $n_k = m_i$, $\Phi_k := \Phi_i = (c_{i1}, c_{i2}, \cdots, c_{im_i})$. Then we obtain a class of Moran sets associated with $\omega \in A_\eta^{\mathbb{N}}$. Let us denote such a Moran set by $E(\omega)$.

Note 2.2. Let $\omega = (s_1, s_2, \cdots) \in A^{\mathbb{N}}$. For the rest of the paper we will keep our Moran set $E(\omega)$ fixed and assume its structure satisfies the strong open set condition as defined earlier.

2.3. **Moran measure.** Let J_σ be a basic element of order k of the Moran set $E(\omega)$. Note that by construction, if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k$, $k \geq 1$, and $s_n = a_i \in A$ $(1 \leq i \leq m)$, then $\sigma_n \in \{1, 2, \cdots, m_i\}$ $(1 \leq n \leq k)$. For such a $\sigma \in D_k$ $(k \geq 1)$ let us define $\sigma(a_1)\sigma(a_2)\cdots\sigma(a_m)$ as follows: Fix i $(1 \leq i \leq m)$ and calculate $\|\omega|_k\|_{a_i}$. If $\|\omega|_k\|_{a_i} = 0$ assume $\sigma(a_i) = \emptyset$. If $\|\omega|_k\|_{a_i} \neq 0$, let $i_1, i_2, \cdots, i_{\|\omega|_k\|_{a_i}}$ be the positions where the symbol a_i appears in the word $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)$. Note that $\sigma_{i_j} \in \{1, 2, \cdots, m_i\}$ for $1 \leq j \leq \|\omega|_k\|_{a_i}$. Write $\sigma(a_i) = (\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_{\|\omega|_k\|_{a_i}}})$. Then $\sigma(a_1)\sigma(a_2)\cdots\sigma(a_m)$ is another arrangement of $(\sigma_1, \sigma_2, \cdots, \sigma_k)$. For every $a_i \in A$, let $P_{a_i} = (p_{i_1}, p_{i_2}, \cdots, p_{i_{m_i}})$ $(1 \leq i \leq m)$ be probability vectors, i.e., $p_{i_j} > 0$ and $\sum_{j=1}^{m_i} p_{i_j} = 1$ $(1 \leq i \leq m)$. Define $p_{\sigma(a_i)} = p_{i\sigma_{i_1}} p_{i\sigma_{i_2}} \cdots p_{i\sigma_{i_{\|\omega|_k\|_{a_i}}}}$, $1 \leq i \leq m$, where $p_{i\sigma_{i_j}} \in \{p_{i_1}, p_{i_2}, \cdots, p_{i_{m_i}}\}$ for $1 \leq j \leq \|\omega|_k\|_{a_i}$. If $\sigma(a_i) = \emptyset$ assume that $p_{\sigma(a_i)} = 1$. We define

$$p_\sigma := p_{\sigma(a_1)} p_{\sigma(a_2)} \cdots p_{\sigma(a_m)}.$$

It is easy to see that $\sum_{\sigma\in D_k} p_\sigma = \sum_{\sigma\in D_k} \prod_{i=1}^m p_{\sigma(a_i)} = 1$ for any $k \geq 1$.

Suppose μ is a mass distribution on $E(\omega)$, denoted by μ_ω, such that for any J_σ $(\sigma \in D_k)$, $\mu(J_\sigma) = p_{\sigma(a_1)} p_{\sigma(a_2)} \cdots p_{\sigma(a_m)}$ and $\mu(\sum_{\sigma\in D_k} J_\sigma) = 1$. Measure μ_ω is a Moran measure and $\text{supp}\mu_\omega = E(\omega)$, which is an extension of a self-similar measure in [12]. Note that for $E(\omega)$, there exists $D_{E(\omega)} \subset D_\infty$ such that $\pi(D_{E(\omega)}) = E(\omega)$. Define the left shift map $T : D_{E(\omega)} \to D_{E(\omega)}$ by $T(\sigma) = T(\sigma_1, \sigma_2, \cdots) = (\sigma_2, \sigma_3, \cdots)$ for $\sigma = (\sigma_1, \sigma_2, \cdots) \in D_{E(\omega)}$. For each $i \in \{1, 2, \cdots, m\}$, $i_j \in \{1, 2, \cdots, m_i\}$, we define a right shift map $\tau_{i_j} : D_{E(\omega)} \to D_{E(\omega)}$ as follows: $\tau_{i_j}(\sigma) = i_j\sigma$, for all $\sigma = \sigma_1\sigma_2\cdots \in D_{E(\omega)}$. For the probability vectors $P_{a_1}, P_{a_2}, \cdots, P_{a_m}$ we denote by ν_ω the corresponding product measure on $D_{E(\omega)}$; then $\mu_\omega = \nu_\omega \circ \pi^{-1}$. Moreover, we have

