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FREE CENTRAL EXTENSIONS OF GROUPS AND MODULAR LIE POWERS OF RELATION MODULES

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ABSTRACT. The most prominent special case of our main result is that the free centre-by-(nilpotent of class $(c-1)$)-by-abelian groups $F/[\gamma_c(F'), F]$ are torsion-free whenever c is divisible by at least two distinct primes. This is in stark contrast to the case where c is a prime or $c = 4$, where these relatively free groups contain non-trivial elements of finite order.

1. INTRODUCTION

Let F be a non-cyclic free group and N a normal subgroup of F, set $G = F/N$, and let $\gamma_c N$ denote the c-th term of the lower central series of N. The quotient

$$F/[\gamma_c N, F], \tag{1.1}$$

where $c \geq 2$, is a free central extension,

$$1 \to \gamma_c N/[\gamma_c N, F] \to F/[\gamma_c N, F] \to F/\gamma_c N \to 1,$$

of $F/\gamma_c N$, which is in turn an extension of G with free nilpotent (of class $(c-1)$) kernel $N/\gamma_c N$. It is well known [11] that the group $F/\gamma_c N$ is always torsion-free. However, elements of finite order may occur in the central subgroup $\gamma_c N/[\gamma_c N, F]$. The most striking instance of this phenomenon is the presence of torsion in the free centre-by-metabelian group $F/[F'', F]$, that is (1.1) with $c = 2$ and $N = F'$. This was discovered in 1973 by C.K. Gupta [5], who proved by purely group theoretic means that the quotient $F''/[F'', F]$ contains an elementary abelian 2-group of rank $\binom{r}{4}$, where r is the rank of F. Yu.V. Kuz′min [9] noticed that Gupta's torsion subgroup may be identified with $H_4(F/F') \otimes \mathbb{Z}/2\mathbb{Z}$, the fourth integral homology group of the free abelian group F/F' reduced modulo 2, and gave an alternative proof of Gupta's result using homological methods. Gupta's original result on torsion in $F/[F'', F]$ combined with Kuz′min's discovery that this is linked to homology of groups created considerable interest in studying the appearance of torsion in central extensions of the form (1.1). It turned out that for certain values of c the torsion subgroup $\tau_c N$ of (1.1), which is of course contained in the central subgroup $\gamma_c N/[\gamma_c N, F]$, can be identified with a homology group: *Suppose that G*

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has no elements of order dividing c; then

$$\tau_c N \cong \begin{cases} H_4(G, \mathbb{Z}/c\mathbb{Z}), & \text{if } c \text{ is a prime;} \\ H_6(G, \mathbb{Z}/2\mathbb{Z}), & \text{if } c = 4. \end{cases} \tag{1.2}$$

This was proved in [12] for the case where c is a prime, and in [13] for $c = 4$. A partial result for the case $c = 2$ was earlier established by Kuz′min [10]. Recently, the authors [7] have proved that, in startling contrast to the results (1.2), there is no torsion in (1.1) if $c = 6$ (under the assumption that G has no 6-torsion). In the present paper we extend this result to arbitrary composite numbers which are not prime powers.

Theorem 1. *Let F be a non-cyclic free group, N a normal subgroup of F and $G = F/N$. If c is a positive integer that is divisible by at least two distinct primes and G has no non-trivial elements of order dividing c, then $F/[\gamma_c N, F]$ is torsion-free.*

In the special case where $N = F'$ this gives the following.

Corollary 1. *The free centre-by-(nilpotent of class $(c-1)$)-by abelian group $F/[\gamma_c(F'), F]$ is torsion-free whenever c is divisible by at least two distinct primes.*

Theorem 1 will be deduced from a result on modular Lie powers of relation modules. The abelianization $N_{ab} = N/N'$ of the normal subgroup N with G-action given by conjugation in F is known as the relation module stemming from the free presentation $G = F/N$. Reduction modulo p, where p is a prime, turns the relation module into the $(\mathbb{Z}/p\mathbb{Z})G$-module $M_p = N_{ab} \otimes (\mathbb{Z}/p\mathbb{Z})$. The n-th Lie power $\mathcal{L}_n(M_p)$ is the degree n homogeneous component of the free Lie algebra $\mathcal{L}(M_p)$ on the module M_p, regarded as a $(\mathbb{Z}/p\mathbb{Z})G$-module with action induced by the G-action on M_p.

