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EPSILON-HYPERCYCLIC OPERATORS ON A HILBERT SPACE

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Abstract. For every fixed $\varepsilon > 0$, we construct a bounded linear operator on the separable Hilbert space having an orbit which intersects every cone of aperture $\varepsilon > 0$, but such that every orbit avoids a certain ball of positive radius (which depends on the orbit) and a fixed centre.

1. Introduction

Let X be a (real or complex) Banach space and let $T \in \mathfrak{L}(X)$ be a bounded operator on X. T is called *hypercyclic* provided there exists a vector $x \in X$ such that its T-orbit $O(x,T)$ defined by $O(x,T) = \{T^n x \mid n \geq 0\}$ is dense in X. Hypercyclicity is a part of linear dynamics, a rapidly evolving branch of functional analysis. For a complete account on this topic, we refer to the recent book [2].

One theme of linear dynamics is the study of orbits which satisfy a property weaker than denseness. Does this imply that the orbit is itself dense? Does this imply that the operator is hypercyclic? Let us mention here three results:

(1) A somewhere dense orbit must be dense: this is a beautiful result of P. Bourdon and N. Feldman [3].
(2) Let $d > 0$ be such that $O(x,T)$ meets every ball of radius d. Then $O(x,T)$ is not necessarily dense; however, T is hypercyclic, so that there exists a dense orbit. This a result of N. Feldman [5].
(3) Even if T admits a weakly dense orbit, T does not need to be hypercyclic; examples are given in [4], [6] and in [2].

In a recent paper [1], C. Badea, S. Grivaux and V. Müller have investigated a weaker version of Feldman's result:

Definition 1.1. Let $\varepsilon \in (0,1)$. A vector $x \in X$ is called an *ε-hypercyclic vector* for $T \in \mathfrak{L}(X)$ provided that for every nonzero vector $y \in X$, there exists an integer $n \in \mathbb{N}$ such that $\|T^n x - y\| \leq \varepsilon \|y\|$. The operator T is called *ε-hypercyclic* if it admits an ε-hypercyclic vector.

Thus, an ε-hypercyclic operator admits an orbit which intersects every cone of aperture ε. In [1], the following is proved.

Theorem 1.2. *For every $\varepsilon \in (0,1)$, there exists an ε-hypercyclic operator on the space $\ell^1(\mathbb{N})$ which is not hypercyclic.*

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It is rather natural that it is easier to produce such an operator on ℓ^1. Indeed this space often plays an extremal role in linear dynamics (see for instance Chapters 2, 4 and 12 of [2]). In [1], the authors asked for the existence of an ε-hypercyclic operator, yet not hypercyclic, on the separable *Hilbert* space. The aim of this paper is to answer this question positively.

Theorem 1.3. *For every $\varepsilon \in (0,1)$, there exists an ε-hypercyclic operator on the separable Hilbert space which is not hypercyclic.*

Of course, the existence of such an operator on *every* separable Banach space remains open. The rest of this paper will be devoted to the proof of Theorem 1.3. It follows the lines of [1], with several modifications due to the Hilbertian nature of the example. However, because of the technical difficulties of these proofs, we provide a completely self-contained argument. See Remark 5.1 for an overview of the differences with [1].

Let us conclude this introduction by showing that Theorem 1.3 is in some sense optimal.

Proposition 1.4. *Let $F : (0,+\infty) \to (0,+\infty)$ be such that $F(r)/r \to 0$ as $r \to \infty$. Let $T \in \mathfrak{L}(X)$ and $x \in X$ be such that for any nonzero vector $y \in X$, one may find $n \in \mathbb{N}$ such that $\|T^n x - y\| \leq F(\|y\|)$. Then T is hypercyclic.*

Proof. Let $\varepsilon \in (0,1)$, $r_0 > 0$. Also let $x \in X$ be as in the statement of the proposition. There exists $M \in (0,\infty)$ such that $F(Mr)/Mr < \varepsilon$ for any $r \geq r_0$. Let $y \in X$ with $\|y\| \geq r_0$ and $n \in \mathbb{N}$ be such that $\|T^n x - My\| \leq F(M\|y\|)$. Then, setting $z := M^{-1}x$, one obtains

$$\|T^n z - y\| \leq \frac{1}{M} F(M\|y\|) \leq \varepsilon \|y\|.$$

Thus, T has the following property: for any $\varepsilon > 0$ and any $r_0 > 0$, there exists a vector $z \in X$ such that for any $y \in X$ with $\|y\| \geq r_0$, we may find an integer $n \in \mathbb{N}$ such that $\|T^n z - y\| \leq \varepsilon\|y\|$. A look at the proof of Theorem 1.4 of [1] shows that this forces T to be hypercyclic. □

