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A NEW CROSS THEOREM FOR SEPARATELY HOLOMORPHIC FUNCTIONS

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ABSTRACT. We prove a new cross theorem for separately holomorphic functions.

1. Introduction. Main result

Throughout the paper we will work in the following geometric context; details may be found in [5].

We fix an integer $N \geq 2$ and let D_j be a (connected) *Riemann domain* over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$. Let $\varnothing \neq A_j \subset D_j$ be *locally pluriregular*, $j = 1, \dots, N$.

We will use the following conventions: $A'_j := A_1 \times \cdots \times A_{j-1}$, $j = 2, \dots, N$, $A''_j := A_{j+1} \times \cdots \times A_N$, $j = 1, \dots, N-1$. Analogously, a point $a = (a_1, \dots, a_N) \in D_1 \times \cdots \times D_N$ may be written as $a = (a'_j, a_j, a''_j)$, where $a'_j := (a_1, \dots, a_{j-1})$, $a''_j := (a_{j+1}, \dots, a_N)$ (with obvious exceptions for $j \in \{1, N\}$).

We define an N*–fold cross*

$$\boldsymbol{X} = \boldsymbol{X}((D_j, A_j)_{j=1}^N) := \bigcup_{j=1}^N A'_j \times D_j \times A''_j.$$

One may easily prove that $\boldsymbol{X}$ is connected.

We say that a function $f : \boldsymbol{X} \longrightarrow \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{X}$ (we write $f \in \mathcal{O}_s(\boldsymbol{X})$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j$, the function $D_j \ni z_j \longmapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$ is holomorphic in D_j.

Let h_{A_j, D_j} denote the relative extremal function of A_j in D_j, $j = 1, \dots, N$. Recall that $h_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \leq 1,\ u|_A \leq 0\}$, $A \subset D$ (cf. [6], § 4.5). Put

$$\widehat{\boldsymbol{X}} := \{(z_1, \dots, z_N) \in D_1 \times \cdots \times D_N : h^*_{A_1, D_1}(z_1) + \cdots + h^*_{A_N, D_N}(z_N) < 1\},$$

where * stands for the upper semicontinuous regularization. One may prove that $\widehat{\boldsymbol{X}}$ is connected and $\boldsymbol{X} \subset \widehat{\boldsymbol{X}}$.

The *classical cross theorem* is the following result:

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Theorem 1 ([12], [13], [15], [14], [8], [9], [10], [7], [1], [16]). *For each $f \in \mathcal{O}_s(\boldsymbol{X})$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{X}$ and* $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{X}} |f| \leq +\infty$.

The aim of this paper is to extend the above theorem to a class of more general objects, namely (N,k)–*crosses* $\boldsymbol{X}_{N,k}$ defined for $k \in \{1,\dots,N\}$ as follows:

$$\boldsymbol{X}_{N,k} = \mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N) := \bigcup_{\substack{\alpha_1,\dots,\alpha_N \in \{0,1\} \\ \alpha_1+\dots+\alpha_N=k}} \mathfrak{X}_\alpha,$$

where

$$\mathfrak{X}_\alpha := \mathfrak{X}_{1,\alpha_1} \times \dots \times \mathfrak{X}_{N,\alpha_N}, \quad \mathfrak{X}_{j,\alpha_j} := \begin{cases} D_j, & \text{if } \alpha_j = 1, \\ A_j, & \text{if } \alpha_j = 0. \end{cases}$$

Notice that N–fold crosses are just $(N,1)$–crosses in the above terminology. Obviously, $\boldsymbol{X}_{N,N} = D_1 \times \dots \times D_N$. Thus, if $N = 2$, then in fact we have only $\boldsymbol{X}_{2,1}$.

Recall that the theory of extension of separately holomorphic functions had been first developed for $N = 2$. Then the N–fold case (obtained via induction) was considered as a natural generalization of $\boldsymbol{X}_{2,1}$. In our opinion, each of the crosses $\boldsymbol{X}_{N,k}$ may be considered as a *natural* generalization of $\boldsymbol{X}_{2,1}$. Consequently, one should try to find an analogous version of the cross theorem for all (N,k)–crosses.