$$\nu_\omega = \sum_{j=1}^{m_i} p_{i_j} \nu_\omega \circ {\tau_{i_j}}^{-1} \text{ and } \mu_\omega = \sum_{j=1}^{m_i} p_{i_j} \mu_\omega \circ S_{ij}^{-1}, \tag{2}$$

where S_{ij} is the similarity mapping corresponding to the similarity ratio c_{ij} for the Moran set $E(\omega)$ ($1 \leq i \leq m$ and $1 \leq j \leq m_i$). For this measure μ_ω we will determine the quantization dimension and its relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis.

2.4. **Topological pressure.** For $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k$ let us write

$$c_\sigma = \begin{cases} c_{1\sigma_1} c_{2\sigma_2} \cdots c_{k\sigma_k} & \text{if } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases}$$

For $q, t \in \mathbb{R}$, let us now define the function

$$Z_n(q,t) = \sum_{\sigma \in D_n} p_\sigma^q c_\sigma^t = \prod_{i=1}^m \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right)^{\|\omega|_n\|_{a_i}} .$$

For $n \in \mathbb{N}$ let us write $c_n(q,t) = \frac{\log Z_n(q,t)}{n}$. Then for $n, p \in \mathbb{N}$ with $n > p \geq 1$ we have

$$\begin{aligned} |c_n(q,t) - c_p(q,t)| &= \left| \sum_{i=1}^m \frac{1}{n} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right)^{\|\omega|_n\|_{a_i}} - \sum_{i=1}^m \frac{1}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right)^{\|\omega|_p\|_{a_i}} \right| \\ &= \left| \sum_{i=1}^m \frac{\|\omega|_n\|_{a_i}}{n} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right) - \sum_{i=1}^m \frac{\|\omega|_p\|_{a_i}}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right) \right| \\ &\leq \left| \sum_{i=1}^m \frac{\|\omega|_n\|_{a_i}}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right) - \sum_{i=1}^m \frac{\|\omega|_p\|_{a_i}}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right) \right| \\ &= \left| \sum_{i=1}^m \frac{\|s_{p+1} s_{p+2} \cdots s_n\|_{a_i}}{p} \log \left(\sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t \right) \right|, \end{aligned}$$

which tends to zero as p tends to infinity. Hence the sequence $\{c_n(q,t)\}$ is a Cauchy sequence and so by the 'Cauchy convergence criteria' $\{c_n(q,t)\}_{n=1}^\infty$ is a convergent sequence. Let us now define the function $P(q,t)$ as follows:

$$P(q,t) = \lim_{n\to\infty} c_n(q,t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma \in D_n} p_\sigma^q c_\sigma^t = \sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{i_j}^q c_{ij}^t. \tag{3}$$

The function $P(q,t)$ is called the *topological pressure* corresponding to the given Moran construction.

The following proposition states the well-known properties of the function $P(q,t)$ (cf. [6, 18]).

Proposition 2.5. *(i)* $P(q,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(ii) $P(q,t)$ *is strictly decreasing in each variable separately.*

(iii) For fixed q *we have*

$$\lim_{t\to+\infty} P(q,t) = -\infty \text{ and } \lim_{t\to-\infty} P(q,t) = +\infty.$$

(iv) $P(q,t)$ *is convex: if* $q_1, q_2, t_1, t_2 \in \mathbb{R}$, $a_1, a_2 \geq 0, a_1 + a_2 = 1$, *then*

$$P(a_1 q_1 + a_2 q_2, a_1 t_1 + a_2 t_2) \leq a_1 P(q_1, t_1) + a_2 P(q_2, t_2).$$

Now for fixed q, the function $P(q,t)$ is a continuous function of t. Its value ranges from $-\infty$ (when $t \to +\infty$) to $+\infty$ (when $t \to -\infty$). Therefore, by the intermediate value theorem there is a real number β such that $P(q,\beta) = 0$. The solution β is unique, since $P(q,\cdot)$ is strictly decreasing. This defines β implicitly as a function of q: for each q there is a unique $\beta = \beta(q)$ such that $P(q,\beta(q)) = 0$.

The following proposition gives the well-known properties of the function $\beta(q)$ (cf. [6, 18]).

