PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 11, November 2010, Pages 4119–4121
S 0002-9939(2010)10450-1
Article electronically published on June 29, 2010

ON KHINTCHINE INEQUALITIES WITH A WEIGHT

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(Communicated by Nigel J. Kalton)

ABSTRACT. In this paper we prove a weighted version of the Khintchine inequalities.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(r_n)_{n\geq 1}$ be a Rademacher sequence. For a random variable $\xi : \Omega \to \mathbb{R}$ and $p > 0$ we write $\|\xi\|_p = (\mathbb{E}|\xi|^p)^{1/p}$. Our main result is the following weighted version of Khintchine's inequality. We also allow the weight to be zero on a set of positive measure.

Theorem 1. *Let $0 < p < \infty$, let $w \in L^q(\Omega)$ for some $q > p$, and assume $s := \mathbb{P}(w \neq 0) > 2/3$. Let $\xi = \sum_{n\geq 1} r_n x_n$ with $\sum_{n\geq 1} x_n^2 < \infty$. Then there exist constants $C_1 := C_1(p, w), C_2 := C_2(p, w) > 0$ such that*

$$C_1^{-1}\Big(\sum_{n\geq 1} x_n^2\Big)^{\frac{1}{2}} \leq \|w\xi\|_p \leq C_2\Big(\sum_{n\geq 1} x_n^2\Big)^{\frac{1}{2}}. \tag{1}$$

Consequently, the p-th moments for $0 < p < q$ are all comparable.

If $w \equiv 1$, the result reduces to the Khintchine inequalities [4]. The nontrivial part in (1) is the first estimate. It is somewhat remarkable since the weight is not assumed to be bounded from below. To our knowledge (1) was not known before and is potentially useful. Moreover, several experts expected the result to be false. For the proof we need a well-known L^0-version of Khintchine's inequality. We provide the details to obtain explicit constants.

Proposition 2. *For all $a \in [0, 1)$ and for all $(x_n)_{n\geq 1}$ in ℓ^2, one has*

$$\mathbb{P}\Big(\Big|\sum_{n\geq 1} r_n x_n\Big| > a\Big) < (1 - a^2)^2/3 \ \Rightarrow\ \sum_{n\geq 1} |x_n|^2 \leq 1.$$

We need the Paley-Zygmund inequality (see [2, Corollary 3.3.2]), which says that for a positive nonzero random variable $\xi : \Omega \to \mathbb{R}$ and $q \in (2, \infty)$ one has

$$\mathbb{P}(\xi > \lambda\|\xi\|_2) \geq \Big[(1 - \lambda^2)\frac{\|\xi\|_2^2}{\|\xi\|_q^2}\Big]^{q/(q-2)}, \quad \lambda \in [0, 1].$$

Received by the editors January 13, 2010 and, in revised form, February 16, 2010.
2010 *Mathematics Subject Classification.* Primary 60E15; Secondary 46B09, 60G50.
Key words and phrases. Khintchine inequalities, weight.
The author was supported by a VENI subsidy 639.031.930 of the Netherlands Organization for Scientific Research (NWO).

Proof. Assume $\sum_{n\geq 1} x_n^2 > 1$. Let $\xi = \left|\sum_{n\geq 1} r_n x_n\right|$ and $m := \|\xi\|_2 > 1$. Recall the following case of Khintchine's inequality: $\mathbb{E}\xi^4 \leq 3(\mathbb{E}\xi^2)^2$ (see [2, Section 1.3]). Therefore, the Paley-Zygmund inequality implies that

$$\mathbb{P}(\xi > a) \geq \mathbb{P}(\xi > a\|\xi\|_2) \geq (1-a^2)^2 \frac{(\mathbb{E}\xi^2)^2}{\mathbb{E}\xi^4} \geq (1-a^2)^2/3. \qquad \square$$

We will also need the following lemma.

Lemma 3. *Let $\eta = \sum_{n\geq 1} r_n x_n$, with $\sum_{n\geq 1} x_n^2 \in (0,\infty)$. Then $\mathbb{P}(\eta = 0) \leq 1 - 2e^{-2+\gamma} \approx 0.517$, where γ is Euler's constant.*

Note that for $\eta = r_1 + r_2$ one has $\mathbb{P}(\eta = 0) = 1/2$, which shows that the lemma is close to optimal.

