REALIZATION OF THE MAPPING CLASS GROUP OF HANDLEBODY BY DIFFEOMORPHISMS

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Abstract. For the oriented 3-dimensional handlebody constructed from a 3-ball by attaching $g$ 1-handles, it is shown that the natural surjection from the group of orientation-preserving diffeomorphisms of it to the mapping class group of it has no section when $g$ is at least 6.

Let $M$ be an $n$-dimensional compact oriented manifold and $S$ be a subset of $\partial M$. We denote the group of orientation-preserving diffeomorphisms of $M$ whose restrictions on $S$ are the identity by $\text{Diff}(M, S)$, the subgroups of them consisting of elements that are isotopic to the identity by $\text{Diff}_0(M, S)$, and the quotient group $\text{Diff}(M, S)/\text{Diff}_0(M, S)$ by $\mathcal{M}(M, S)$. For an element $f$ of $\text{Diff}(M, S)$, let $[f]$ be the element of $\mathcal{M}(M, S)$ represented by $f$. The homomorphism $\pi_{M, S}$ from $\text{Diff}(M, S)$ to $\mathcal{M}(M, S)$ defined by $\pi_{M, S}(f) = [f]$ is a surjection. Let $\Gamma$ be a subgroup of $\mathcal{M}(M, S)$. We call a homomorphism $s$ from $\Gamma$ to $\text{Diff}(M, S)$ which satisfies $\pi_{M, S} \circ s = \text{id}_{\Gamma}$ a section for $\pi_{M, S}$ over $\Gamma$. Morita [7] showed that the natural surjection from $\text{Diff}^2(\Sigma_g)$ to the mapping class group $\mathcal{M}(\Sigma_g)$ of $\Sigma_g$ has no section over $\mathcal{M}(\Sigma_g)$ when $g \geq 5$. Markovic [5] (when $g \geq 6$) and Markovic and Saric [6] (when $g \geq 2$) showed that the natural surjection from $\text{Homeo}(\Sigma_g)$ to $\mathcal{M}(\Sigma_g)$ has no section over $\mathcal{M}(\Sigma_g)$. By using the different method from them, Franks and Handel [2] showed that the natural surjection from $\text{Diff}(\Sigma_g)$ to $\mathcal{M}(\Sigma_g)$ has no section over $\mathcal{M}(\Sigma_g)$ when $g \geq 3$.

Let $H_g$ be an oriented 3-dimensional handlebody of genus $g$ which is an oriented 3-manifold constructed from a 3-ball by attaching $g$ 1-handles. Let $\Sigma_g$ be an oriented closed surface of genus $g$; then $\partial H_g = \Sigma_g$. The restriction to the boundary defines a homomorphism $\rho_\partial : \text{Diff}(H_g) \to \text{Diff}(\Sigma_g)$, and $\rho_\partial$ induces an injection $\mathcal{M}(H_g) \to \mathcal{M}(\Sigma_g)$ since $H_g$ is an irreducible 3-manifold. We will show:

**Theorem 1.** If $g \geq 6$, there is no section for $\pi_{H_g} : \text{Diff}(H_g) \to \mathcal{M}(H_g)$ over $\mathcal{M}(H_g)$.

For contradiction, we assume that there is a section $s : \mathcal{M}(H_g) \to \text{Diff}(H_g)$. Let $\Gamma$ be a subgroup of $\mathcal{M}(H_g)$, and $i_\Gamma$ be the inclusion from $\Gamma$ to $\mathcal{M}(H_g)$. Then $\Gamma$ is a
subgroup of \( \mathcal{M}(\Sigma_g) \), and the composition \( \rho_0 \circ s \circ \iota_1 \) is a section for \( \pi_{\Sigma_g} : \text{Diff}(\Sigma_g) \to \mathcal{M}(\Sigma_g) \) over \( \Gamma \). Therefore, if we can find a subgroup \( \Gamma \) of \( \mathcal{M}(H_g) \), over which there is no section for \( \pi_{\Sigma_g} \), then Theorem 1 follows.