We say that a function $f : \boldsymbol{X}_{N,k} \longrightarrow \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$) if for all $a = (a_1,\dots,a_N) \in A_1 \times \dots \times A_N$ and $\alpha = (\alpha_1,\dots,\alpha_N) \in \{0,1\}^N$ with $|\alpha| = k$, the function

$$D^\alpha := \prod_{\substack{j\in\{1,\dots,N\}: \\ \alpha_j=1}} D_j \ni z \longmapsto f(i_{a,\alpha}(z))$$

is holomorphic, where $i_{a,\alpha} : D^\alpha \longrightarrow \mathfrak{X}_\alpha$,

$$i_{a,\alpha}(z) := (w_1,\dots,w_N), \quad w_j := \begin{cases} z_j, & \text{if } \alpha_j = 1, \\ a_j, & \text{if } \alpha_j = 0. \end{cases}$$

Put

$$\widehat{\boldsymbol{X}}_{N,k} = \widehat{\mathbb{X}}_{N,k}((A_j,D_j)_{j=1}^N) := \Big\{(z_1,\dots,z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N h^*_{A_j,D_j}(z_j) < k\Big\}.$$

Note that $\widehat{\boldsymbol{X}}_{N,N} = D_1 \times \dots \times D_N$.

Let $\varphi_j : D_j \longrightarrow \widetilde{D}_j$ be the envelope of holomorphy (cf. [4], Definition 1.8.1). Observe that since φ_j is locally biholomorphic, the set $\widetilde{A}_j := \varphi_j(A_j) \subset \widetilde{D}_j$ is locally pluriregular, $j = 1,\dots,N$. Let

$$\widetilde{\boldsymbol{X}}_{N,k} := \mathbb{X}_{N,k}((\widetilde{A}_j,\widetilde{D}_j)_{j=1}^N), \quad \widehat{\widetilde{\boldsymbol{X}}}_{N,k} := \widehat{\mathbb{X}}_{N,k}((\widetilde{A}_j,\widetilde{D}_j)_{j=1}^N).$$

Put

$$\varphi : D_1 \times \dots \times D_N \longrightarrow \widetilde{D}_1 \times \dots \times \widetilde{D}_N, \quad \varphi(z_1,\dots,z_N) := (\varphi_1(z_1),\dots,\varphi_N(z_N)).$$

Note that:

- $\varphi(\boldsymbol{X}_{N,k}) \subset \widetilde{\boldsymbol{X}}_{N,k}$,

- $\varphi(\widehat{\boldsymbol{X}}_{N,k}) \subset \widehat{\widetilde{\boldsymbol{X}}}_{N,k}$ (because $h^*_{\widetilde{A}_j,\widetilde{D}_j} \circ \varphi_j \le h^*_{A_j,D_j}$, $j = 1,\dots,N$).

Our main result is the following *cross theorem for (N,k)–crosses*.

Theorem 2. *For every $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{\widetilde{\boldsymbol{X}}}_{N,k})$ such that $\widehat{f} \circ \varphi = f$ on $\boldsymbol{X}_{N,k}$ and $\sup_{\widehat{\widetilde{\boldsymbol{X}}}_{N,k}} |\widehat{f}| = \sup_{\boldsymbol{X}_{N,k}} |f|$.*

The proof will be presented in §5 and will be based on Theorem 1 and the following lemmas (which might also be useful in other applications).

Lemma 3. *Let G be a Riemann domain over $\mathbb{C}^n$, let $D \Subset G$ be a Riemann domain of holomorphy, and let $A \subset D$ be non-pluripolar. Put*

$$\varDelta(\mu) := \{z \in D : h^*_{A,D}(z) < \mu\}, \quad 0 < \mu \le 1.$$

Then

$$h^*_{\varDelta(r),\varDelta(s)} = \max\Big\{0, \frac{h^*_{A,D} - r}{s - r}\Big\} \text{ on } \varDelta(s), \quad 0 < r < s \le 1.$$

Lemma 4. *Assume additionally that $D_1,\dots,D_N$ are Riemann domains of holomorphy. Then*

$$h^*_{\widehat{\boldsymbol{X}}_{N,k-1},\widehat{\boldsymbol{X}}_{N,k}}(z) = \max\Big\{0, \sum_{j=1}^{N} h^*_{A_j,D_j}(z_j) - k + 1\Big\},$$
$$z = (z_1,\dots,z_N) \in \widehat{\boldsymbol{X}}_{N,k}, \quad k \in \{2,\dots,N\}.$$

To our knowledge the formula from Lemma 3 seems to be new. We do not know whether Lemmas 3, 4 are true for arbitrary Riemann domains.

2. Basic properties of (N,k)–crosses

Remark 5. (a) $A_1 \times \dots \times A_N \subset \boldsymbol{X}_{N,k} \subset \widehat{\boldsymbol{X}}_{N,k}$.