2. Strategy

We start with the Hilbert space H which is the ℓ^2-direct sum of countably many copies of $\ell^2(\mathbb{N})$. We denote by $(e_n)_{n\geq 0}$ the canonical basis of $\ell^2(\mathbb{N})$. Let T be the "simplest" hypercyclic operator on H, namely the backward weighted shift with operator weights

$$T(x_0, x_1, \dots) = (T_1 x_1, T_2 x_2, \dots),$$

where $T_i = 2I$. If $(y^{(k)})$ is a dense sequence of H of finitely-supported vectors (with support say in $\{0, \dots, k-1\}$), it is well-known to specialists that the sum

$$x = \sum_k (T^{-1})^{n_k} (y^{(k)}) \tag{1}$$

defines a hypercyclic vector for T, provided (n_k) is sufficiently fast increasing. Here, $T^{-1}(x_0, x_1, \dots)$ means $(0, T_1^{-1}x_0, T_2^{-1}x_1, \dots)$.

We will modify slightly the operators T_i to be sure that T is not hypercyclic, but remains very close to a hypercyclic operator. For instance, we can ask that $T_j(e_0) = e_0$ and $T_j(e_n) = 2e_n$ for any $n \geq 1$. Modified in this way, T becomes nonhypercyclic and the series (1) is not convergent.

We then arrange T so that it remains ε-hypercyclic. We first set

$$z^{(k)} = y^{(k)} + \big(c_0^{(k)} e_{p_k}, \dots, c_{k-1}^{(k)} e_{p_k}, 0, \dots\big),$$

where $c_j^{(k)}$ is controlled by $\|y_j^{(k)}\|$ and p_k is a large integer. We will modifiy T_j for j near n_k and just on the vector e_{p_k} so that $(T^{-1})^{n_k}(z^{(k)})$ becomes small. Thus the series $x = \sum_k (T^{-1})^{n_k}(z^{(k)})$ will be convergent and will give an ε-hypercyclic vector.

3. Outline

Let a be a positive integer so that $2^{-a} < \varepsilon$. Let X be the Hilbert space $\ell^2(\mathbb{N})$ endowed with the canonical basis $(e_n)_{n\geq 0}$. Let H be the ℓ^2-direct sum of countably many copies of X. Let $(y^{(k)})_{k\geq 1}$ be a sequence of vectors of H satisfying the following properties:

(i) the set $\{y^{(k)} \mid k \geq 1\}$ is dense in H;

(ii) each $y^{(k)}$ can be written as a sequence $y^{(k)} = (y_0^{(k)}, \dots, y_{k-1}^{(k)}, 0, \dots)$, where each $y_j^{(k)}$ is a vector of ℓ^2 which is in the linear span of the vectors e_i, $0 \leq i \leq k-1$;

(iii) $\|y^{(k)}\| \leq 2^k$.

We shall construct by induction on $k \geq 1$ two increasing sequences of integers (n_k) and (n'_k) such that $n'_{k-1} < n_k \leq n'_k$ for every $k \geq 1$, a sequence $(S_j)_{j\geq 1}$ of bounded operators on X (at step k, we will produce the operators $S_{n'_{k-1}+1}, \dots, S_{n'_k}$) and a sequence of vectors $(z^{(k)}) \in H$, $z^{(k)} = (z_0^{(k)}, \dots, z_{k-1}^{(k)}, 0, \dots)$ satisfying the following properties:

(P1): For any $k \geq 1$, $\|z^{(k)} - y^{(k)}\| \leq 2^{-a}\|y^{(k)}\|$.

(P2): Each operator S_j is bounded, invertible, upper triangular with $\|S_j^{-1}\| \leq 2$.

(P3): $\|S_j S_{j-1} \dots S_1\| \leq M(a)$ for every $j \in \mathbb{N}$, where $M(a)$ is a constant which just depends on a.

(P4): $S_j e_0 = e_0$ for every $j \geq 1$.

