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WEIGHTED-L^2 INTERPOLATION ON NON-UNIFORMLY SEPARATED SEQUENCES

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ABSTRACT. We define a weighted-ℓ^2-norm associated to a discrete sequence Γ in $\mathbb{C}$ and a weight function φ. We then give a sufficient condition which ensures that we can always extend weighted-ℓ^2 data to global holomorphic functions which are also weighted-L^2. The condition is such that the so-called upper density of Γ is strictly less than one.

1. Introduction

Let Γ be a discrete sequence in $\mathbb{C}$ and let $\varphi : \mathbb{C} \to \mathbb{R}$ be a fixed $\mathscr{C}^2$-smooth function such that for some constants $M > m > 0$,

$$m \leq \Delta\varphi \leq M.$$

We have set $\Delta := \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}$, which is off by a factor of 4π from the usual Laplacian. This convention is convenient in the formulation of our results. In this paper we consider the problem of extending weighted-ℓ^2 data from Γ to an entire holomorphic function lying in the space

$$\mathscr{H}^2_\varphi(\mathbb{C}) := \left\{ F \in \mathcal{O}(\mathbb{C}) \ : \ ||F||^2_\varphi := \int_{\mathbb{C}} |F|^2 e^{-\varphi} dA < +\infty \right\}.$$

In one sense, this problem has been completely solved in a series of papers by a number of authors, namely Berndtsson, Ortega-Cerdà, Seip and Wallstén [7, 9, 1, 5]. The results we refer to can be stated as follows.

Theorem 1.1 (Berndtsson, Ortega-Cerdà, Seip, Wallstén). *The following statements are equivalent:*

(1) *For any $\{a_\gamma\}_{\gamma \in \Gamma} \in \ell^2_\varphi(\Gamma)$, i.e., satisfying*

$$\sum_{\gamma \in \Gamma} |a_\gamma|^2 e^{-\varphi(\gamma)} < +\infty,$$

there exists $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(\gamma) = a_\gamma$ for all $\gamma \in \Gamma$.

(2) *The sequence Γ is uniformly separated and satisfies*

$$D^+_\varphi(\Gamma) := \limsup_{r \to \infty} \sup_{z \in \mathbb{C}} \frac{n_\Gamma(z, r)}{\int_{B_r(z)} \Delta\varphi} < 1.$$

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In the last statement and from now on, $B_r(z)$ denotes the disk of radius r centered around z, and $n_\Gamma(z,r)$ the cardinality of the set $\Gamma_r(z) := B_r(z) \cap \Gamma$. The number $D^+_\varphi(\Gamma)$ is called the *upper density* of Γ. The above theorem is accompanied by a long history that the reader can find in the papers where the theorem is proved. Some work in higher dimensions has also been done [4, 6]. There is also an analogous set of problems that can be formulated in the unit disk. See, for example, [8] or the book [2].

In the present article, our goal is to investigate what happens when the sequence Γ is no longer uniformly separated. Because the theorem of Berndtsson et al. provides necessary and sufficient conditions for extension, something must be changed in the formulation of the problem we consider. We have chosen to replace the space $\ell^2_\varphi(\Gamma)$ with another weighted-ℓ^2 space and seek a positive result.

We associate to every $\gamma \in \Gamma$ the following numbers:

$$\sigma_\gamma := \inf_{\substack{\tilde\gamma\in\Gamma\\ \tilde\gamma\neq\gamma}} |\gamma - \tilde\gamma|, \quad \rho_\gamma := \min\left(\frac{\sigma_\gamma}{2}, 1\right) \quad \text{and} \quad n_\gamma := n_\Gamma(\gamma, 1).$$

Definition 1.2. We define the Hilbert space

$$\mathfrak{H}^2_\varphi(\Gamma) := \left\{ \{a_\gamma\} : \ \|\{a_\gamma\}\|^2_\varphi := \sum_\gamma |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} < +\infty \right\}.$$

With this notation, we are ready to state our main result.