(b) $\boldsymbol{X}_{N,k-1} \subset \boldsymbol{X}_{N,k}$, $\widehat{\boldsymbol{X}}_{N,k-1} \subset \widehat{\boldsymbol{X}}_{N,k}$, $k = 2,\dots,N$.

(c) $\boldsymbol{X}_{N,k} = (\boldsymbol{X}_{N-1,k-1} \times D_N) \cup (\boldsymbol{X}_{N-1,k} \times A_N)$, $k = 2,\dots,N-1$, $N \ge 3$.

(d) $\boldsymbol{X}_{N,k}$ and $\widehat{\boldsymbol{X}}_{N,k}$ are connected.

(e) If $(D_{j,k})_{k=1}^\infty$ is a sequence of subdomains of D_j such that $D_{j,k} \nearrow D_j$, $D_{j,k} \supset A_{j,k} \nearrow A_j$, $j = 1,\dots,N$, then $\mathbb{X}_{N,k}((A_{j,k},D_{j,k})_{j=1}^N) \nearrow \boldsymbol{X}_{N,k}$ and
$$\widehat{\mathbb{X}}_{N,k}((A_{j,k},D_{j,k})_{j=1}^N) \nearrow \widehat{\boldsymbol{X}}_{N,k}.$$

(f) If $D_1,\dots,D_N$ are domains of holomorphy, then $\widehat{\boldsymbol{X}}_{N,k}$ is a domain of holomorphy.

3. Proof of Lemma 3

Let

$$L := h^*_{\varDelta(r),\varDelta(s)}, \quad R := \max\Big\{0, \frac{h^*_{A,D} - r}{s - r}\Big\}.$$

Put $\varDelta[r] := \{z \in D : h^*_{A,D}(z) \le r\}$. It is clear that

$$(*) \qquad L = h_{\varDelta(r),\varDelta(s)} \ge h^*_{\varDelta[r],\varDelta(s)} \ge h_{\varDelta[r],\varDelta(s)} \ge R,$$
$$L = R = 0 \text{ on } \varDelta(r), \quad R = 0 \text{ on } \varDelta[r].$$

Step 1. Reduction to the case $s = 1$.

Suppose that $0 < r < s < 1$. Observe that $\Delta(s)$ is a Riemann region of holomorphy. Moreover, $h^*_{A\cap\Delta(s),\Delta(s)} = (1/s)h^*_{A,D}$ on $\Delta(s)$.

Indeed, it obvious that $h^*_{A\cap\Delta(s),\Delta(s)} \geq (1/s)h^*_{A,D}$ on $\Delta(s)$. Let $u \in \mathcal{PSH}(\Delta(s))$, $u \leq 1$, $u \leq 0$ on $A \cap \Delta(s)$. Observe that for every $z_0 \in D \cap \partial(\Delta(s))$ we have $\limsup_{z\to z_0} u(z) \leq 1 \leq (1/s)h^*_{A,D}(z_0)$. Thus, the function

$$v := \begin{cases} \max\{su, h^*_{A,D}\} & \text{on } \Delta(s), \\ h^*_{A,D} & \text{on } D \setminus \Delta(s) \end{cases}$$

is plurisubharmonic on D. It is known that there exists a pluripolar set $P \subset A$ such that $h^*_{A,D} = 0$ on $A \setminus P$ (cf. [6]). Hence, $A \setminus P \subset \Delta(s)$, $v \leq h^*_{A\setminus P,D} = h^*_{A,D}$, and therefore, $h^*_{A\cap\Delta(s),\Delta(s)} \leq (1/s)h^*_{A,D}$ on $\Delta(s)$.

In particular, $A \cap S$ is not pluripolar for every connected component S of $\Delta(s)$. Hence,

$$L = h_{\Delta(r),\Delta(s)} = h_{\{h^*_{A,\Delta(s)}<r/s\},\Delta(s)}, \quad R = \max\left\{0, \frac{h^*_{A,\Delta(s)} - r/s}{1 - r/s}\right\}.$$

Thus the problem for (D, A, r, s) reduces to $(S, A\cap S, r/s, 1)$, where S is a connected component of $\Delta(s)$.

From now on we assume that $s = 1$.

Step 2. Approximation. Let $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \subset D_\nu$ is non-pluripolar, $\nu \in \mathbb{N}$. Suppose that the formula holds for each (D_ν, A_ν, r). Then it holds for (D, A, r).