Proof. By scaling we can assume $\|\eta\|_2 = 1$. By the Paley-Zygmund inequality applied with $\xi = |\eta|$ and $\lambda = 0$, together with the best constant in the Khintchine inequality (see [3]) one sees that for all $q > 2$,

$$\mathbb{P}(|\eta| > 0) = \mathbb{P}(\xi > 0) \geq \Big[B_q^{-2}\Big]^{q/(q-2)},$$

where $B_q = \sqrt{2}\Big(\frac{\Gamma((q+1)/2)}{\sqrt{\pi}}\Big)^{1/q}$. An elementary calculation for Γ-functions shows that $B_q^{-2q/(q-2)} \to 2e^{-2+\gamma}$ as $q \downarrow 2$, and this implies the result. $\square$

Proof of Theorem 1. The second estimate follows from Hölder's inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and the unweighted Khintchine inequality with constant $k_{r,2}$:

$$\|w\xi\|_p \leq \|w\|_q \|\xi\|_r \leq \|w\|_q k_{r,2} \Big(\sum_{n\geq 1} x_n^2\Big)^{\frac{1}{2}},$$

where we used $q > p$ to ensure that $r \in (0,\infty)$.

Next we prove the first estimate. If all the x_n are zero, there is nothing to prove. If not, then by Lemma 3 and the assumption, we have $\mathbb{P}(w\xi \neq 0) = \mathbb{P}(w \neq 0, \xi \neq 0) > 0$, and therefore $\|w\xi\|_p > 0$. To complete the proof we can assume that $\|w\xi\|_p = 1$, as follows by a scaling argument. Moreover, by replacing w by $|w|$ if necessary, we can assume that w is nonnegative.

Choose $a \in (0,1)$ so small that $b := (1-a^2)^2/3 > 1-s$, where $s = \mathbb{P}(w \neq 0)$. (For example, take a such that $b = (1-a^2)^2/3 = [(1-s)+1/3]/2$.) Let

$$\delta_0 = \sup\{\delta > 0 : \mathbb{P}(w > \delta) \geq (s+1-b)/2\}.$$

Since $\mathbb{P}(w > 0) = s > (s+1-b)/2$ we have $\delta_0 > 0$. Clearly, $A = \{w \geq \delta_0\}$ satisfies $\mathbb{P}(A) \geq (s+1-b)/2$. It follows that for all $t > 0$,

$$\begin{aligned}\mathbb{P}(\{|\xi| > t\} \cap A) &= \mathbb{P}(\mathbf{1}_A|\xi| > t) \leq t^{-p}\mathbb{E}(\mathbf{1}_A|\xi|^p)\\ &\leq t^{-p}\delta_0^{-p}\mathbb{E}(w^p\mathbf{1}_A|\xi|^p) \leq t^{-p}\delta_0^{-p}\mathbb{E}(|w\xi|^p) = t^{-p}\delta_0^{-p}.\end{aligned}$$

Therefore,

$$\mathbb{P}(\{|\xi| > t\}) \leq \mathbb{P}(\{|\xi| > t\} \cap A) + \mathbb{P}(\Omega \setminus A) < t^{-p}\delta_0^{-p} + 1 - (s+1-b)/2.$$

Now with $t = \delta_0^{-1}\Big(b - 1 + (s+1-b)/2\Big)^{-\frac{1}{p}}$ it follows that $\mathbb{P}(|\xi| > t) < b$. Let $y_n = \frac{a x_n}{t}$ and $\eta = \sum_{n\geq 1} r_n y_n$. Then $\mathbb{P}(|\eta| > a) = \mathbb{P}(|\xi| > t) < b$. Therefore, Proposition 2 gives that $\sum_{n\geq 1} y_n^2 \leq 1$. In other words, $\sum_{n\geq 1} x_n^2 \leq \frac{t^2}{a^2}$ and the result follows with $C_1 = a/t$. $\square$

Remark 4. Some possible extensions are:

(1) A more sophisticated application of the Paley-Zygmund inequality in Proposition 2 shows that in the theorem it suffices to assume that $\mathbb{P}(w \neq 0) > 1 - 2e^{-2+\gamma} \approx 0.517$. This is close to optimal, as can be seen by taking $w = \mathbf{1}_{r_1+r_2\neq 0}$ and $\xi = r_1 + r_2$, for which the weighted inequality (1) does not hold.

(2) The integrability condition on w used for the second estimate of (1) can be improved. However, the general function space for w is difficult to describe and not even rearrangement invariant (cf. [1]).

(3) With a similar technique one can obtain versions of Theorem 1 for Gaussian random variables, q-stable random variables, etc.

(4) The case where the x_n take values in a normed space X can also be considered. Then $\left(\sum_{n\geq 1} x_n^2\right)^{\frac{1}{2}}$ has to be replaced by the L^2-norm $\|\xi\|_2$, where $\xi = \sum_{n\geq 1} r_n x_n$. Note that Lemma 3 extends to this setting, as follows by applying Lemma 3 with $\eta = \langle \xi, x^*\rangle$ for a functional $x^* \in X^*$ for which $\langle \xi, x^*\rangle$ is nonzero. Also the constants in Proposition 2 can be taken as before. This follows from the fact that, also in the vector-valued setting, $\|\xi\|_4 \leq 3^{1/4}\|\xi\|_2$ (see [5]).

Acknowledgment

The author thanks the referee for helpful comments.

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