Theorem 1.3. *Let Γ and φ be as above and suppose that $D^+_\varphi(\Gamma) < 1$. Then the restriction map*

$$R_\Gamma : \mathscr{H}^2_\varphi(\mathbb{C}) \to \mathfrak{H}^2_\varphi(\Gamma),$$

which sends f to its restriction to Γ, is surjective.

Remark. Note that we do not claim that the map R_Γ is well-defined; in fact Proposition 2.5 shows that it is defined and bounded on all of $\mathscr{H}^2_\varphi(\mathbb{C})$ if and only if the sequence Γ is *uniformly separated*; i.e., there exists $\varepsilon > 0$ such that

$$\rho_\gamma \geq \varepsilon \quad \text{for all } \gamma \in \Gamma.$$

Remark. If the restriction map is surjective, we say that Γ is *interpolating*. Note that the property of being interpolating is always relative to the norms of the Hilbert spaces under consideration.

Remark. In the definition of $n_\gamma := n_\Gamma(\gamma, 1)$, the choice of 1 as the radius was arbitrary and we could have used instead $n_{\gamma,\epsilon} := n_\Gamma(\gamma,\epsilon)$ for any fixed $\epsilon > 0$ and the result still holds. This may enlarge the set of sequences that are interpolating.

Remark. Our theorem does not provide any necessary conditions for interpolation. It is not known at this time (and is likely not the case) whether the density condition is also necessary and not just sufficient.

Throughout, the notation $f \lesssim g$ will be used to mean that there exists a constant $\tilde C > 0$ independent from f and g such that $f \leq \tilde C g$ and $f \simeq g$ will mean that $f \lesssim g$ and $g \lesssim f$. Furthermore, C will be used to denote an arbitrary positive constant whose value could change from one occurrence to the next. We will sometimes write C_r when we wish to emphasize a dependence on some particular parameter r. Finally, we will use $\lesssim_r$ and $\simeq_r$ to emphasize that the constants involved might depend on the parameter r.

2. Preliminary facts

2.1. **Local estimates.** The following lemmas can be found, in one form or another, in the papers [1] and [5]. We fill in the details here for the reader's convenience.

Lemma 2.1. *Let φ be as above. Take any $z \in \mathbb{C}$ and any $0 < r \leq 1$. Then there exists a holomorphic function H_z defined in $B_r(z)$, with $H_z(z) = 0$, and a constant C independent of z and r such that*

$$|\varphi(z) - \varphi(w) + 2\,\mathrm{Re}\, H_z(w)| \leq C$$

for all $w \in B_r(z)$.

Proof. If we define h_z in $B_r(z)$ by

$$h_z(w) := \varphi(w) - \varphi(z) + \int_{B_r(z)} \Big(\ln|z-\xi| - \ln|w-\xi|\Big) \Delta\varphi(\xi) dA_\xi, \tag{2.1}$$

then h_z is harmonic and $h_z(z) = 0$. Since $B_r(z)$ is simply connected there exists a holomorphic function H_z such that $2\,\mathrm{Re}\, H_z = h_z$ and $\mathrm{Im}\ H_z(z) = 0$.

We have the following estimates:

(1) $-\frac{M\pi}{2} \leq \int_{B_r(z)} \ln|z-\xi| \Delta\varphi(\xi) dA_\xi \leq 0$,

(2) $-M\pi r^2 \ln 2r \leq -\int_{B_r(z)} \ln|w-\xi| \Delta\varphi(\xi) dA_\xi \leq \frac{\pi M}{2}$.

