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A NOTE ON COMPLETE RESOLUTIONS

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Abstract. It is shown that the Eckmann-Shapiro Lemma holds for complete cohomology if and only if complete cohomology can be calculated using complete resolutions. It is also shown that for an $\mathbf{LH}\mathfrak{F}$-group G the kernels in a complete resolution of a $\mathbb{Z}G$-module coincide with Benson's class of cofibrant modules.

1. Introduction

Let G be a group and $\mathbb{Z}G$ its integral group ring. A $\mathbb{Z}G$-module M is said to admit a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ of coincidence index n if there is an acyclic complex $\mathcal{F} = \{(F_i, \vartheta_i) |\ i \in \mathbb{Z}\}$ of projective modules and a projective resolution $\mathcal{P} = \{(P_i, d_i) |\ i \in \mathbb{Z}, i \geq 0\}$ of M such that $\mathcal{F}$ and $\mathcal{P}$ coincide in dimensions greater than n; that is,

$$\begin{array}{l} \mathcal{F} : \cdots \to F_{n+1} \to F_n \xrightarrow{\vartheta_n} F_{n-1} \to \ \cdots \ \to F_0 \to F_{-1} \to F_{-2} \to \cdots \\ \qquad\quad\ \ \| \qquad\quad \| \\ \mathcal{P} : \cdots \to P_{n+1} \to P_n \xrightarrow{d_n} P_{n-1} \to \ \cdots \ \to P_0 \to M \to 0 \end{array}$$

A $\mathbb{Z}G$-module M is said to admit a complete resolution in the strong sense if there is a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ with $\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}, Q)$ acyclic for every $\mathbb{Z}G$-projective module Q.

It was shown by Cornick and Kropholler in [7] that if M admits a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ in the strong sense, then

$$\widehat{\mathrm{Ext}}^*_{\mathbb{Z}G}(M, B) \simeq H^*(\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}, B))$$

where $\widehat{\mathrm{Ext}}^*_{\mathbb{Z}G}(M, \_)$ is the P-completion of $\mathrm{Ext}^*_{\mathbb{Z}G}(M, \_)$, defined by Mislin for any group G [13] as

$$\widehat{\mathrm{Ext}}^k_{\mathbb{Z}G}(M, B) = \lim_{\substack{\longrightarrow \\ r>k}} S^{k-r}\mathrm{Ext}^r_{\mathbb{Z}G}(M, B)$$

where $S^{-m}T$ is the m-th left satellite of a functor T.

Alternative but equivalent definitions of the complete $\widehat{\mathrm{Ext}}$-groups were given by Benson and Carlson [5] and Vogel [9].

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Complete cohomology $\hat{H}^*(G,\_) = \widehat{\mathrm{Ext}}^*_{\mathbb{Z}G}(\mathbb{Z},\_)$, where $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module, generalizes the Tate cohomology defined for finite groups and the Farrell-Tate cohomology defined for groups that have a finite-index subgroup of finite cohomological dimension.

A group G is said to admit a complete resolution if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a complete resolution.

It turns out that G admits a complete resolution in the strong sense if and only if the generalized cohomological dimension $\underline{\mathrm{cd}}G$ is finite [3], where

$$\underline{\mathrm{cd}}G = \sup\{n \in \mathbb{N} |\ \exists M\ \mathbb{Z}\text{-free},\ \exists F\ \mathbb{Z}G\text{-free}:\ \mathrm{Ext}^n_{\mathbb{Z}G}(M,F) \neq 0\}$$

was defined by Ikenaga in his study of generalized Farrell-Tate cohomology in [10].

Note that $\underline{\mathrm{cd}}G = \mathrm{Gcd}_{\mathbb{Z}}G$ [3], the Gorenstein projective dimension of G, which is defined via resolutions of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ by Gorenstein projective modules and is related to the G-dimension defined by Auslander in [1] (see also [2] and [6]).

A $\mathbb{Z}G$-module M is said to be Gorenstein projective if it admits a complete resolution in the strong sense of coincidence index 0, i.e. if M is a kernel in a complete resolution in the strong sense.

Complete resolutions do not always exist; for example, if G has a free abelian subgroup of infinite rank, then G does not admit a complete resolution [14].

If the complete cohomology can be calculated using complete resolutions, then one has certain advantages such as the Eckmann-Shapiro Lemma and certain spectral sequences.