Proposition 2.6. *Let $\beta = \beta(q)$ be defined by $P(q,\beta(q)) = 0$. Then*

(i) β is a continuous function of the real variable q.

(ii) β is strictly decreasing: if $q_1 < q_2$, then $\beta(q_1) > \beta(q_2)$.

(iii) $\lim_{q\to-\infty} \beta(q) = +\infty$ and $\lim_{q\to+\infty} \beta(q) = -\infty$.

(iv) β is a convex function: if $q_1, q_2, a_1, a_2 \in \mathbb{R}$ with $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$, then

$$\beta(a_1q_1 + a_2q_2) \leq a_1\beta(q_1) + a_2\beta(q_2).$$

The function $\beta(q)$ is sometimes denoted by $T(q)$ and called the *temperature function*. A more general discussion of this function can be found in [11], where our $\beta(q)$ function would correspond to $-\tau(q)$ in their notation.

Remark 2.7. Note that

$$\sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} c_{ij}^{\beta(0)} = \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} p_{i_j}^0 c_{ij}^{\beta(0)} = P(0,\beta(0)) = 0.$$

Hence, $\beta(0)$ gives the Hausdorff dimension $\mathrm{Dim}_H(E(\omega))$ of the Moran set $E(\omega)$ (cf. [13]). Note that

$$P(1,0) = \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} p_{i_j} = \sum_{i=1}^{m} \eta_i \log 1 = 0,$$

and hence $\beta(1) = 0$ (see Figure 1).

Remark 2.8. If $p_{i_j} = p_j$, $c_{ij} = c_j$ and and $m_i = m$ for $1 \leq j \leq m_i$, $1 \leq i \leq m$, then the Moran set $E(\omega)$ reduces to the self-similar set E, and the Moran measure μ_ω reduces to the self-similar measure μ generated by the contractive similarity mappings $S_1, S_2, \cdots, S_m$ with the similarity ratios $c_1, c_2, \cdots, c_m$, for which Graf and Luschgy determined the quantization dimension (cf. [8, 10]).

3. Main result

The relationship between the quantization dimension function D_r and the temperature function $\beta(q)$ for the probability measure μ_ω is given by the following theorem. For a graphical description see Figure 1.

Theorem 3.1. *Let μ_ω be the Moran measure on the Moran set $E(\omega)$. Let $\beta = \beta(q)$ be the temperature function of the thermodynamic formalism. For each $r \in (0,+\infty)$ choose q_r such that $\beta(q_r) = rq_r$. Then the quantization dimension for the probability measure μ_ω is given by*

$$D_r = \frac{\beta(q_r)}{1-q_r}.$$

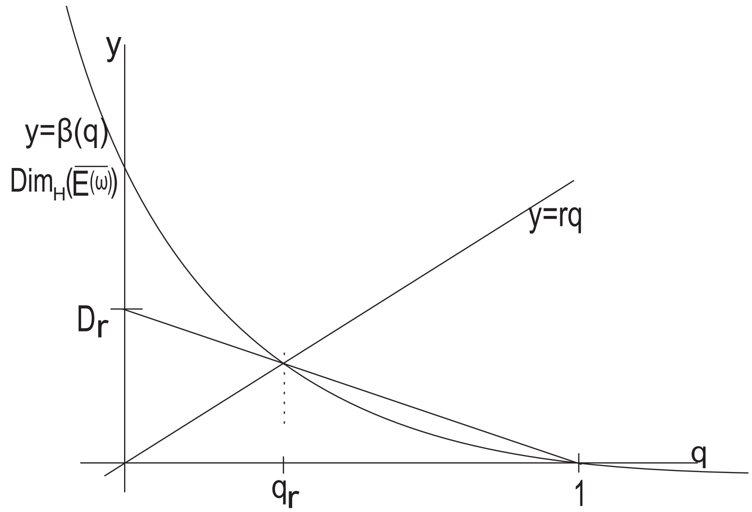

FIGURE 1. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y-intercept of the line through this point and the point (1, 0).

Lemma 3.2. *Let $0 < r < +\infty$ be fixed. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that*

$$\sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} = 0.$$

Proof. By Equation (3), we have

$$P(t, rt) = \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^t.$$

Proposition 2.5 says that $P(t, rt)$ is continuous, convex and strictly decreasing, and hence there exists a unique $t \in \mathbb{R}$ such that $P(t, rt) = 0$.