The upper estimate in (1) follows from the subharmonicity of φ and the fact that $\ln|z-\xi| \leq 0$ for $\xi \in B_r(z)$. The lower estimate holds because

$$\begin{aligned}
\int_{B_r(z)} \ln|z-\xi| \Delta\varphi(\xi) dA_\xi &\geq \int_{B_1(z)} \ln|z-\xi| \Delta\varphi(\xi) dA_\xi \\
&\geq M \int_{B_1(z)} \ln|z-\xi| dA_\xi \\
&= 2\pi M \int_0^1 u \ln u \; du \\
&= -\frac{\pi M}{2}.
\end{aligned}$$

To see (2), let $\Delta_1 = B_r(z) \cap B_1(w)$ and $\Delta_2 = B_r(z) - \Delta_1$. Then

$$\begin{aligned}
-\int_{B_r(z)} \ln|w-\xi| \Delta\varphi(\xi) dA_\xi &= \left(-\int_{\Delta_1} \ln|w-\xi| \Delta\varphi(\xi) dA_\xi\right) \\
&\quad + \left(-\int_{\Delta_2} \ln|w-\xi| \Delta\varphi(\xi) dA_\xi\right) \\
&= I_1 + I_2.
\end{aligned}$$

Note that $|w-\xi| \leq 1$ for $\xi \in \Delta_1$ and $1 \leq |w-\xi| \leq 2r$ for $\xi \in \Delta_2$. Then just as above we have the estimates

$$\begin{aligned}
0 \leq I_1 &\leq -\int_{B_1(w)} \ln|w-\xi| \Delta\varphi(\xi) dA_\xi \\
&\leq -2\pi M \int_0^1 u \ln u \; du \\
&= \frac{\pi M}{2}
\end{aligned}$$

and

$$\begin{aligned}0 \geq I_2 &\geq -\int_{\Delta_2} \ln 2r \Delta\varphi(\xi) dA_\xi \\ &\geq -M \ln 2r \int_{B_r(z)} dA_\xi \\ &= -M\pi r^2 \ln 2r.\end{aligned}$$

Hence

$$-M\pi r^2 \ln 2r \leq I_1 + I_2 \leq \frac{\pi M}{2},$$

which is what (2) claims. Therefore, since $r \leq 1$ and $r^2 \ln 2r \to 0$ as $r \to 0$, the result follows. □

Lemma 2.2. *Let φ be as above. Take any $z \in \mathbb{C}$ and any $0 < r \leq 1$. Then given any $F \in \mathscr{H}^2_\varphi(\mathbb{C})$, the following estimate holds:*

$$|F(z)|^2 e^{-\varphi(z)} \lesssim \frac{1}{\pi r^2} \int_{B_r(z)} |F(w)|^2 e^{-\varphi(w)} dA_w \lesssim \|F\|^2_\varphi.$$

Proof. Take any H_z that satisfies the conclusions of Lemma 1.1. Using Cauchy's Integral Formula and Lemma 2.1, we get the estimate

$$\begin{aligned}|F(z)|^2 = |F(z)e^{-H_z(z)}|^2 &\leq \frac{1}{\pi r^2} \int_{B_r(z)} |F(w)|^2 e^{-2\operatorname{Re} H_z(w)} dA_w \\ &\lesssim \frac{1}{\pi r^2} \int_{B_r(z)} |F(w)|^2 e^{-\varphi(w)} dA_w e^{\varphi(z)},\end{aligned}$$

which implies the result. □

2.2. **Density and separation.** First note that, by definition, $D^+_\varphi(\Gamma) < \alpha$ if and only if there exists some $\delta > 0$ such that for all $z \in \mathbb{C}$ and all r sufficiently large, $n_\Gamma(z,r) < (\alpha - \delta)\int_{B_r(z)} \Delta\varphi$.

Proposition 2.3. *A sequence Γ is a finite union of uniformly separated sequences if and only if $D^+_\varphi(\Gamma) < +\infty$.*

Proof. First suppose that Γ is not a finite union of uniformly separated sequences. Then for any $r > 0$ and any integer m there exists a point $z_m \in \mathbb{C}$ such that $n_\Gamma(z_m, r) > m$. But then

$$\sup_{z\in\mathbb{C}} \frac{n_\Gamma(z,r)}{\int_{B_r(z)} \Delta\varphi} = +\infty,$$

which in turn implies that $D^+_\varphi(\Gamma) = +\infty$.