Here we show that the validity of the Eckmann-Shapiro Lemma for complete cohomology actually implies that complete resolutions exist and that complete cohomology can be calculated using complete resolutions:

Theorem A. *The following are equivalent for a group G.*

(1) *The Eckmann-Shapiro Lemma holds for complete cohomology.*
(2) *G has a complete resolution and every complete resolution of G is a complete resolution in the strong sense.*
(3) *Complete cohomology can be calculated using complete resolutions.*

We also show the following:

Theorem B. *If a $\mathbb{Z}G$-module M is a kernel in a complete resolution and A is a $\mathbb{Z}G$-module that is $\mathbb{Z}F$-projective for every finite subgroup F of G, then $M \otimes A$ (with diagonal G-action) is projective as a $\mathbb{Z}H$-module for every $\mathbf{LH}\mathfrak{F}$-subgroup H of G.*

Theorem B in particular implies the two results below:

Corollary C. *If G is an $\mathbf{LH}\mathfrak{F}$-group, then the kernels of a complete resolution of a $\mathbb{Z}G$-module coincide with the cofibrant modules.*

Corollary D. *If G is an $\mathbf{LH}\mathfrak{F}$-group, then every complete resolution of a ZG-module M is a complete resolution in the strong sense.*

Cofibrant modules were introduced by Benson in [4] for his study of $\mathbb{Z}G$-modules that admit a projective resolution by finitely generated projective modules, when G is an $\mathbf{LH}\mathfrak{F}$-group. They are defined as follows: for any group G, a $\mathbb{Z}G$-module M is said to be cofibrant if $M \otimes B(G,\mathbb{Z})$ is a projective $\mathbb{Z}G$-module, where $B(G,\mathbb{Z})$ is the set of bounded functions from G to $\mathbb{Z}$.

The class $\mathbf{H}\mathfrak{F}$ was defined by Kropholler in [11] as the smallest class of groups which contains the class of finite groups and is such that whenever a group G admits a finite dimensional contractible G-CW-complex with stabilizers in $\mathbf{H}\mathfrak{F}$, we have that G is in $\mathbf{H}\mathfrak{F}$. $\mathbf{LH}\mathfrak{F}$ is the class of groups G such that every finitely generated subgroup of G is in $\mathbf{H}\mathfrak{F}$. The class $\mathbf{LH}\mathfrak{F}$ includes, for example, all soluble-by-finite groups and all groups with a faithful representation as endomorphisms of a Noetherian module over a commutative ring, and is extension-closed, closed under ascending unions and closed under amalgamated free products and HNN extensions.

In [3] it was proved that if G is an $\mathbf{LH}\mathfrak{F}$-group and M a $\mathbb{Z}G$-module that admits a projective resolution by finitely generated projective modules, then M is a Gorenstein projective $\mathbb{Z}G$-module if and only if it is a cofibrant module.

Corollary C shows that for $\mathbf{LH}\mathfrak{F}$-groups the Gorenstein projective modules coincide with the cofibrant modules. We believe the following to be true:

Conjecture A. *For any group G the Gorenstein projective modules coincide with the cofibrant modules.*

Conjecture B (see also Conj. B in [15])**.** *A $\mathbb{Z}G$-module M admits a complete resolution if and only if it admits a complete resolution in the strong sense.*

Conjecture B in particular implies that Gorenstein projectivity is a subgroup-closed property. Here we show that this is so if the subgroup is in $\mathbf{LH}\mathfrak{F}$ (Corollary 2.2).

2. Proof of the results

Proof of Theorem A. (1) $\Rightarrow$ (2): Assume that the Eckmann-Shapiro Lemma holds for the complete cohomology of G, i.e.

$$\widehat{\operatorname{Ext}}^*_{\mathbb{Z}G}(A, \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B)) \simeq \widehat{\operatorname{Ext}}^*_{\mathbb{Z}H}(A|_{\mathbb{Z}H}, B)$$

for every $\mathbb{Z}G$-module A, every $\mathbb{Z}H$-module B and every subgroup $H \leq G$. Applying the Eckmann-Shapiro Lemma with the trivial subgroup $H = 1$, we have that $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(A, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, B)) \simeq \widehat{\operatorname{Ext}}^0_{\mathbb{Z}}(A|_{\mathbb{Z}}, B)$ for every $\mathbb{Z}G$-module A and every $\mathbb{Z}$-module B. We have that $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}}(A|_{\mathbb{Z}}, B) = 0$, since A and B have finite projective dimension over $\mathbb{Z}$ [11, 4.2].