Indeed, we know that $h^*_{A_\nu,D_\nu} \searrow h^*_{A,D}$. Hence $\{h^*_{A_\nu,D} < r\} \nearrow \Delta(r)$. Thus $h^*_{\{h^*_{A_\nu,D}<r\},D} \searrow h^*_{\Delta(r),D}$.

Step 3. The case where D is hyperconvex, A is compact, and $h^*_{A,D}$ is continuous.

Let $u \in \mathcal{PSH}(D)$, $u \leq 1$, $u \leq 0$ on $\Delta[r]$. Using continuity of $h^*_{A,D}$ and [6], Proposition 4.5.2, we easily conclude that $\Delta[r]$ is compact. Let $U := D \setminus \Delta[r]$. Observe that for $z_0 \in \partial U$ we get

$$\liminf_{U\ni z\to z_0} (h^*_{A,D}(z) - (1-r)u(z) - r) \geq 0.$$

Hence, by the domination principle (cf. [6], Corollary 3.7.4), $(1-r)u + r \leq h^*_{A,D}$ in U. This shows that $h_{\Delta[r],D} \leq R$. Thus, by (*), we get $h^*_{\Delta[r],D} \equiv R$ for all $0 < r < 1$. Observe that $\Delta[r_\nu] \nearrow \Delta(r)$ for $0 < r_\nu \nearrow r$. Consequently, $L \equiv R$.

Step 4. The case where D is hyperconvex and A is compact.

Let $A^{(\varepsilon)} := \bigcup_{a\in A} \overline{\widehat{\mathbb{P}}(a,\varepsilon)}$, where $\widehat{\mathbb{P}}(a,\varepsilon)$ stands for the "polydisc" in the sense of the Riemann domain D ($A^{(\varepsilon)}$ is defined for small $\varepsilon > 0$). By [6], Corollary 4.5.9, we know that $h_{A^{(\varepsilon)},D} = h^*_{A^{(\varepsilon)},D}$ is continuous. Thus, using Step 3 and (*), we get

$$h_{\{h_{A^{(\varepsilon)},D}\leq r\},D} = \max\left\{0, \frac{h_{A^{(\varepsilon)},D} - r}{1 - r}\right\}, \quad 0 < \varepsilon \ll 1.$$

By [6], Proposition 4.5.10, we have $h_{A^{(\varepsilon)},D} \nearrow h_{A,D}$ as $\varepsilon \searrow 0$. In particular,

$$\{h_{A^{(\varepsilon)},D} \leq r\} \searrow \{h_{A,D} \leq r\} \text{ as } \varepsilon \searrow 0.$$

Hence, once again by [6], Proposition 4.5.10,

$$h_{\{h_{A^{(\varepsilon)},D}\leq r\},D} \nearrow h_{\{h_{A^{(\varepsilon)},D}\leq r\},D} \text{ as } \varepsilon \searrow 0.$$

Consequently,

$$h_{\{h_{A,D}\le r\},D} = \max\Big\{0, \frac{h_{A,D}-r}{1-r}\Big\} \le R.$$

Thus $h^*_{\{h_{A,D}\le r\},D} \le R$. Observe that the set $\{h_{A,D} \le r\} \setminus \varDelta[r]$ is pluripolar. Consequently, $h^*_{\varDelta[r],D} \le R$. We finish the proof as in Step 3.

Step 5. The case where A is open.

We use Step 4 and approximation (Step 2) with $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \Subset D_\nu$ is compact non-pluripolar and D_ν is hyperconvex, $\nu \in \mathbb{N}$.

Step 6. The case where D is hyperconvex and $A \Subset D$ is non-pluripolar.

By Step 5 we have

$$h^*_{\{h^*_{\varDelta(\varepsilon),D}<r\},D} = \max\Big\{0, \frac{h^*_{\varDelta(\varepsilon),D}-r}{1-r}\Big\}, \quad 0<\varepsilon<1.$$

By [2] we get

$$\frac{h^*_{A,D}-\varepsilon}{1-\varepsilon} \le h^*_{\varDelta(\varepsilon),D} \le h^*_{A,D};$$

in particular, $h^*_{\varDelta(\varepsilon),D} \nearrow h^*_{A,D}$ as $\varepsilon \searrow 0$. Moreover,

$$\{h^*_{\varDelta(\varepsilon),D} < \frac{r-\varepsilon}{1-\varepsilon}\} \subset \varDelta(r) \subset \{h^*_{\varDelta(\varepsilon),D} < r\}, \quad 0<\varepsilon<r.$$

Consequently,

$$\max\Big\{0, \frac{h^*_{\varDelta(\varepsilon),D} - \frac{r-\varepsilon}{1-\varepsilon}}{1-\frac{r-\varepsilon}{1-\varepsilon}}\Big\} = h^*_{\{h^*_{\varDelta(\varepsilon),D}<\frac{r-\varepsilon}{1-\varepsilon}\},D}$$
$$\ge h^*_{\varDelta(r),D} \ge h^*_{\{h^*_{\varDelta(\varepsilon),D}<r\},D} = \max\Big\{0, \frac{h^*_{\varDelta(\varepsilon),D}-r}{1-r}\Big\}, \quad 0<\varepsilon<r.$$

Letting $\varepsilon \searrow 0$, we get the required formula.