It $t = 0$, then $P(0, 0) = \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} 1 = \sum_{i=1}^{m} \eta_i \log m_i > 0$; if $t = 1$, then

$$\begin{aligned} P(1, r1) &= \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r) \\ &\leq \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{m_i} p_{i_j} c_{\max}^r \\ &= \sum_{i=1}^{m} \eta_i (\log \sum_{j=1}^{m_i} p_{i_j} + \log c_{\max}^r) \\ &= r \log c_{\max} < 0, \end{aligned}$$

where $c_{\max} = \max\{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq m_i\}$. Therefore by the intermediate value theorem, the unique $t \in \mathbb{R}$ for which $P(t, rt) = 0$ must lie between 0 and 1. Then $\kappa_r = \frac{rt}{1-t}$ satisfies the conclusion of the lemma. □

We call $\Gamma \subset D$ a *finite maximal antichain* if Γ is a finite set of words in D, such that every sequence in D_∞ is an extension of some word in Γ, but no word of Γ is an extension of another word in Γ. Of course, this requires that the index set $\{1, 2, \cdots, m_i : 1 \leq i \leq m\}$ is finite. We will make this assumption in the remainder of this paper. By $|\Gamma|$ we denote the cardinality of Γ. Note that from the definition of Γ it follows that a finite maximal antichain does not contain the empty word $\emptyset$ as all words are extensions of $\emptyset$.

Remark 3.3. If Γ is a finite maximal antichain and $n = \max\{|\sigma| : \sigma \in \Gamma\}$, then every word $\sigma \in D$ with $|\sigma| \geq n$ is an extension of some word in Γ.

Lemma 3.4. *Let Γ be a finite maximal antichain. Then $\mu_\omega = \sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}$.*

Proof. For $\sigma \in D_k$ let us write $E_\sigma(\omega) = S_\sigma(E(\omega))$, which is called a cylinder set in $E(\omega)$ of length $k \geq 0$, where S_σ are the similarity mappings that arise in the construction of $E(\omega)$. The Borel σ-algebra on $E(\omega)$ is generated by all sets of the form $E_\sigma(\omega)$ for $\sigma \in D$.

Let $n \in \mathbb{N}$ and $n \geq \max\{|\sigma| : \sigma \in \Gamma\}$. As the Moran set $E(\omega)$ satisfies the invariance equality (1), it is enough to prove that for any cylinder set $E_\tau(\omega)$ of length n,

$$\mu_\omega(E_\tau(\omega)) = \sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}(E_\tau(\omega)).$$

Since Γ is a finite maximal antichain, for $\tau \in D_n$ there exists $x \in \Gamma$ such that $\tau = xy$ for some $y \in D$. Then $E_\tau(\omega) = E_{xy}(\omega) = S_{xy}(E(\omega)) = S_x(S_y(E(\omega))) = S_x(E_y(\omega))$, and

$$\begin{aligned}&\sum_{\sigma \in \Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}(E_\sigma(\omega)) \\ &\quad = p_x \mu_\omega \circ S_x^{-1}(S_x(E_y(\omega))) = p_x \mu_\omega(E_y(\omega)) = p_x p_y = p_\tau = \mu_\omega(E_\tau(\omega)),\end{aligned}$$

and thus the proof is complete. □

Lemma 3.5. *Let $0 < r < +\infty$ and κ_r be as in Lemma 3.2 and let Γ be a finite maximal antichain. Then $\sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq 1$.*

Proof. For any fixed positive integer n by Lemma 3.2, we have

(4)
$$0 = \lim_{p \to \infty} \frac{1}{np} \log \sum_{\sigma \in D_{np}} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} = \lim_{p \to \infty} \frac{1}{np} \sum_{i=1}^{m} \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right)^{\|\sigma|_{np}\|_{a_i}}.$$

Note that $\|\sigma|_{np}\|_{a_i} = \|\sigma|_n\|_{a_i} + \|T^p(\sigma)|_n\|_{a_i} + \cdots + \|T^{(n-1)p}(\sigma)|_n\|_{a_i}$. Let

$$L := L(n, p) = \min\left\{1, \frac{\|T^p(\sigma)|_n\|_{a_i}}{\|\sigma|_n\|_{a_i}}, \cdots, \frac{\|T^{(n-1)p}(\sigma)|_n\|_{a_i}}{\|\sigma|_n\|_{a_i}}\right\},$$

and then $\|\sigma|_{np}\|_{a_i} \geq pL\|\sigma|_n\|_{a_i}$. Hence from (4) we have