It follows that $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(A, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)) = 0$ for any $\mathbb{Z}G$-module A. For any $\mathbb{Z}G$-module A there is a canonical injection $A \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$. So, if I is injective, then it is a direct summand of $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, I)$. It follows that $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(I, I) = 0$, and thus I has finite projective dimension [11, 4.2]. It is easy to see that the supremum of the projective dimensions of the injective $\mathbb{Z}G$-modules, $\operatorname{spli}\mathbb{Z}G$, is finite; hence (2) follows from [8, §4].

(2) $\Rightarrow$ (3) by Theorem 1.2 of [7].

(3) $\Rightarrow$ (1): It follows from the implication (4) $\Rightarrow$ (2) of Theorem 2.2 in [15] and [7] that every $\mathbb{Z}G$-module M admits a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ in the strong sense and $\widehat{\operatorname{Ext}}^*_{\mathbb{Z}G}(M, B) \simeq H^n(\operatorname{Hom}_{\mathbb{Z}G}(\mathcal{F}, B))$. Clearly the computation of the complete cohomology via complete resolutions implies the validity of the Eckmann-Shapiro Lemma. □

Lemma 2.1. *Let G be a group.*

(a) *If $M_i, i \in I$, is a family of $\mathbb{Z}G$-modules that admit complete resolutions (in the strong sense) of coincidence index 0, then the direct sum $\bigoplus_{i\in I} M_i$ admits a complete resolution (in the strong sense) of coincidence index 0.*

(b) *If M admits a complete resolution of coincidence index n and A is a $\mathbb{Z}$-free $\mathbb{Z}G$-module, then $M \otimes A$ (with diagonal G-action) admits a complete resolution of coincidence index n.*

(c) *Let $\{(F_i, \vartheta_i), i \in \mathbb{Z}\}$ be an acyclic complex of projective $\mathbb{Z}G$-modules with kernels $M_i, i \in \mathbb{Z}$. If there is a bound on the projective dimensions of the M_i's, then M_i is projective for every $i \in \mathbb{Z}$.*

Proof. The proofs of (a) and (b) are straightforward.

(c) Let $\mathrm{pd}_{\mathbb{Z}G} M_r = k > 0$ for some $r \in \mathbb{Z}$. It follows from the short exact sequence $0 \to M_r \to F_r \to M_{r-1} \to 0$ that $\mathrm{pd}_{\mathbb{Z}G} M_{r-1} = k+1$ and thus inductively $\mathrm{pd}_{\mathbb{Z}G} M_{r-s} = k+s$ for all $s \in \mathbb{N}$, which is a contradiction since there is a bound on the projective dimensions of the M_i's. □

Proof of Theorem B. We first prove the theorem for $\mathbf{H}\mathfrak{F}$-subgroups, using transfinite induction on the ordinal number α such that the subgroup H belongs to $\mathbf{H}_\alpha\mathfrak{F}$. For $\alpha = 0$, $M \otimes A$ is projective over any finite subgroup of G, because M is $\mathbb{Z}$-free and A is projective over the finite subgroups of G. Assume that the result is true for all $\mathbf{H}_\beta\mathfrak{F}$-subgroups of G for all $\beta < \alpha$, and let H be an $\mathbf{H}_\alpha\mathfrak{F}$-subgroup of G. There is an exact sequence of $\mathbb{Z}H$-modules

$$0 \to C_r \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

where each C_i is a direct sum of modules of the form $\mathbb{Z}[H/F]$ with $F \leq H$ and $F \in \mathbf{H}_\beta\mathfrak{F}$ for $\beta < \alpha$. By the inductive hypothesis, $M \otimes A$ is projective over each $\mathbb{Z}F$, so each $\mathbb{Z}[H/F] \otimes M \otimes A$ is projective over $\mathbb{Z}H$ and thus each $C_i \otimes M \otimes A$ is projective over $\mathbb{Z}H$. Tensoring the above sequence with $M \otimes A$ gives us an exact sequence

$$0 \to C_r \otimes M \otimes A \to \cdots \to C_0 \otimes M \otimes A \to M \otimes A \to 0.$$