To see the other direction, suppose that Γ is a finite union of uniformly separated sequences. Then for any $\delta > 0$ there exists some integer N_δ such that any disk of radius δ contains at most N_δ points of Γ, i.e. $n_\Gamma(z,\delta) \leq N_\delta$ for all $z \in \mathbb{C}$. Now, any disk of radius r can be covered by a union of $\lceil 2r \rceil^2$ disks of radius $\frac{1}{\sqrt{2}}$, where $\lceil x \rceil$ denotes the ceiling function. If we let $\delta = \frac{1}{\sqrt{2}}$ we have the estimates

$$D^+_\varphi(\Gamma) \leq \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{N_\delta \lceil 2r \rceil^2}{\int_{B_r(z)} \Delta\varphi} \leq \limsup_{r\to\infty} \frac{N_\delta (2r+1)^2}{m\pi r^2} = \frac{2N_\delta}{m\pi} < +\infty. \qquad \square$$

The following lemma says that the so-called *one point interpolation problem* can always be solved when the weight φ is as above.

Lemma 2.4. *Take any $u, a \in \mathbb{C}$. Then there exists an $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(u) = a$ and*

$$\int_{\mathbb{C}} |F|^2 e^{-\varphi} dA \lesssim |a|^2 e^{-\varphi(u)}.$$

Lemma 2.4 follows from Theorem 1 since any point $u \in \mathbb{C}$ is trivially a sequence with density zero and $\rho_u = 1$. The existence of the estimate can be seen from the proof of Theorem 1.

Using Lemmas 2.2 and 2.4 we can prove

Proposition 2.5. *The map R_Γ is defined and bounded on $\mathscr{H}^2_\varphi(\mathbb{C})$ if and only if the sequence Γ is uniformly separated.*

Proof. First suppose that Γ is uniformly separated, i.e., that there exists $\epsilon > 0$ such that $\rho_\gamma \geq \epsilon$ for all $\gamma \in \Gamma$. Then there exists an integer N such $n_\gamma \leq N$ for all $\gamma \in \Gamma$. Using Lemma 2.2 and the fact that $B_\epsilon(\gamma)$ are disjoint for all $\gamma \in \Gamma$, we have that for any $F \in \mathscr{H}^2_\varphi(\mathbb{C})$,

$$\begin{aligned} R_\Gamma(F) = \sum_\gamma |F(\gamma)|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} &\leq \frac{1}{\epsilon^{2N}} \sum_\gamma |F(\gamma)|^2 e^{-\varphi(\gamma)} \\ &\lesssim \sum_\gamma \int_{B_\epsilon(\gamma)} |F(w)|^2 e^{-\varphi(w)} dA_w \lesssim \|F\|^2_\varphi. \end{aligned}$$

It then follows that R_Γ is defined and bounded on $\mathscr{H}^2_\varphi(\mathbb{C})$.

Now suppose that R_Γ is defined and bounded on $\mathscr{H}^2_\varphi(\mathbb{C})$. Take any $\gamma \in \Gamma$. By Lemma 2.4 there exist $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(\gamma) = e^{\frac{\varphi(\gamma)}{2}}$ and $\|F\|_\varphi \leq C$, where the constant is independent of γ. Since $\rho_\gamma \leq 1$, the following estimate holds:

$$\frac{1}{\rho_\gamma} \leq \sum_\gamma |F(\gamma)|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} \leq C.$$

Thus Γ is uniformly separated. $\square$

2.3. **Singularization of φ.** We will want to modify the weight φ to introduce singularities at the points of Γ. Toward this end, we introduce the function $s_r : \mathbb{C} \to [-\infty, +\infty)$ defined by

$$s_r(z) := \sum_{\gamma \in \Gamma} (\log|z-\gamma|^2 - \frac{1}{\pi r^2} \int_{B_r(z)} \log|\xi - \gamma|^2 dA_\xi).$$