$$\begin{aligned}
0 &= \lim_{p\to\infty} \frac{\|\sigma|_{np}\|_{a_i}}{np} \sum_{i=1}^{m} \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right) \\
&\geq \lim_{p\to\infty} \frac{pL\|\sigma|_n\|_{a_i}}{np} \sum_{i=1}^{m} \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right) \\
&= L \sum_{i=1}^{m} \frac{\|\sigma|_n\|_{a_i}}{n} \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right) = L \sum_{i=1}^{m} \log \left(\sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right)^{\frac{\|\sigma|_n\|_{a_i}}{n}} \\
&= L \log \prod_{i=1}^{m} \left(\sum_{j=1}^{m_i} (p_{i_j} c_{ij}^r)^{\frac{\kappa_r}{r+\kappa_r}} \right)^{\frac{\|\sigma|_n\|_{a_i}}{n}} = L \log \sum_{\sigma\in D_n} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}},
\end{aligned}$$

and so,

$$\sum_{\sigma\in D_n} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq 1. \tag{5}$$

Now let $\ell = \min\{|\sigma| : \sigma \in \Gamma\}$. Since Γ does not contain the empty word, we have $\ell \geq 1$. Then for each $\sigma \in \Gamma$ there exists $\tau(\sigma) \in D$ with $|\tau(\sigma)| = \ell$ and $\tau(\sigma) \prec \sigma$; i.e., there exists $x(\sigma) \in D$ such that $\sigma = \tau(\sigma)x(\sigma)$. Since $p_\sigma \leq p_{\tau(\sigma)}$ and $c_\sigma \leq c_{\tau(\sigma)}$ for $\sigma \in \Gamma$, using (5) we have

$$\sum_{\sigma\in\Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq \sum_{\sigma\in\Gamma} (p_{\tau(\sigma)} c_{\tau(\sigma)}^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq \sum_{\tau\in D_\ell} (p_\tau c_\tau^r)^{\frac{\kappa_r}{r+\kappa_r}} \leq 1. \qquad \square$$

Lemma 3.6. *Let $\Gamma \subset D$ be a finite maximal antichain, $n \in \mathbb{N}$ with $n \geq |\Gamma|$, and $0 < r < +\infty$. Then $V_{n,r}(\mu_\omega) \leq \inf\left\{\sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r V_{n_\sigma,r}(\mu_\omega) : 1 \leq n_\sigma,\ \sum_{\sigma\in\Gamma} n_\sigma \leq n\right\}$.*

Proof. Suppose $n_\sigma \geq 1$ for each $\sigma \in \Gamma$, and $\sum_{\sigma\in\Gamma} n_\sigma \leq n$. For each $\sigma \in \Gamma$ let α_σ be an n_σ-optimal set for $V_{n_\sigma,r}(\mu_\omega)$. Since $|\bigcup_{\sigma\in\Gamma} S_\sigma(\alpha_\sigma)| \leq n$, $\mu_\omega = \sum_{\sigma\in\Gamma} p_\sigma \mu_\omega \circ S_\sigma^{-1}$, we have

$$\begin{aligned}
V_{n,r}(\mu_\omega) &\leq \int d(x, \bigcup S_\sigma(\alpha_\sigma))^r d\mu_\omega(x) \\
&= \sum_{\sigma\in\Gamma} p_\sigma \int d(x, \bigcup S_\sigma(\alpha_\sigma))^r d(\mu_\omega \circ S_\sigma^{-1})(x) \\
&\leq \sum_{\sigma\in\Gamma} p_\sigma \int d(S_\sigma(x), S_\sigma(\alpha_\sigma))^r d\mu_\omega(x) \\
&= \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r \int d(x, \alpha_\sigma)^r d\mu_\omega(x) \\
&= \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r V_{n_\sigma,r}(\mu_\omega),
\end{aligned}$$

which implies the lemma. $\square$

Proposition 3.7. *Let $0 < r < +\infty$ and let κ_r be as in Lemma 3.2. Then* $\limsup_{n\to\infty} ne_{n,r}^{\kappa_r} < +\infty$.

Proof. Let $q_r = \frac{\kappa_r}{r+\kappa_r}$. Then $\beta(q_r) = rq_r$. Choose ϵ_0 so that $0 < \epsilon_0 < 1$. Fix $m \in \mathbb{N}$. Choose any $n \in \mathbb{N}$ so that $\frac{m}{n} < \epsilon_0$, and set $\epsilon = \epsilon_0 \frac{m}{n}$, and then $0 < \epsilon < 1$. Let $\Gamma = \Gamma(\epsilon) = \{\sigma \in D : (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} < \epsilon \le (p_{\sigma^-} c_{\sigma^-}^r)^{\frac{\kappa_r}{r+\kappa_r}}\}$, where σ^- is the word obtained by deleting the last letter of σ; i.e., if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)$, then $\sigma^- = (\sigma_1, \sigma_2, \cdots, \sigma_{k-1})$ for $k \ge 2$, and $\sigma^- = \emptyset$ if $k = 1$. Since the index set $\{1, 2, \cdots, m_i : 1 \le i \le m\}$ is finite, Γ is a finite maximal antichain.