(P5): $\|S_{n_k+j-p} \dots S_{j+1} z_j^{(k)}\| \leq 2^{-k}$ for every $k \geq 1$, every $j = 0, \dots, k-1$ and every $p \leq n_{k-1}$.

(Q1): $S_{n'_k} \dots S_2 S_1 = I$ for every $k \in \mathbb{N}$.

(Q2): Let $k \geq 1$, $p > k^2$ and i belong to $\{n'_{k-1}, \dots, n'_k - 1\}$. Then $S_1^{-1} \dots S_i^{-1} e_p = 2^{i-n'_{k-1}} e_p$.

Once this has been done, we define T on H by setting

$$T(x_0, x_1, \dots) = (S_1^{-1}x_1, S_2^{-1}x_2, \dots).$$

By **(P2)**, T is well-defined, bounded, with $\|T\| \leq 2$. We then set

$$x^{(k)} = (\underbrace{0, \dots, 0}_{n_k}, S_{n_k} \dots S_1 z_0^{(k)}, S_{n_k+1} \dots S_2 z_1^{(k)}, \dots, S_{n_k+k-1} \dots S_k z_{k-1}^{(k)}, 0, \dots).$$

By **(P5)** applied with $p = 0$, $\|x^{(k)}\| \leq k2^{-k}$, so that one may define the vector

$$x = \sum_{k=1}^{+\infty} x^{(k)}.$$

We claim that x is an ε-hypercyclic vector for T. Indeed, let us fix $k \geq 1$. Then

$$T^{n_k}x^{(j)} = \begin{cases} 0 & \text{if } j < k \\ z^{(k)} & \text{if } j = k \\ (\underbrace{0,\dots,0}_{n_j-n_k}, S_{n_j-n_k}\dots S_1 z_0^{(k)},\dots,S_{n_j-n_k+k-1}\dots S_k z_{k-1}^{(k)},0,\dots) & \text{otherwise.} \end{cases}$$

By **(P1)** and **(P5)**, this yields

$$\begin{aligned} \|T^{n_k}x - y^{(k)}\| &\leq \|z^{(k)} - y^{(k)}\| + \sum_{j>k} j2^{-j} \\ &\leq 2^{-a}\|y^{(k)}\| + \varepsilon_k, \end{aligned}$$

with $\varepsilon_k \to 0$ as $k \to +\infty$. Let us now fix $y \in H\backslash\{0\}$, and let us consider a subsequence $(y^{(p_k)})$, with $p_k \to +\infty$, such that $y^{(p_k)} \to y$. We get, provided k is large enough,

$$\begin{aligned} \|T^{n_{p_k}}x - y\| &\leq \|T^{n_{p_k}}x - y^{(p_k)}\| + \|y^{(p_k)} - y\| \\ &\leq 2^{-a}\|y^{(p_k)}\| + \varepsilon_{p_k} + \|y^{(p_k)} - y\| \\ &\leq 2^{-a}\|y\| + 2^{-a}\|y^{(p_k)} - y\| + \varepsilon_{p_k} + \|y^{(p_k)} - y\| \\ &\leq \varepsilon\|y\| \end{aligned}$$

since $2^{-a} < \varepsilon$. Thus, x is an ε-hypercyclic vector for T. On the other hand, T is not hypercyclic. More precisely, for $w \neq (e_0, 0, \dots)$, $(e_0, 0, \dots)$ does not belong to the closure of the orbit of w. Otherwise, suppose that we can find an increasing sequence (m_k) such that $T^{m_k}w$ goes to $(e_0, 0, \dots)$. Taking the first coordinate, we get that $S_1^{-1}\dots S_{m_k}^{-1}w_{m_k}$ tend to e_0. Now, if we write by **(P3)** and **(P4)**,

$$\begin{aligned} \|w_{m_k} - e_0\| &= \|S_{m_k}\dots S_1(S_1^{-1}\dots S_{m_k}^{-1}w_{m_k} - e_0)\| \\ &\leq M(a)\|S_1^{-1}\dots S_{m_k}^{-1}w_{m_k} - e_0\|, \end{aligned}$$

then we find that (w_{m_k}) goes to e_0. This contradicts that $w \in H$.

Observe that during the proof that T is ε-hypercyclic yet not hypercyclic, we did not use properties **(Q1)** and **(Q2)**. They are just useful inside the inductive process. Observe also that the deduction of Theorem 1.3 from properties **(Px)** is completely similar to the process followed in [1].