Theorem 2. *Let n be a positive integer that is divisible by at least two distinct primes, let p be a prime divisor of n, and suppose that G has no elements of order p. Then the Lie power $\mathcal{L}_n(M_p)$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module.*

This extends a recent result on projectivity of modular Lie powers of relation modules: *For an arbitrary positive integer $n \geq 2$, the Lie power $\mathcal{L}_n(M_p)$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module for all primes p that do not divide n* [8, Corollary]. Notice that this result holds without any restriction on the group G. The paper [8] was written with applications to (1.1) in mind. Combining the result of [8] with Theorem 2 yields the following corollary, which is exactly what is required to deduce Theorem 1.

Corollary 2. *Let n be a positive integer that is divisible by at least two distinct primes, and suppose that G has no elements of order dividing n. Then the Lie power $\mathcal{L}_n(M_p)$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module for all primes p.*

The paper is organized as follows. In Section 2 we deduce Theorem 1 from Corollary 2, and the remaining sections are devoted to the proof of Theorem 2. For the latter, we expand the results of [8] to a point that makes it possible to invoke a deep result on modular Lie powers, the Bryant-Schocker decomposition, a special case of which was also the key ingredient in [7].

2. Proof of Theorem 1

Let $G = F/N$ as before and let $N_{ab} = N/N'$ denote the relation module stemming from this free presentation. As was explained in the Introduction, we need to show that the central subgroup $\gamma_c N/[\gamma_c N, F]$ is torsion-free. Let $\mathcal{L}(N_{ab})$ denote the free Lie ring on N_{ab} and $\mathcal{L}_n(N_{ab})$ its degree n homogeneous component, viewed as a G-module with action induced by the action of G on N_{ab}. Then there is an isomorphism

$$\gamma_n N/[\gamma_n N, F] \cong \mathcal{L}_n(N_{ab}) \otimes_G \mathbb{Z}, \tag{2.1}$$

where $\mathbb{Z}$ denotes the ring of integers regarded as a trivial G-module (see [1]). In fact, the lower central quotient $\gamma_n N/\gamma_{n+1} N$ is isomorphic to the Lie power $\mathcal{L}_n(N_{ab})$, and trivialising the G-action on both sides gives (2.1).

Now let p be an arbitrary prime and consider the short exact sequence

$$0 \to \mathcal{L}_n(N_{ab}) \xrightarrow{p} \mathcal{L}_n(N_{ab}) \to \mathcal{L}_n(M_p) \to 0,$$

where the maps are given by multiplication by p and reduction modulo p respectively and $M_p = N_{ab} \otimes \mathbb{Z}/p\mathbb{Z}$. Then part of the associated long exact homology sequence is

$$\cdots \to H_1(G, \mathcal{L}_n(M_p)) \to \mathcal{L}_n(N_{ab}) \otimes_G \mathbb{Z} \xrightarrow{p} \mathcal{L}_n(N_{ab}) \otimes_G \mathbb{Z} \to \cdots, \tag{2.2}$$

and its exactness yields that the subgroup of elements of order dividing p in the tensor product $\mathcal{L}_n(N_{ab}) \otimes_G \mathbb{Z}$ is the image of the homology group $H_1(G, \mathcal{L}_n(M_p))$ under the connecting homomorphism in (2.2). In particular, if this homology group is trivial, then there are no elements of order p in the groups (2.1).

Now suppose that n is a positive integer with at least two distinct prime divisors and that G has no elements of order dividing n. By Corollary 2 we have that $\mathcal{L}_n(M_p)$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module for all primes p. Thus, the homology groups $H_k(G, \mathcal{L}_n(M_p))$ are trivial for all $k \geq 1$ and for all primes p. It now follows from (2.1) and (2.2) that $\gamma_n N/[\gamma_n N, F]$ (and hence $F/[\gamma_n N, F]$) contains no non-trivial elements of finite order. This completes the proof of Theorem 1.