First note that s_r is well defined since $\log|\xi - \gamma|^2$ is locally integrable and harmonic for $\xi \neq \gamma$ and so by the mean value property for harmonic functions,

$$s_r(z) = \sum_{\gamma \in \Gamma_r(z)} (\log|z-\gamma|^2 - \frac{1}{\pi r^2} \int_{B_r(z)} \log|\xi - \gamma|^2 dA_\xi).$$

Recall that $\Gamma_r(z) = \Gamma \cap B_r(z)$. Similarly, since $\log|z-\gamma|^2$ is subharmonic for all $z \in \mathbb{C}$, we have by the sub-mean value property for subharmonic functions that

$$s_r(z) \leq 0.$$

Note that $e^{-s_r(z)}$ is not locally integrable at any $\gamma \in \Gamma$.

From the distributional equation $\Delta_z \log|z-\gamma|^2 = \delta_\gamma$ we have that

$$\Delta_z s_r(z) = \sum_{\gamma \in \Gamma_r(z)} \delta_\gamma - \frac{n_\Gamma(z,r)}{\pi r^2} \geq -\frac{n_\Gamma(z,r)}{\pi r^2},$$

where δ_γ is the point mass distribution centered at γ and the inequality is meant in the sense of positive distributions.

Let $T_\gamma := \left\{ z \in \mathbb{C} \ : \frac{\rho_\gamma}{4} \leq |z-\gamma| \leq \ \frac{3\rho_\gamma}{4} \right\}$. Then for $z \in T_\gamma$ we have

$$\begin{aligned}
\sum_{\tilde\gamma \in \Gamma_r(z)} \log|z-\tilde\gamma|^2 \quad &= \quad \sum_{\tilde\gamma \in \Gamma_r(z) \cap \Gamma_1(\gamma)} \log|z-\tilde\gamma|^2 \\
&\quad + \sum_{\tilde\gamma \in \Gamma_r(z) - \Gamma_1(\gamma)} \log|z-\tilde\gamma|^2 \\
&\geq \quad n_\gamma \log\left(\frac{\rho_\gamma}{4}\right)^2 + \log\left(\frac{1}{16}\right)(n_\Gamma(z,r) - n_\gamma) \\
&\geq \quad n_\gamma \log \rho_\gamma^2 - C n_\Gamma(z,r)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\gamma \in \Gamma_r(z)} -\frac{1}{\pi r^2} \int_{B_r(z)} \log|\xi-\gamma|^2 dA_\xi \quad &\geq \quad \sum_{\gamma \in \Gamma_r(z)} -\frac{1}{\pi r^2} \int_{B_r(z)-B_1(\gamma)} \log|\xi-\gamma|^2 dA_\xi \\
&\geq \quad \sum_{\gamma \in \Gamma_r(z)} -\frac{1}{\pi r^2} \int_{B_r(z)-B_1(\gamma)} \log(2r)^2 dA_\xi \\
&\geq \quad -2n_\Gamma(z,r) \log 2r.
\end{aligned}$$

Thus we have that for $z \in T_\gamma$,

$$s_r(z) \geq n_\gamma \log \rho_\gamma^2 - C_r n_\Gamma(z,r).$$

Define

$$\varphi_r(z) := \frac{1}{\pi r^2} \int_{B_r(z)} \varphi(\xi) dA_\xi.$$

By using the decomposition (2.1) for φ in $B_r(z)$ it can be seen that $|\varphi - \varphi_r| \leq C_r$ or equivalently that $e^{-\varphi} \simeq_r e^{\varphi_r}$. This in particular implies that the spaces $L^2_\varphi(\mathbb{C})$ and $L^2_{\varphi_r}(\mathbb{C})$ are the same with equivalent norms. The same is true of $\mathfrak{H}^2_\varphi(\Gamma)$ and $\mathfrak{H}^2_{\varphi_r}(\Gamma)$.