4. The construction

So, let us start with the construction by setting $n_0 = n'_0 = 0$. Let $k \geq 1$ and let us assume that the construction has been carried out until step $k-1$. Namely, we suppose that we have constructed $z^{(1)}, \dots, z^{(k-1)}$, $n_0, \dots, n_{k-1}$, $n'_0, \dots, n'_{k-1}$, $S_1, \dots, S_{n'_{k-1}}$ so that all the properties **(P1)** to **(P6)**, **(Q1)** and **(Q2)** are satisfied for the integers for which they are meaningful. Let Δ_k be a very large integer. We set

$$\begin{cases} n_k &= n'_{k-1} + (a+1) + \Delta_k \\ \\ n'_k &= n_k + (a+1) + \Delta_k. \end{cases}$$

Let us fix $j \leq k-1$ and let us set $u_j^{(k)} = S_1^{-1}\dots S_j^{-1}y_j^{(k)}$. This is indeed possible since $k-1 \leq n'_{k-1}$ and each S_l has been supposed to be invertible for $l \leq n'_{k-1}$. Let us also consider l_j to be the unique integer such that $n'_{l_j} \leq j < n'_{l_j+1}$. As before, $l_j < k$. Writing by **(Q1)** $u_j^{(k)} = S_{n'_{l_j}+1}^{-1} \dots S_j^{-1} y_j^{(k)}$ and using **(P2)**, we get $\|u_j^{(k)}\| \le 2^{j-n'_{l_j}} \|y_j^{(k)}\|$. We now decompose $u_j^{(k)}$ as

$$u_j^{(k)} := c_j^{(k)} e_0 + w_j^{(k)}, \text{ with } w_j^{(k)} \in \operatorname{span}(e_1, \dots, e_{k-1})$$

(recall that the operators S_j, hence S_j^{-1}, are upper-triangular). This leads us to the following definition:

$$z_j^{(k)} := y_j^{(k)} + 2^{-(j-n'_{l_j})} 2^{-a} c_j^{(k)} e_{k^2}.$$

In particular, $\|z_j^{(k)} - y_j^{(k)}\| \le 2^{-a} \|y_j^{(k)}\|$, which shows that **(P1)** is satisfied by the very definition of $z^{(k)} := (z_0^k, \dots, z_{k-1}^{(k)}, 0, \dots)$.

We now define the operators S_j, $n'_{k-1} < j \le n'_k$, by giving their action on the vectors of the Hilbertian basis $(e_i)_{i\ge 0}$ of ℓ^2.

- For $i = 0$, $S_j e_0 = e_0$.
- For $i = k^2$,

$$S_j e_{k^2} = \begin{cases} 2e_{k^2} & j = n'_{k-1}+1, \dots, n'_{k-1}+a \\ -e_0 + e_{k^2} & j = n'_{k-1}+a+1 \\ \frac{1}{2} e_{k^2} & j = n'_{k-1}+a+2, \dots, n'_{k-1}+(a+1)+\Delta_k = n_k \\ 2e_k & j = n_k+1, \dots, n_k+\Delta_k \\ e_0 + e_{k^2} & j = n_k+\Delta_k+1 \\ \frac{1}{2} e_k & j = n_k+\Delta_k+2, \dots, n_k+\Delta_k+a+1 = n'_k. \end{cases}$$

- For $i \ne 0, k^2$,

$$S_j e_i = \begin{cases} \frac{1}{2} e_i & j < n'_k \\ 2^{n'_k - n'_{k-1} - 1} e_i & j = n'_k. \end{cases}$$

The successive values of $S_j \dots S_1 e_{k^2}$ for $j = n'_{k-1}, \dots, n'_k$ are very important. Using **(Q1)** again to replace $S_j \dots S_1 e_{k^2}$ by $S_j \dots S_{n'_{k-1}+1} e_{k^2}$, we find that they are equal to

$$\underset{}{e_{k^2}, \dots, 2^a e_{k^2},} \quad \underset{\substack{\uparrow \\ j = n_k - \Delta_k}}{2^a e_{k^2} - 2^a e_0, \dots,} \quad \underset{\substack{\uparrow \\ j = n_k}}{2^{-\Delta_k + a} e_{k^2} - 2^a e_0, \dots,} \quad \underset{\substack{\uparrow \\ j = n_k + \Delta_k}}{2^a e_{k^2} - q 2^a e_0, \; 2^a e_{k^2}, \dots, e_{k^2}}$$

It remains to prove that all the properties are satisfied with this construction. Before going into any detail, let us notice that, since **(Q1)** is true at step $k-1$, the formal properties **(P4)**, **(Q1)** and **(Q2)** are verified.