Let \( D \) be a 2-disk in \( \Sigma_g \), and \( \Sigma_{g,1} \) be \( \Sigma_g \setminus \text{int} \, D \). Let \( c \) be an essential simple closed curve on \( \Sigma_g \) such that \( \Sigma_g \setminus c \) is not connected. Then the closure of one component of \( \Sigma_g \setminus c \) is diffeomorphic to \( \Sigma_{g,1} \), and the closure of the other component of \( \Sigma_g \setminus c \) is diffeomorphic to \( \Sigma_{g,1} \). We remark that \( g = g_1 + g_2 \) and \( g_1, g_2 \geq 1 \). These diffeomorphisms induce injections \( \mathcal{M}(\Sigma_{g_1,1}, \partial \Sigma_{g_1,1}) \to \mathcal{M}(\Sigma_g) \) and \( \mathcal{M}(\Sigma_{g_2,1}, \partial \Sigma_{g_2,1}) \to \mathcal{M}(\Sigma_g) \) (see [8]). By these injections, we consider \( \mathcal{M}(\Sigma_{g_1,1}, \partial \Sigma_{g_1,1}) \) and \( \mathcal{M}(\Sigma_{g_2,1}, \partial \Sigma_{g_2,1}) \) as subgroups of \( \mathcal{M}(\Sigma_g) \). From Theorem 1.6 in [2] proved by Franks and Handel, we see:

**Theorem 2** ([2]). Let \( \Gamma_1 \) be a nontrivial finitely generated subgroup of \( \mathcal{M}(\Sigma_{g_1,1}, \partial \Sigma_{g_1,1}) \) such that \( H^1(\Gamma_1, \mathbb{R}) = 0 \) and \( \mu \) be an element of \( \mathcal{M}(\Sigma_{g_2,1}, \partial \Sigma_{g_2,1}) \) which is represented by a pseudo-Anosov homeomorphism on \( \partial \Sigma_{g_1,1} \). Then there is no section for \( \pi_{\Sigma_g} : \text{Diff}(\Sigma_g) \to \mathcal{M}(\Sigma_g) \) over \( (\Gamma_1, \mu) \), where \( (\Gamma_1, \mu) \) is a subgroup of \( \mathcal{M}(\Sigma_g) \) generated by elements of \( \Gamma_1 \) and \( \mu \).

We assume \( g \geq 6 \). The 3-manifold \( \Sigma_{2,1} \times [0, 1] \) is diffeomorphic to \( H_4 \). Let \( D_1 \) be a 2-disk in \( \text{int} \, (\partial \Sigma_{2,1} \times [0, 1]) \subset \partial (\Sigma_{2,1} \times [0, 1]) \). \( D_2 \) and \( D_3 \) be disjoint 2-disks on \( \partial H_{g-6} \), and \( D_4 \) be a 2-disk on \( \partial H_2 \). Along these 2-disks, we glue \( \Sigma_{2,1} \times [0, 1] \) to \( H_g \). These inclusions induce natural homomorphisms \( i_1 : \mathcal{M}(\Sigma_{2,1} \times [0, 1], \partial \Sigma_{2,1} \times [0, 1]) \to \mathcal{M}(H_g) \) and \( i_2 : \mathcal{M}(H_2, D_4) \to \mathcal{M}(H_g) \). If \( [h] \) is in \( \mathcal{M}(\Sigma_{2,1} \times [0, 1], \partial \Sigma_{2,1} \times [0, 1]) \) (resp. \( \mathcal{M}(H_2, D_4) \)) represented by \( h \in \text{Diff}(\Sigma_{2,1} \times [0, 1], \partial \Sigma_{2,1} \times [0, 1]) \) (resp. \( \text{Diff}(H_2, D_4) \)), then \( i_1([h]) \) (resp. \( i_2([h]) \)) is represented by the diffeomorphism obtained by extending \( h \) to \( H_g \) using the identity mapping on \( H_g \setminus \Sigma_{2,1} \times [0, 1] \) (resp. \( H_g \setminus H_2 \)).