Step 7. The general case.

We use Step 6 and approximation (Step 2) with $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \Subset D_\nu$ is non-pluripolar and D_ν is hyperconvex, $\nu \in \mathbb{N}$.

The proof of Lemma 3 is completed.

4. Proof of Lemma 4

By Remark 5(e), we may assume that $A_j \Subset D_j \Subset G_j$, where G_j is a Riemann domain over $\mathbb{C}^{n_j}$, $j=1,\dots,N$. Fix $2\le k\le N$. Let

$$h_j := h^*_{A_j,D_j},\ j=1,\dots,N, \quad h(z_1,\dots,z_N) := h_1(z_1)+\dots+h_N(z_N).$$

Let

$$L_{N,k} := h^*_{\widehat{\boldsymbol{X}}_{N,k-1},\widehat{\boldsymbol{X}}_{N,k}}, \quad R_{N,k}(z) = \max\Big\{0, \sum_{j=1}^N h^*_{A_j,D_j}(z_j) - k + 1\Big\},$$
$$z=(z_1,\dots,z_N)\in\widehat{\boldsymbol{X}}_{N,k}.$$

It is clear that $L_{N,k} \ge R_{N,k}$ and $L_{N,k} = R_{N,k} = 0$ on $\widehat{\boldsymbol{X}}_{N,k-1}$. Fix an $a = (a_1,\dots,a_N) \in \widehat{\boldsymbol{X}}_{N,k}\setminus\widehat{\boldsymbol{X}}_{N,k-1}$. We may assume that $h_1(a_1)\le\dots\le h_N(a_N)$. Suppose that $h_1(a_1)=\dots=h_s(a_s)=0$ and $h_{s+1}(a_{s+1}),\dots,h_N(a_N)>0$ for an

$s \in \{0, \dots, N\}$. Since $h(a) \geq k-1$, we see that in fact $s \leq N-k \leq N-2$. In particular, if $N=2$, then $s=0$.

Let $\widehat{\boldsymbol{Y}}_{N-s,p} = \widehat{\mathbb{X}}_{N-s,p}((A_j, D_j)_{j=s+1}^N)$, $p \in \{k-1, k\}$. Observe that

$$\{a_1, \dots, a_s\} \times \widehat{\boldsymbol{Y}}_{N-s,p} \subset \widehat{\boldsymbol{X}}_{N,p}, \quad p \in \{k-1, k\}.$$

Consequently,

$$h^*_{\widehat{\boldsymbol{X}}_{N,k-1}, \widehat{\boldsymbol{X}}_{N,k}}(a) \leq h^*_{\widehat{\boldsymbol{Y}}_{N-s,k-1}, \widehat{\boldsymbol{Y}}_{N-s,k}}(a_{s+1}, \dots, a_N).$$

Thus, if we know that $L_{N-s,k}(a_{s+1}, \dots, a_N) \leq R_{N-s,k}(a_{s+1}, \dots, a_N)$, then

$$L_{N,k}(a) \leq R_{N-s,k}(a_{s+1}, \dots, a_N) = R_{N,k}(a).$$

This reduces the proof to the case $s=0$, i.e. $h_j(a_j) > 0$, $j = 1, \dots, N$.

Put

$$\varDelta_{j,t} := \{z_j \in D_j : h_j(z_j) < t\}, \quad j = 1, \dots, N.$$

Take $0 < r_j < s_j \leq 1$, $j = 1, \dots, N$, such that $r_1 + \dots + r_N = k-1$ and $s_1 + \dots + s_N = k$. Observe that

$$\varDelta_{1,r_1} \times \dots \times \varDelta_{N,r_N} \subset \widehat{\boldsymbol{X}}_{N,k-1}, \quad \varDelta_{1,s_1} \times \dots \times \varDelta_{N,s_N} \subset \widehat{\boldsymbol{X}}_{N,k}.$$