Hence by the previous lemma we have

$$\begin{aligned}
V_{n,r}(\mu_\omega) &\le \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r V_{m,r}(\mu_\omega) \\
&= \sum_{\sigma\in\Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} (p_\sigma c_\sigma^r)^{\frac{r}{r+\kappa_r}} V_{m,r}(\mu_\omega) \\
&< \sum_{\sigma\in\Gamma} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu_\omega) \\
&\le \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu_\omega) \quad \text{(by Lemma 3.5)} \\
&= \epsilon_0^{\frac{r}{\kappa_r}} (\frac{m}{n})^{\frac{r}{\kappa_r}} V_{m,r}(\mu_\omega) \\
\Longrightarrow nV_{n,r}^{\frac{\kappa_r}{r}}(\mu_\omega) &\le \epsilon_0 m V_{m,r}^{\frac{\kappa_r}{r}}(\mu_\omega).
\end{aligned}$$

Since the inequality holds for all but a finite number of n we have

$$\limsup_{n\to\infty} ne_{n,r}^{\kappa_r} \le \epsilon_0 m e_{m,r}^{\kappa_r} < +\infty. \qquad \square$$

Lemma 3.8. *Let $\Gamma \subset D$ be a finite maximal antichain. Then there exists $n_0 = n_0(\Gamma)$ such that for every $n \ge n_0$ there exists a set of positive integers $\{n_\sigma := n_\sigma(n)\}_{\sigma\in\Gamma}$ such that $\sum_{\sigma\in\Gamma} n_\sigma \le n$ and*

$$u_{n,r} \ge \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r u_{n_\sigma,r}.$$

Proof. Let U be the open set from the strong open set condition. Then there exists $\tau \in D$ such that $S_\tau(J) \subset U$. Note that here the similarity mappings S_σ for $\sigma \in D$ are the similarity mappings as defined in $(M2)$ of the Moran set construction. Let $\epsilon = d(S_\tau(J), U^c)$ and $\lambda = \min\{c_\sigma : \sigma \in \Gamma\}$. Then for $\sigma \in \Gamma$ we have $d(S_\sigma S_\tau(J), S_\sigma(U^c)) \ge \lambda d(S_\tau(J), U^c) = \lambda\epsilon$, which implies $d(x, U^c) \ge d(x, S_\sigma(U^c)) \ge \lambda\epsilon$ for any $x \in S_\sigma(S_\tau(J))$. For each n, let α_n be an n-optimal set for $u_{n,r}$ and let $\delta_n = \max\{d(x, \alpha_n \cup U^c) : x \in E(\omega)\}$. Since $\delta_n \to 0$ as $n \to \infty$ we can choose n_0 such that $\delta_n < \lambda\epsilon$ for all $n \ge n_0$. Suppose $n \ge n_0$ and $x \in S_\sigma(S_\tau(E(\omega)))$. Then there exists $a \in \alpha_n \cup U^c$ such that $d(x, \alpha_n \cup U^c) = d(x, a) \le \delta_n < \lambda\epsilon$, and so $a \in S_\sigma(U)$. Therefore, letting $\alpha_{n_\sigma} = \alpha_n \cap S_\sigma(U)$, we have $n_\sigma := |\alpha_{n_\sigma}| \ge 1$

and $\sum_{\sigma\in\Gamma} n_\sigma \leq n$. It can be proved that for any $x \in E(\omega)$, $d(S_\sigma(x), \alpha_n \cup U^c) \geq d(S_\sigma(x), \alpha_n \cup S_\sigma(U^c))$ and $d(S_\sigma(x), \alpha_n \cup S_\sigma(U^c)) = d(S_\sigma(x), \alpha_{n_\sigma} \cup S_\sigma(U^c))$. Hence,

$$\begin{aligned}
u_{n,r} &= \int d(x, \alpha_n \cup U^c)^r d\mu_\omega(x) \\
&= \sum_{\sigma\in\Gamma} p_\sigma \int d(S_\sigma(x), \alpha_n \cup U^c)^r d\mu_\omega(x) \\
&\geq \sum_{\sigma\in\Gamma} p_\sigma \int d(S_\sigma x, \alpha_n \cup S_\sigma(U^c))^r d\mu_\omega(x) \\
&= \sum_{\sigma\in\Gamma} p_\sigma \int d(S_\sigma x, \alpha_{n_\sigma} \cup S_\sigma(U^c))^r d\mu_\omega(x) \\
&= \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r \int d(x, S_\sigma^{-1}(\alpha_{n_\sigma}) \cup U^c)^r d\mu_\omega(x) \\
&\geq \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r u_{n_\sigma,r}. \qquad \square
\end{aligned}$$