We define homomorphisms \( \Pi : \text{Diff}(\Sigma_{2,1}, \partial \Sigma_{2,1}) \to \text{Diff}(\Sigma_{2,1} \times [0, 1], \partial \Sigma_{2,1} \times [0, 1]) \) by \( \Pi(h) = h \times \text{id}_{[0,1]} \), and \( I_1 : \text{Diff}(\Sigma_{2,1} \times [0, 1], \partial \Sigma_{2,1} \times [0, 1]) \to \text{Diff}(H_2) \) by the identity on \( H_2 \setminus \Sigma_{2,1} \times [0, 1] \). Then the composition \( I_1 \circ \Pi \) induces a homomorphism \( P : \mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1}) \to \mathcal{M}(H_2) \). By applying Corollary 4.2 of [8] to the subsurface \( \Sigma_{2,1} \times [0, 1] \subset \partial H_g \), the injectivity of \( P \) is shown. Korkmaz [4] showed that \( H_1(\mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1}), \mathbb{Z}) = \mathbb{Z}/10\mathbb{Z} \); hence \( H^1(\mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1}), \mathbb{R}) = 0 \). Therefore, \( \Gamma_1 = P(\mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1})) \) satisfies the assumption of Theorem 2 when \( g_1 = g - 2, g_2 = 2 \).

Fathi and Laudenbach [3] constructed a pseudo-Anosov homeomorphism \( \phi \) on \( \partial(H_2) \) which is a restriction of a homeomorphism on \( H_2 \). The definitions of pseudo-Anosov homeomorphisms and terminologies (e.g., singular foliation) related to them can be found in [1]. Any pseudo-Anosov homeomorphism preserves the set of singular points of the singular foliation which is preserved by this homeomorphism. Since the number of singular points of the singular foliation is finite, a proper power of \( \phi \), say \( \phi^n \), fixes some points. Let \( p \) be a point fixed by \( \phi^n \). Then \( \phi^n \) defines a pseudo-Anosov homeomorphism on \( \partial(H_2) \setminus p = \text{int} \, \Sigma_{2,1} \). Let \( \mu \) be an element of \( \mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1}) \subset \mathcal{M}(\Sigma_g) \) represented by this homeomorphism. Then \( \mu \) is an element of \( \mathcal{M}(H_g) \) and satisfies the assumption of Theorem 2 when \( g_1 = g - 2, g_2 = 2 \).
Then \( \langle P(\mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1})), \mu \rangle \) is a subgroup of \( \mathcal{M}(H_g) \) and, by Theorem 2, there is no section over \( \langle P(\mathcal{M}(\Sigma_{2,1}, \partial \Sigma_{2,1})), \mu \rangle \). Therefore, there is no section for \( \pi_{H_g} : \text{Diff}(H_g) \to \mathcal{M}(H_g) \) over \( \mathcal{M}(H_g) \).

**Remark 3.** Two subgroups \( G_1 \) and \( G_2 \) of \( \mathcal{M}(\Sigma_g) \) are **conjugate** if there is an element \( h \in \mathcal{M}(\Sigma_g) \) such that \( hG_1h^{-1} = G_2 \). When two subgroups \( G_1 \) and \( G_2 \) of \( \mathcal{M}(\Sigma_g) \) are conjugate, there is a section for \( \pi_{\Sigma_g} \) over \( G_1 \) if and only if there is a section for \( \pi_{\Sigma_g} \) over \( G_2 \). In the above proof of Theorem 1, it is shown that there is no section for \( \pi_{\Sigma_g} \) over \( \mathcal{M}(H_g) \) under any identification of \( \partial H_g \) with \( \Sigma_g \), since, under the different identifications of \( \partial H_g \) with \( \Sigma_g \), \( \mathcal{M}(H_g) \) is regarded as a conjugate subgroup of \( \mathcal{M}(\Sigma_g) \).

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