3. Some preliminary results

We shall use the following notation. For a commutative ring K with 1, a group G and a K-free KG-module A, we let $T^n(A)$, A^n, $\mathcal{L}_n(A)$ and $\mathcal{M}_n(A)$ denote n-th tensor, symmetric, Lie and metabelian Lie powers of A, respectively. Thus $\mathcal{L}_n(A)$ is the degree n homogeneous component of the free Lie algebra $\mathcal{L}(A)$ over K on A, and $\mathcal{M}_n(A)$ is the degree n homogeneous component of the free metabelian Lie algebra $\mathcal{M}(A) = \mathcal{L}(A)/\mathcal{L}''(A)$, the quotient of $\mathcal{L}(A)$ by its second derived algebra.

3.1. The (A, C)-filtration of B^n.

Let

$$0 \to A \to B \xrightarrow{\beta} C \to 0 \tag{3.1}$$

be a short exact sequence of K-free KG-modules, and identify A with its image in B. For $n > m \geq 0$ let $K_B^{n,m}$ denote the submodule of B^n spanned by the elements

$$a_1 \circ \cdots \circ a_{m+1} \circ b_{m+2} \circ \cdots \circ b_n$$

where $a_1, \ldots, a_{m+1} \in A$ and $b_{m+2}, \ldots, b_n \in B$. It will also be convenient to set $K_B^{n,-1} = B^n$. These submodules form a filtration

$$0 < A^n = K_B^{n,n-1} < K_B^{n,n-2} < \cdots < K_B^{n,0} < K_B^{n,-1} = B^n$$

of B^n. We shall also require a number of exact sequences from [8].

Lemma 3.1. (i) *If $n > m \geq 0$, then there is a short exact sequence*

$$0 \to K_B^{n,m} \to K_B^{n,m-1} \to A^m \otimes C^{n-m} \to 0. \tag{3.2}$$

(ii) *If K has no additive torsion, $n > m \geq 1$ and $n - m + 1$ is invertible in K, then there is a four-term exact sequence*

$$0 \to K_B^{n,m} \to K_B^{n,m-2} \to K_B^{m,m-2} \otimes C^{n-m} \to A^{m-1} \otimes \mathcal{M}_{n-m+1}(C) \to 0. \tag{3.3}$$

(iii) *For all $n \geq 2$ there is a short exact sequence*

$$0 \to \mathcal{M}_n(A) \to B \otimes A^{n-1} \to K_B^{n,n-2} \to 0. \tag{3.4}$$

Parts (i) and (iii) of the lemma are simply [8, Equation (3.2)] and [8, Lemma 3.3] respectively, while part (ii) may be obtained by combining Lemmas 3.2 and 3.4 of [8].

3.2. **The Bryant-Schocker decomposition.** One of the key elements in the proof of Theorem 2 is the Bryant-Schocker Decomposition Theorem for modular Lie powers.

Theorem 3.2. *Let K be a field of characteristic $p > 0$, let G be a group, and let A be a KG-module. Let d be a positive integer not divisible by p. For every $m \geq 0$ there is a submodule $B_{p^m d}$ of $\mathcal{L}_{p^m d}(A)$ such that $B_{p^m d}$ is a direct summand of $T^{p^m d}(A)$ and*

$$\mathcal{L}_{p^m d}(A) = \mathcal{L}_{p^m}(B_d) \oplus \mathcal{L}_{p^{m-1}}(B_{pd}) \oplus \cdots \oplus \mathcal{L}_1(B_{p^m d}).$$

Moreover, $B_{p^m d}$ is isomorphic to a direct summand of $\mathcal{L}_d(T^{p^m}(A))$.

The statements in Theorem 3.2 were originally established in [3, Theorem 4.4] and [4, Theorem 4.2] for finite-dimensional modules. These results have now been generalised in [2] to allow for arbitrary modules. We will refer to the modules $B_{p^m d}$ with $m \geq 0$ as the Bryant-Schocker modules for $\mathcal{L}(A)$. In [7] it was shown that the Bryant-Schocker modules for $\mathcal{L}(M_p)$ have trivial homology in all positive dimensions. In fact, as we shall see, these modules are projective $(\mathbb{Z}/p\mathbb{Z})G$-modules.