Proposition 3.9. *Let the Moran set construction satisfy the strong open set condition and let* $0 < r < +\infty$. *Moreover, let* κ_r *be as in Lemma* 3.2 *and let* $0 < \ell < \kappa_r$. *Then* $\liminf_{n\to\infty} n e_{n,r}^\ell > 0$.

Proof. Since $0 < \ell < \kappa_r$ and κ_r is unique for which

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma\in D_n} (p_\sigma c_\sigma^r)^{\frac{\kappa_r}{r+\kappa_r}} = 0,$$

we have

$$\sum_{\sigma\in D_n} (p_\sigma c_\sigma^r)^{\frac{\ell}{r+\ell}} \to \infty \text{ as } n \to \infty.$$

Choose $t \in \mathbb{N}$ so that $\sum_{\sigma\in D_t} (p_\sigma c_\sigma^r)^{\frac{\ell}{r+\ell}} \geq 1$. Let $\Gamma = \{\sigma \in D : |\sigma| = t\}$. Then Γ is a finite maximal antichain. The previous lemma yields an n_0, and for $n \geq n_0$ the numbers $\{n_\sigma := n_\sigma(n)\}_{\sigma\in\Gamma}$ which satisfy the conclusion of that lemma. Set $c = \min\{n^{r/\ell} u_{n,r} : n \leq n_0\}$. Clearly each $u_{n,r} > 0$ and hence $c > 0$. Suppose $n \geq n_0$ and $k^{r/\ell} u_{k,r} \geq c$ for all $k < n$. Using the previous lemma we have

$$\begin{aligned}
n^{r/\ell} u_{n,r} &\geq n^{r/\ell} \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r u_{n_\sigma,r} \\
&= n^{r/\ell} \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r (n_\sigma(n))^{-r/\ell} (n_\sigma(n))^{r/\ell} u_{n_\sigma,r} \\
&\geq c \sum_{\sigma\in\Gamma} p_\sigma c_\sigma^r (\frac{n_\sigma(n)}{n})^{-r/\ell}.
\end{aligned}$$

Using Hölder's inequality (with exponents less than 1) we have

$$n^{r/\ell} u_{n,r} \geq c \left(\sum_{\sigma\in\Gamma} (p_\sigma c_\sigma^r)^{\ell/(r+\ell)} \right)^{(1+r/\ell)} \left(\sum_{\sigma\in\Gamma} (\frac{n_\sigma(n)}{n})^{(-r/\ell)(-\ell/r)} \right)^{-r/\ell}.$$

By our choice of Γ, which depends only on ℓ and not on n, and the fact that $\sum_{\sigma\in\Gamma} n_\sigma(n) \leq n$, we see that $n^{r/\ell} u_{n,r} \geq c$. Hence by induction,

$$\liminf_{n\to\infty} n u_{n,r}^{\ell/r} \geq c^{\ell/r} > 0, \text{ i.e., } \liminf_{n\to\infty} n e_{n,r}^{\ell} > 0. \qquad \square$$

Proof of Theorem 3.1. From Proposition 11.3 of [8] we know:

(a) If $0 \leq t < \underline{D}_r < s$, then

$$\lim_{n\to\infty} n e_{n,r}^t = +\infty \text{ and } \liminf_{n\to\infty} n e_{n,r}^s = 0.$$

(b) If $0 \leq t < \overline{D}_r < s$, then

$$\limsup_{n\to\infty} n e_{n,r}^t = +\infty \text{ and } \lim_{n\to\infty} n e_{n,r}^s = 0.$$

From (a) and Proposition 3.9, we have $\ell \leq \underline{D}_r$ whenever $\ell < \kappa_r$. Hence, $\kappa_r \leq \underline{D}_r$. From ($b$) and Proposition 3.7, we have $\overline{D}_r \leq \kappa_r$. Hence, $\kappa_r \leq \underline{D}_r \leq \overline{D}_r \leq \kappa_r$; i.e., the quantization dimension D_r exists and $D_r = \kappa_r$. Note that for $q_r = \frac{\kappa_r}{r+\kappa_r}$ and $\beta(q_r) = rq_r$ we have $D_r = \frac{\beta(q_r)}{1-q_r}$. Hence the proof of the theorem.