It follows that $\mathrm{pd}_{\mathbb{Z}H} M \otimes A < \infty$ for any $\mathbb{Z}G$-module M that has a complete resolution of coincidence index 0. If there is not a bound on these projective dimensions, there is a family of $\mathbb{Z}G$-modules $M_n, n \in \mathbb{N}$, that have complete resolutions of coincidence index 0 and are such that $\mathrm{pd}_{\mathbb{Z}H} M_n \otimes A \geq n$ for all $n \in \mathbb{N}$. Then the direct sum $M = \bigoplus_{n\in\mathbb{N}} M_n$ has a complete resolution of coincidence index 0 by Lemma 2.1(a), and $\mathrm{pd}_{\mathbb{Z}H} M \otimes A \geq \mathrm{pd}_{\mathbb{Z}H} M_n \otimes A \geq n$ for all $n \in \mathbb{N}$ so that $\mathrm{pd}_{\mathbb{Z}H} M \otimes A$ is infinite, which is a contradiction. It follows that there is a bound on the projective dimension $\mathrm{pd}_{\mathbb{Z}H} M \otimes A$ where M is a $\mathbb{Z}G$-module that has a complete resolution of coincidence index 0. From Lemma 2.1 parts (b) and (c), we obtain that $M \otimes A$ is projective as a $\mathbb{Z}H$-module.

Next we prove the result for $\mathbf{LH}\mathfrak{F}$-subgroups, using induction on the cardinality of the subgroup. If H is a countable subgroup, then it belongs to $\mathbf{H}\mathfrak{F}$, so the result has been proved. Assume that H is uncountable and that the result is true for any subgroup with cardinality strictly smaller than $|H|$, and let M be a $\mathbb{Z}G$-module that has a complete resolution of coincidence index 0. Since H is uncountable it can be expressed as the union of an ascending chain of subgroups H_α, $\alpha < \gamma$, for some ordinal number γ such that each H_α has cardinality strictly smaller than $|H|$. By the inductive hypothesis $M \otimes A$ is projective over each $\mathbb{Z}H_\alpha$, so by Lemma 5.6 in [4] we have $\mathrm{pd}_{\mathbb{Z}H} M \otimes A \leq 1$. Thus we have proved that $M \otimes A$ has finite projective dimension over $\mathbb{Z}H$, for any $\mathbb{Z}G$-module M that has a complete resolution of coincidence index 0. It follows from the above argument that $M \otimes A$ is projective over H, for any $\mathbb{Z}G$-module M that has a complete resolution of coincidence index 0. □

Proof of Corollary C. Let P_* be a complete resolution with kernels $M_i, i \in \mathbb{Z}$. It is known [12] that $B(G, \mathbb{Z})$ is a projective $\mathbb{Z}F$-module for any finite subgroup F of G. Hence it follows from Theorem B that each M_i is cofibrant. The converse was shown in [7]. □

Proof of Corollary D. Let P_* be a complete resolution with kernels $M_i, i \in \mathbb{Z}$. It follows from Corollary C that each M_i is a cofibrant module. There is a $\mathbb{Z}$-split $\mathbb{Z}G$-monomorphism $\mathbb{Z} \hookrightarrow B(G, \mathbb{Z})$; hence if Q is a projective $\mathbb{Z}G$-module, then Q is a $\mathbb{Z}G$-direct summand of $\mathrm{Hom}_{\mathbb{Z}}(B(G, \mathbb{Z}), Q)$ (with diagonal action), and so the complex $\mathrm{Hom}_{\mathbb{Z}G}(P_*, Q)$ is a direct summand of the complex

$$\mathrm{Hom}_{\mathbb{Z}G}(P_*, \mathrm{Hom}_{\mathbb{Z}}(B(G, \mathbb{Z}), Q)) \simeq \mathrm{Hom}_{\mathbb{Z}G}(P_* \otimes B(G, \mathbb{Z}), Q).$$

This complex is exact because $M_i \otimes B(G, \mathbb{Z})$ are projective $\mathbb{Z}G$-modules, so $P_* \otimes B(G, \mathbb{Z})$ splits. □

It is not known whether Gorenstein projectivity is a subgroup-closed property. Here we show that this is so if the subgroup is an $\mathbf{LH}\mathfrak{F}$-group.

Corollary 2.2. *If G is a group and M is a Gorenstein projective module, then $M|_H$ is Gorenstein projective for any $\mathbf{LH}\mathfrak{F}$-subgroup H of G.*

Proof. Since M is Gorenstein projective it admits a complete resolution in the strong sense of coincidence index 0 over $\mathbb{Z}G$, which is a complete resolution of coincidence index 0 over $\mathbb{Z}H$ for any subgroup H of G. If the subgroup is in $\mathbf{LH}\mathfrak{F}$, then the resolution is a complete resolution in the strong sense by Corollary D. Hence $M|_H$ is a Gorenstein projective module. □

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