4. Two technical lemmas

Let p be an arbitrary, but fixed, prime. From now on we work with the coefficient ring $K = \mathbb{Z}_{(p)}$, that is, the ring of integers localised at p. We write R for the group ring $\mathbb{Z}_{(p)}G$. The following results are generalisations of Lemmas 4.1 and 4.2 of [8]. In the proofs of these results we use repeatedly, without reference, the following well-known and easily verified facts. First, let

$$0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$$

be an exact sequence of R-modules in which $A_2, \ldots, A_n$ are all known to be projective; then A_1 is also projective. Second, the tensor product of a projective R-module and a $\mathbb{Z}_{(p)}$-free R-module is always R-projective. Hence, if A is a $\mathbb{Z}_{(p)}$-free projective R-module, we have that the tensor power $T^n(A)$ is also projective for every positive integer n. This is not always the case for other polylinear powers. However, by [8, Lemma 2.1] we have: *If A is a $\mathbb{Z}_{(p)}$-free projective R-module, then A^n, $\mathcal{L}_n(A)$ and $\mathcal{M}_n(A)$ are all projective R-modules too, provided either $p \nmid n$ or G has no elements of order p.*

Lemma 4.1. *Let* (3.1) *be a short exact sequence of* $\mathbb{Z}_{(p)}$*-free* R*-modules such that*
(i) B *is a projective* R*-module and*
(ii) *the metabelian Lie power* $\mathcal{M}_n(C)$ *is a projective* R*-module for all* $n \geq 2$ *with* $n \not\equiv 0 \bmod p$.

If $n > m \geq 1$*, then* $K_B^{n,m}$ *is a projective* R*-module whenever* n *and* m *are not divisible by* p.

Proof. We prove the result by induction on m. If $m = 1$, by Lemma 3.1 (ii) there is an exact sequence

$$0 \to K_B^{n,1} \to B^n \to B \otimes C^{n-1} \to \mathcal{M}_n(C) \to 0. \tag{4.1}$$

Since n is not divisible by p, the three terms to the right of $K_B^{n,1}$ in (4.1) are all projective R-modules; B^n is projective by [8, Lemma 2.1], $B \otimes C^{n-1}$ is projective because of the projective tensor factor B, and $\mathcal{M}_n(C)$ is projective by condition (ii). Hence, $K_B^{n,1}$ is a projective R-module.

For the inductive step, let $m > 1$. First suppose that $m \not\equiv 0 \bmod p$ and $m - 1 \not\equiv 0 \bmod p$ and consider the short exact sequence (3.2). Here, $K_B^{n,m-1}$ is projective by induction and $A^m \otimes C^{n-m}$ is projective, since the tensor factor $A^m = K^{m,m-1}$ is also projective by induction. It follows that $K_B^{n,m}$ is a projective R-module. It remains to deal with the case where $m \equiv 1 \bmod p$. Since $n - m + 1 \not\equiv 0 \bmod p$, by Lemma 3.1 (ii) we have the exact sequence (3.3). Here the three terms to the right of $K_B^{n,m}$ are all projective; $K_B^{n,m-2}$ and $K_B^{m,m-2}$ are projective by induction and $\mathcal{M}_{n-m+1}(C)$ is projective by condition (ii). It follows that $K_B^{n,m}$ is a projective R-module. □

Corollary 4.2. *Under the conditions of Lemma* 4.1*,*
(i) A^n *is a projective* R*-module whenever* $n \not\equiv 0, 1 \bmod p$*,*
(ii) $\mathcal{M}_n(A)$ *is a projective* R*-module whenever* $n \not\equiv 0, 2 \bmod p$.

Proof. (i) This follows immediately from Lemma 4.1 since $A^n = K^{n,n-1}$.

(ii) Consider the short exact sequence (3.4). Here the middle and right-hand terms are projective R-modules; the former is projective because of the tensor factor B, and the latter is projective by Lemma 4.1. □

Lemma 4.3. *Let* (3.1) *be a short exact sequence of* $\mathbb{Z}_{(p)}$*-free* R*-modules such that*
(i) B *is a projective* R*-module,*
(ii) *the symmetric power* C^n *is a projective* R*-module for all* $n \geq 2$ *with* $n \not\equiv 0, 1 \bmod p$*, and*
(iii) *the metabelian Lie power* $\mathcal{M}_n(C)$ *is a projective* R*-module for all* $n \geq 2$ *with* $n \not\equiv 0, 2 \bmod p$.