Hence, using the product property for the relative extremal function (cf. [3], Theorem 4.1) and Lemma 3, we get

$$\begin{aligned} L_{N,k}(z) &\leq h^*_{\varDelta_{1,r_1} \times \dots \times \varDelta_{N,r_N}, \varDelta_{1,s_1} \times \dots \times \varDelta_{N,s_N}}(z) \\ &= \max\{h^*_{\varDelta_{1,r_1}, \varDelta_{1,r_1}}(z_1), \dots, h^*_{\varDelta_{N,r_N}, \varDelta_{N,r_N}}(z_N)\} \\ &= \max\Big\{0, \frac{h_1(z_1) - r_1}{s_1 - r_1}, \dots, \frac{h_N(z_N) - r_N}{s_N - r_N}\Big\}, \\ &\qquad\qquad\qquad\qquad z = (z_1, \dots, z_N) \in \varDelta_{1,s_1} \times \dots \times \varDelta_{N,s_N}. \end{aligned}$$

Observe that there exist numbers $s_1, \dots, s_N \in (0,1]$ such that $s_1 + \dots + s_N = k$ and

$$h_j(a_j) < s_j < \frac{h_j(a_j)}{h(a) - k + 1}, \quad j = 1, \dots, N.$$

Indeed, since the case where $h(a) = k-1$ is trivial, we may assume that $h(a) > k-1$. Note that $h_j(a_j) < \frac{h_j(a_j)}{h(a)-k+1}$, $j = 1, \dots, N$. Suppose that

$$\frac{h_j(a_j)}{h(a) - k + 1} \leq 1, \ j = 1, \dots, \sigma, \quad \frac{h_j(a_j)}{h(a) - k + 1} > 1, \ j = \sigma + 1, \dots, N,$$

for a $\sigma \in \{0, \dots, N\}$. Observe that

$$\sum_{j=1}^N \frac{h_j(a_j)}{h(a) - k + 1} = \frac{h(a)}{h(a) - k + 1} > k,$$

so the case $\sigma = N$ is simple. Thus, assume that $\sigma \leq N-1$. We only need to show that

$$\Big(\sum_{j=1}^{\sigma} \frac{h_j(a_j)}{h(a) - k + 1}\Big) + N - \sigma > k.$$

The case where $\sigma \le N-k$ is obvious. Thus assume that $\sigma \ge N-k+1$. We have to show that

$$\sum_{j=1}^{\sigma} h_j(a_j) > (h(a)-k+1)(k-N+\sigma)$$
$$= (k-1-N+\sigma)h(a) + \Big(\sum_{j=1}^{\sigma} h_j(a_j)\Big) + \Big(\sum_{j=\sigma+1}^{N} h_j(a_j)\Big) + (-k+1)(k-N+\sigma)$$

or, equivalently,

$$(k-1-N+\sigma)h(a) + \Big(\sum_{j=\sigma+1}^{N} h_j(a_j)\Big) < (k-1)(k-N+\sigma).$$

We have

$$(k-1-N+\sigma)h(a) + \Big(\sum_{j=\sigma+1}^{N} h_j(a_j)\Big)$$
$$< (k-1-N+\sigma)k + N - \sigma \le (k-1)(k-N+\sigma),$$

which gives the required inequality.

Now, define

$$r_j := \frac{h_j(a_j) - s_j(h(a)-k+1)}{k-h(a)}, \quad j = 1,\dots,N.$$

Then:

- $r_j > 0$ because $s_j < \frac{h_j(a_j)}{h(a)-k+1}$,
- $r_j < s_j$ because $h_j(a) < s_j$,
- $r_1 + \cdots + r_N = k-1$,
- $\frac{h_j(a_j)-r_j}{s_j-r_j} = h(a)-k+1$, $j = 1,\dots,N$.

Thus

$$L_{N,k}(a) \le \max\Big\{0, \frac{h_1(a_1)-r_1}{s_1-r_1}, \dots, \frac{h_N(a_N)-r_N}{s_N-r_N}\Big\}$$
$$= \max\{0, h(a)-k+1\} = R_{N,k}(a).$$

The proof of Lemma 4 is completed.

5. Proof of Theorem 2

First we prove that for each function $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ there exists exactly one $\widetilde{f} \in \mathcal{O}_s(\widetilde{\boldsymbol{X}}_{N,k})$ such that $\widetilde{f} \circ \varphi \equiv f$ and $\sup_{\widetilde{\boldsymbol{X}}_{N,k}} |\widetilde{f}| = \sup_{\boldsymbol{X}_{N,k}} |f|$.