References

[1] R. Cawley and R. D. Mauldin, *Multifractal decomposition of Moran fractals*, Advances in Mathematics, 92 (1992), 196-236. MR1155465 (93b:58085)

[2] M. Dai and Y. Jiang, *Fractal dimension and measure of the subset of Moran set*, Chaos, Solitons and Fractals, 40 (2009), 190-196. MR2517925

[3] M. Dai and D. Liu, *The local dimension of Moran measures satisfying the strong separation condition*, Chaos, Solitons and Fractals, 38 (2008), 1025-1030. MR2435601 (2009g:28015)

[4] M. Dai and X. Tan, *Quantization dimension of random self-similar measures,* J. Math. Anal. Appl., 362 (2010), 471-475.

[5] K. J. Falconer, *Techniques in fractal geometry*, Chichester: Wiley, 1997. MR1449135 (99f:28013)

[6] K. J. Falconer, *The multifractal spectrum of statistically self-similar measures*, Journal of Theoretical Probability, Vol. 7, No. 3, 681-702, 1994. MR1284660 (95m:60076)

[7] A. Gersho and R. M. Gray, *Vector quantization and signal compression*, Kluwer Academic Publishers: Boston, 1992.

[8] S. Graf and H. Luschgy, *Foundations of quantization for probability distributions*, Lecture Notes in Mathematics, 1730, Springer, Berlin, 2000. MR1764176 (2001m:60043)

[9] S. Graf and H. Luschgy, *Asymptotics of the quantization errors for self-similar probabilities*, Real Anal. Exchange, 26, 795-810, 2001. MR1844394 (2002g:28011)

[10] S. Graf and H. Luschgy, *The Quantization Dimension of Self-Similar Probabilities*, Math. Nachr., 241 (2002), 103-109. MR1912380 (2003j:28025)

[11] T. Halsey, M. Jensen, L. Kadanoff and I. Procaccia, *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. A (3), 33, 1141-51, 1986. MR823474 (87h:58125a)

[12] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., 30, 713-747, 1981. MR625600 (82h:49026)

[13] S. Hua, H. Rao, Z. Wen, J. Wu, *On the structures and dimensions of Moran sets*, Science of China (Series A), 43 (8) (2000), 836-852. MR1799919 (2002a:28011)

[14] A. Käenmäki and M. Vilppolainen, *Separation conditions on controlled Moran constructions*, Fund. Math., 200 (2008), 69-100. MR2415480 (2010a:28010)

[15] W. Kreitmeier, *Optimal quantization for dyadic homogeneous Cantor distributions*, Math. Nachr., 281, No. 9, 1307-1327 (2008). MR2442708 (2010c:94025)

[16] L. J. Lindsay and R. D. Mauldin, *Quantization dimension for conformal iterated function systems*, Institute of Physics Publishing, Nonlinearity, 15 (2002), 189-199. MR1877974 (2002j:28014)

[17] W. Min, *The multifractal spectrum of some Moran measures*, Science in China Ser. A Mathematics, 2005, Vol. 48, No. 8, 1097-1112. MR2180104 (2006j:28022)

[18] N. Patzschke, *Self-conformal multifractal measures*, Adv. in Appl. Math., Volume 19, Issue 4 (1997), 486-513. MR1479016 (99c:28020)

[19] Y. Peres, M. Rams, K. Simon and B. Solomyak, *Equivalence of positive Hausdorff measure and the open set condition for self–conformal sets*, Proc. Amer. Math. Soc., 129 (2001), 2689-2699. MR1838793 (2002d:28004)

[20] M. K. Roychowdhury, *Quantization dimension function and ergodic measure with bounded distortion*, Bulletin of the Polish Academy of Sciences, 57 (2009), 251-262.

[21] A. Schief, *Separation properties for self-similar sets*, Proc. Amer. Math. Soc., 122 (1994), 111-115. MR1191872 (94k:28012)

[22] P. L. Zador, *Development and evaluation of procedures for quantizing multivariate distributions,* PhD thesis, Stanford University (1964).

[23] S. Zhu, *Quantization dimension of probability measures supported on Cantor-like sets*, J. Math. Anal. Appl., vol. 338, no. 1, 2008, 742-750. MR2386455 (2009c:28025)

[24] S. Zhu, *The quantization dimension of the self-affine measures on general Sierpiński carpets*, Monatshefte für Mathematik, published electronically, 27 November 2009.

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