If $n > m \geq 0$ *and* $n \not\equiv 0 \bmod p$ *while* $n - m \not\equiv 1 \bmod p$*, then* $K_B^{n,m}$ *is a projective* R*-module.*

Proof. We prove the result by induction on m. If $m = 0$, (3.2) turns into

$$0 \to K_B^{n,0} \to B^n \to C^n \to 0 \tag{4.2}$$

and our assumptions imply that $n \not\equiv 0, 1 \bmod p$. Then C^n is projective by condition (ii), and B^n is projective by [8, Lemma 2.1], so (4.2) gives that $K_B^{n,0}$ is projective. (Note that this case does not occur when $p = 2$.) If $m = 1$, by Lemma 3.1(ii) we have the exact sequence (4.1) and our assumptions imply that $n \not\equiv 0, 2 \bmod p$. Thus, all terms to the right of $K_B^{n,1}$ in (4.1) are projective, B^n is projective by

[8, Lemma 2.1], $B \otimes C^{n-1}$ is projective because of the tensor factor B, and $\mathcal{M}_n(C)$ is projective by condition (iii).

For the inductive step, let $m > 1$. Suppose first that $n - m \not\equiv 0 \bmod p$. Then $n-(m-1) \not\equiv 1 \bmod p$, and hence $K_B^{n,m-1}$ is projective by the inductive hypothesis. Also, since $n - m \not\equiv 0, 1 \bmod p$, the symmetric power C^{n-m} is projective by condition (ii), and hence $A^m \otimes C^{n-m}$ is projective. Now the exact sequence (3.2) implies that $K_B^{n,m}$ is projective. It remains to deal with the case where $n - m \equiv 0 \bmod p$. Then $n-(m-2) \not\equiv 1 \bmod p$ and $m-(m-2) \not\equiv 1 \bmod p$. Hence $K_B^{n,m-2}$ and $K_B^{m,m-2}$ are projective by the inductive hypothesis. Finally, since $n - m + 1 \equiv 1 \bmod p$, by Lemma 3.1 (ii) we have the exact sequence (3.3), in which all terms to the right of $K_B^{n,m}$ are projective. It follows that $K_B^{n,m}$ is also a projective R-module. □

Corollary 4.4. *Under the conditions of Lemma* 4.3*,* $\mathcal{M}_n(A)$ *is a projective R-module whenever* $n \not\equiv 0 \bmod p$.

Proof. Consider the short exact sequence (3.4). Here the middle and right-hand terms are projective R-modules; the former is projective because of the projective tensor factor B, and the latter is projective by Lemma 4.3. □

5. Lie powers of relation modules

We continue with the notation from Section 3. We denote by Δ the augmentation ideal of R, that is, the kernel of the augmentation map $\varepsilon : R \to \mathbb{Z}_{(p)}$. The augmentation sequence is then the short exact sequence of R-modules

$$0 \to \Delta \to R \xrightarrow{\varepsilon} \mathbb{Z}_{(p)} \to 0, \tag{5.1}$$

where $\mathbb{Z}_{(p)}$ is the trivial R-module. The R-module $M = N_{ab} \otimes \mathbb{Z}_{(p)}$ fits into a short exact sequence

$$0 \to M \to P \to \Delta \to 0, \tag{5.2}$$

where P is a free R-module (see [6, Chapter 6, §6]), usually referred to as the relation sequence.

Lemma 5.1. *For all* $m \geq 1$,

(i) *the metabelian Lie power* $\mathcal{M}_n(T^m(M))$ *is a projective R-module for all* $n \geq 2$ *with* $n \not\equiv 0 \bmod p$,

(ii) *the symmetric power* $(T^m(M) \otimes \Delta)^n$ *is a projective R-module for all* $n \geq 2$ *with* $n \not\equiv 0, 1 \bmod p$, *and the metabelian Lie power* $\mathcal{M}_n(T^m(M) \otimes \Delta)$ *is a projective R-module for all* $n \geq 2$ *with* $n \not\equiv 0, 2 \bmod p$.