Indeed, fix an $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$. Take $a = (a_1,\dots,a_N), b = (b_1,\dots,b_N) \in A_1 \times \cdots \times A_N$ and $\alpha = (\alpha_1,\dots,\alpha_N), \beta = (\beta_1,\dots,\beta_N) \in \{0,1\}^N$ with $|\alpha| = |\beta| = k$. To simplify notation, suppose that $\alpha = (1,\dots,1,0,\dots,0)$.

Observe that if $\varphi_j(a_j) = \varphi_j(b_j)$, $j = k+1,\dots,N$, then $f(\cdot, a_{k+1},\dots,a_N) \equiv f(\cdot, b_{k+1},\dots,b_N)$ on $D_1 \times \cdots \times D_k$.

Indeed, since $\varphi_j : D_j \longrightarrow \widetilde{D}_j$ is the envelope of holomorphy, for each $g_j \in \mathcal{O}(D_j)$, there exists a $\widetilde{g}_j \in \mathcal{O}(\widetilde{D}_j)$ such that $g_j \equiv \widetilde{g}_j \circ \varphi_j$. In particular, if $\varphi(z_j) = \varphi(w_j)$, then $g_j(z_j) = g_j(w_j)$. Take arbitrary $c_j \in A_j$, $j = 1, \dots, k$. Then

$$f(c_1, \dots, c_k, a_{k+1}, \dots, a_N) = f(c_1, \dots, c_k, b_{k+1}, a_{k+2}, \dots, a_N) \\ = \dots = f(c_1, \dots, c_k, b_{k+1}, \dots, b_N).$$

Thus $f(\cdot, a_{k+1}, \dots, a_N) = f(\cdot, b_{k+1}, \dots, b_N)$ on $A_1 \times \dots \times A_k$. It remains to use the identity principle.

Recall that

$$(\varphi_1 \times \dots \times \varphi_k) : D_1 \times \dots \times D_k \longrightarrow \widetilde{D}_1 \times \dots \times \widetilde{D}_k$$

is the envelope of holomorphy (cf. [4], Proposition 1.8.15 (b)). Consequently, the function

$$\widetilde{f}_\alpha(\cdot, \varphi_{k+1}(a_{k+1}), \dots, \varphi_N(a_N)) := ((\varphi_1 \times \dots \times \varphi_k)^*)^{-1}(f(\cdot, a_{k+1}, \dots, a_N))$$

is well defined on

$$\widetilde{\mathfrak{X}}_\alpha := \widetilde{D}_1 \times \dots \times \widetilde{D}_k \times \widetilde{A}_{k+1} \times \dots \times \widetilde{A}_N$$

with $\widetilde{f}_\alpha \circ \varphi = f$ on $\mathfrak{X}_\alpha$ and $\sup_{\widetilde{\mathfrak{X}}_\alpha} |\widetilde{f}_\alpha| = \sup_{\mathfrak{X}_\alpha} |f|$.

In particular, $\widetilde{f}_\alpha \circ \varphi = f = \widetilde{f}_\beta \circ \varphi$ on $A_1 \times \dots \times A_N$. Hence, by the identity principle, $\widetilde{f}_\alpha = \widetilde{f}_\beta$ on $\widetilde{\mathfrak{X}}_\alpha \cap \widetilde{\mathfrak{X}}_\beta$.

Thus, we may replace $((D_j, A_j)_{j=1}^N, \boldsymbol{X}_{N,k}, \widehat{\widetilde{\boldsymbol{X}}}_{N,k})$ by $((\widetilde{D}_j, \widetilde{A}_j)_{j=1}^N, \widetilde{\boldsymbol{X}}_{N,k}, \widehat{\widetilde{\boldsymbol{X}}}_{N,k})$. So we may assume that D_j is a domain of holomorphy and $\varphi_j = \mathrm{id}$, $j = 1, \dots, N$.

Moreover, by Remark 5(e), we may reduce the problem to the case where $A_j \Subset D_j \Subset G_j$, where G_j is a Riemann domain over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$.

The case $k = N$ is trivial. The case $k = 1$ is the classical cross theorem (Theorem 1). In particular, there is nothing to prove for $N = 2$. We apply induction on N. Suppose that the result is true for $N - 1 \geq 2$.

Now, we apply finite induction on k. The case $k = 1$ is known. Suppose that the result is true for $k - 1$ with $2 \leq k \leq N - 1$.