5. Boundedness of S_j and S_j^{-1}: Proof of (P2) and (P3)

Each S_j is the sum of a bounded diagonal operator and of a finite-rank and strictly upper triangular operator. Then, it is bounded and invertible. Moreover, for $j \ne n'_{k-1} + a + 1$ and $j \ne n_k + \Delta_k + 1$, it is even diagonal, with diagonal terms greater than $1/2$. This yields **(P2)** except for the previous critical indices. However, for these indices, it is not hard to compute S_j^{-1}:

$$S_j^{-1}(e_i) = \begin{cases} e_0 & i = 0 \\ e_{k^2} \pm e_0 & i = k^2 \\ 2e_i & \text{otherwise.} \end{cases}$$

It is now straightforward to check that $\|S_j^{-1}\| \leq 2$:

$$\begin{aligned} \|S_j^{-1}(\sum_i c_i e_i)\| &= |c_0 + c_{k^2}|^2 + |c_{k^2}|^2 + 4\sum_{i\neq 0,k^2} |c_i|^2 \\ &\leq 2|c_0|^2 + 3|c_{k^2}| + 4\sum_{i\neq 0,k^2} |c_i|^2 \\ &\leq 4\|\sum_i c_i e_i\|^2. \end{aligned}$$

Remark 5.1. Here is the key point where our proof differs from that of [1]. It is easier to prove that an operator is bounded on ℓ^1 than on ℓ^2. We had to change S_j to ensure that S_j^{-1} admits at most two upper-diagonal coefficients. This implies several modifications elsewhere, as in the definition of z_j, to keep properties **(Px)** and **(Qx)**.

To prove **(P3)** for $j = n'_{k-1}, \dots, n'_k$, we have to take into account the values of $S_j \dots S_1(e_{k^2})$. These values have been computed above and we find

$$S_j \dots S_1(e_i) = \begin{cases} e_0 & i = 0 \\ \alpha e_{k^2} + \beta e_0 & i = k^2 \\ \gamma e_i & \text{otherwise}, \end{cases}$$

with $|\alpha|, |\beta| \leq 2^a$ and $|\gamma| \leq 1$. Arguing as before, we easily get **(P3)**.

6. Action of S_j on $z^{(k)}$: Proof of **(P5)**

We now verify that **(P5)** holds true. We first write

$$\begin{aligned} &S_{n_k+j-p} \dots S_{j+1}(z_j^{(k)}) \\ &\quad = S_{n_k+j-p} \dots S_1(S_1^{-1} \dots S_j^{-1})(z_j^{(k)}) \\ &\quad = S_{n_k+j-p} \dots S_{n'_{k-1}+1}(S_1^{-1} \dots S_j^{-1})\big(y_j^{(k)} + c_j^{(k)} 2^{-(j-n'_{l_j})} 2^{-a} e_{k^2}\big) \\ &\quad = S_{n_k+j-p} \dots S_{n'_{k-1}+1}\big(c_j^{(k)} e_0 + w_j^{(k)} + c_j^{(k)} 2^{-a} e_{k^2}\big), \end{aligned}$$

where the last line comes from **(Q2)**. We now observe that $|j-p| \leq \max(j,p) \leq n'_{k-1}$, and we suppose that Δ_k is very large with respect to n'_{k-1}. This gives

$$\begin{aligned} S_{n_k+j-p} \dots S_{j+1}(z_j^{(k)}) &= c_j^{(k)} e_0 + 2^{-(n_k+j-p-n'_{k-1})} w_j^{(k)} \\ &\quad + c_j^{(k)} 2^{-a}\big(2^{-\Delta_k+a+|j-p|} e_{k^2} - 2^a e_0\big) \\ &= 2^{-\Delta_k-(a+1)-(j-p)} w_j^{(k)} + c_j^{(k)} 2^{-\Delta_k+|j-p|} e_{k^2}. \end{aligned}$$

We can now adjust Δ_k to be large enough so that $\|S_{n_k+j-p} \dots S_{j+1}(z_j^{(k)})\| \leq 2^{-k}$. This achieves the proof of Theorem 1.3.

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