Proof. We prove the result by induction on m. For $m = 1$ part (i) of the lemma is established in [8, Theorem]. For the inductive step, suppose that part (i) holds for some $m \geq 1$. We shall show that part (ii) must also hold for this value of m. The augmentation sequence (5.1) tensored by $T^m(M)$ yields a short exact sequence

$$0 \to T^m(M) \otimes \Delta \to T^m(M) \otimes R \to T^m(M) \to 0.$$

Here the middle term is projective because of the free tensor factor R. By the inductive hypothesis, $\mathcal{M}_n(T^m(M))$ is a projective R-module for all $n \geq 2$ with $n \not\equiv 0 \bmod p$. Hence, we may apply Corollary 4.2 to the above short exact sequence to obtain part (ii) for the given m. It now remains to prove that part (i) holds for $m+1$. The relation sequence (5.2) tensored by $T^m(M)$ yields a short exact sequence

$$0 \to T^{m+1}(M) \to T^m(M) \otimes P \to T^m(M) \otimes \Delta \to 0.$$

Here the middle term is projective because of the free tensor factor P. Since (ii) holds for the given m, Corollary 4.4 applied to the exact sequence above yields part (i) for $m+1$. □

Lemma 5.2. *The Lie power $\mathcal{L}_n(T^m(M))$ is a projective R-module for all $m \geq 1, n \geq 2$ with $n \not\equiv 0 \bmod p$.*

Proof. We follow the arguments of [8, Proof of the Theorem] with M replaced by $T^m(M)$. It is proved in [12, Section 3.1] that the Lie power $\mathcal{L}_n(T^m(M))$ has a finite filtration called the type series, whose quotients can be obtained from the metabelian Lie powers

$$\mathcal{M}_2(T^m(M)), \mathcal{M}_3(T^m(M)), \ldots, \mathcal{M}_n(T^m(M)) \tag{5.3}$$

using the operations of taking metabelian Lie powers, symmetric powers and tensor products. Let $\mathfrak{T}$ denote this class of modules. We show that all modules of degree not divisible by p in $\mathfrak{T}$ are projective via induction on the number of operations required to obtain a module in $\mathfrak{T}$ from the metabelian Lie powers in (5.3). The base of our induction is given in Lemma 5.1(i). For the inductive step, if $V \in \mathfrak{T}$ and $\mathcal{M}_k(V)$ (respectively, V^k) is of degree not divisible by p, then neither k nor the degree of V is divisible by p. Hence V is projective by the inductive hypothesis, and $\mathcal{M}_k(V)$ and V^k are projective by [8, Lemma 2.1]. Now let $V, W \in \mathfrak{T}$ and suppose that $V \otimes W$ is of degree not divisible by p. Then either the degree of V is not divisible by p or the degree of W is not divisible by p. By the inductive hypothesis we have that one of the tensor factors V and W is projective, and it follows that the tensor product $V \otimes W$ must be projective too. □

Reduction modulo p gives the following result.

Corollary 5.3. *Let $M_p = N_{ab} \otimes (\mathbb{Z}/p\mathbb{Z})$. The Lie power $\mathcal{L}_n(T^m(M_p))$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module for all $m \geq 1, n \geq 2$ with $n \not\equiv 0 \bmod p$.*

Proof of Theorem 2. Let p be a prime dividing n and write $n = p^k d$ where $p \nmid d$. By assumption, n has at least two distinct prime divisors and hence $d \geq 2$. Theorem 3.2 applied to $\mathcal{L}_n(M_p)$ gives

$$\mathcal{L}_{p^k d}(M_p) = \mathcal{L}_{p^k}(B_d) \oplus \mathcal{L}_{p^{k-1}}(B_{pd}) \oplus \cdots \oplus \mathcal{L}_1(B_{p^k d}), \tag{5.4}$$

where each of the Bryant-Schocker modules $B_{p^i d}$ in (5.4) is a direct summand of $\mathcal{L}_d(T^{p^i}(M_p))$. Now, by Corollary 5.3, each $\mathcal{L}_d(T^{p^i}(M_p))$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module. Hence, the Bryant-Schocker modules for $\mathcal{L}_{p^k d}(M_p)$ are themselves projective $(\mathbb{Z}/p\mathbb{Z})G$-modules. Since G has no elements of order p, [8, Lemma 2.1] gives that all the Lie powers on the right-hand side of (5.4) are projective too. Hence, $\mathcal{L}_{p^k d}(M_p)$ is a projective R-module. □

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