Fix an $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ and let $C := \sup_{\boldsymbol{X}_{N,k}} |f|$. Recall that

$$\boldsymbol{X}_{N,k} = (\boldsymbol{X}_{N-1,k-1} \times D_N) \cup (\boldsymbol{X}_{N-1,k} \times A_N).$$

For each $z_N \in D_N$ the function $f(\cdot, z_N)$ belongs to $\mathcal{O}_s(\boldsymbol{X}_{N-1,k-1})$. By the inductive assumption there exists a $g_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k-1})$ such that $g_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{X}_{N-1,k-1}$ and $\sup_{\widehat{\boldsymbol{X}}_{N-1,k-1}} |g_{z_N}| \leq C$. Analogously, for each $z_N \in A_N$ there exists an $h_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k})$ such that $h_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{X}_{N-1,k}$ and $\sup_{\widehat{\boldsymbol{X}}_{N-1,k}} |h_{z_N}| \leq C$. Recall that $\widehat{\boldsymbol{X}}_{N-1,k-1} \subset \widehat{\boldsymbol{X}}_{N-1,k}$ and $A_1 \times \dots \times A_{N-1} \subset \boldsymbol{X}_{N-1,k-1} \cap \boldsymbol{X}_{N-1,k}$. Since the set $A_1 \times \dots \times A_{N-1}$ is not pluripolar, we get $g_{z_N} = h_{z_N}$ on $\widehat{\boldsymbol{X}}_{N-1,k-1}$ for $z_N \in A_N$.

Consider the 2–fold cross

$$\boldsymbol{Y} := \mathbb{X}(\widehat{\boldsymbol{X}}_{N-1,k-1}, A_N; \widehat{\boldsymbol{X}}_{N-1,k}, D_N) = (\widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N) \cup (\widehat{\boldsymbol{X}}_{N-1,k} \times A_N)$$

and let $F : \boldsymbol{Y} \longrightarrow \mathbb{C}$,

$$F(z', z_N) := \begin{cases} g_{z_N}(z'), & \text{if } (z', z_N) \in \widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N, \\ h_{z_N}(z'), & \text{if } (z', z_N) \in \widehat{\boldsymbol{X}}_{N-1,k} \times A_N. \end{cases}$$

Obviously, $\sup_{\boldsymbol{Y}} |F| \le C$. To see that $F \in \mathcal{O}_s(\boldsymbol{Y})$, we have to prove that for each $z' \in \widehat{\boldsymbol{X}}_{N-1,k-1}$, the function $D_N \ni z_N \longmapsto F(z', z_N)$ is holomorphic. We know that $F(\cdot, z_N)$ is holomorphic for each $z_N \in D_N$. Let

$$\boldsymbol{Z}_{N-1,k-1} := \mathbb{X}_{N-1,k-1}((A_j, D_j)_{j=2}^N).$$

Analogously as above, for each $z_1 \in D_1$ there exists a $\varphi_{z_1} \in \mathcal{O}(\widehat{\boldsymbol{Z}}_{N-1,k-1})$ such that $\varphi_{z_1} = f(z_1, \cdot)$ on $\boldsymbol{Z}_{N-1,k-1}$. Thus

$$F(z_1, \dots, z_N) = f(z_1, \dots, z_N) = \varphi_{z_1}(z_2, \dots, z_N),$$
$$(z_1, \dots, z_N) \in (\boldsymbol{X}_{N-1,k-1} \times D_N) \cap (D_1 \times \boldsymbol{Z}_{N-1,k-1}) \supset A_1 \times \cdots \times A_{N-1} \times D_N.$$

Consequently, $F(z', \cdot) \in \mathcal{O}(D_N)$ for $z' \in A_1 \times \cdots \times A_{N-1}$ and hence, using Terada's theorem (cf. e.g. [11]), we conclude that $F \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N)$.

Now, by the classical cross theorem (Theorem 1) with $N = 2$, there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ such that $\widehat{f} = F$ on $\boldsymbol{Y}$ (in particular, $\widehat{f} = f$ on $\boldsymbol{X}_{N,k}$) and $\sup_{\widehat{\boldsymbol{Y}}} |\widehat{f}| \le C$. Recall that

$$\widehat{\boldsymbol{Y}} = \{(z', z_N) \in \widehat{\boldsymbol{X}}_{N-1,k} \times D_N : h^*_{\widehat{\boldsymbol{X}}_{N-1,k-1}, \widehat{\boldsymbol{X}}_{N-1,k}}(z') + h^*_{A_N, D_N}(z_N) < 1\}.$$

Thus, it remains to apply Lemma 4.

The proof of Theorem 2 is completed.

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