Also, since

$$\Delta_z \varphi_r(z) = \Delta_z \frac{1}{\pi r^2} \int_{B_r(z)} \varphi(\xi) dA = \frac{1}{\pi r^2} \int_{B_r(z)} \Delta_\xi \varphi(\xi) dA$$

we have that

$$m \leq \Delta \varphi_r \leq M.$$

We now define a new (singular) weight

$$\psi_r := \varphi_r + s_r.$$

What we have shown is:

Lemma 2.6. *The functions ψ_r and φ_r have the following properties:*

(1) $e^{-\varphi_r(z)} \le e^{-\psi_r(z)}$ *for all* $z \in \mathbb{C}$.
(2) *For* $z \in T_\gamma$ *we have that* $e^{-\psi_r(z)} \le \frac{1}{\rho_\gamma^{2n_\gamma}} e^{C_r n_\Gamma(z,r)} e^{-\varphi_r(z)}$.
(3) $e^{-\psi_r(z)}$ *is not locally integrable at any* $\gamma \in \Gamma$.

3. Interpolation

3.1. **Proof of Theorem 1.** Our goal is to take any $\{a_\gamma\} \in \mathfrak{H}^2_\varphi(\Gamma)$ and to construct an $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(\gamma) = a_\gamma$. In order to simplify the notation a little we define $B_\gamma := B_{\rho_\gamma}(\gamma)$. For each $\gamma \in \Gamma$, let H_γ be a function satisfying the conclusions of Lemma 2.1 where $r = \rho_\gamma$. Define functions $F_\gamma : B_\gamma \to \mathbb{C}$ by

$$F_\gamma(z) := a_\gamma e^{H_\gamma(z)}.$$

It then follows that F_γ is holomorphic in B_γ and $F_\gamma(\gamma) = a_\gamma$. Furthermore, we have that

$$\begin{aligned}\int_{B_\gamma} |F_\gamma(z)|^2 e^{-\varphi(z)} dA_z &= \int_{B_\gamma} |a_\gamma|^2 \exp(2 \operatorname{Re} H_\gamma(z) - \varphi(z)) dA_z \\ &\lesssim |a_\gamma|^2 e^{-\varphi(\gamma)} \int_{B_\gamma} dA_z. \\ &\lesssim |a_\gamma|^2 e^{-\varphi(\gamma)} \rho_\gamma^2.\end{aligned}$$

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function which is identically 1 on $[0, \frac{1}{4}]$ and identically 0 on $[\frac{3}{4}, \infty)$. Define the function $\hat{F} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{F}(z) := \sum_\gamma F_\gamma(z) \eta_\gamma(z),$$

where $\eta_\gamma := \eta\left(\frac{|z-\gamma|^2}{\rho_\gamma^2}\right)$. Then $\hat{F}$ is well defined, $\hat{F}(\gamma) = a_\gamma$, and we have the estimates

$$\int_{\mathbb{C}} |\hat{F}(z)|^2 e^{-\varphi(z)} dA_z \le \sum_\gamma \int_{B_\gamma} |F_\gamma(z)|^2 e^{-\varphi(z)} dA_z \lesssim \sum_\gamma |a_\gamma|^2 e^{-\varphi(\gamma)} \rho_\gamma^2, \tag{3.1}$$

and therefore

$$\int_{\mathbb{C}} |\hat{F}(z)|^2 e^{-\varphi(z)} dA_z \lesssim \sum_\gamma |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} < +\infty.$$

Thus $\hat{F}$ is a smooth solution to our problem. We now want to correct $\hat{F}$ in some controlled manner in order to produce a holomorphic solution. There is a standard way to do this, which we now describe.

Our assumption on $D^+_\varphi(\Gamma)$ implies that for r sufficiently large,

(1) $n_\Gamma(z, r) \lesssim r^2$,
(2) there exists $\delta > 0$ such that

$$\Delta\psi_r(z) \ge \Delta\varphi_r(z) - \frac{n_\Gamma(z,r)}{\pi r^2} \ge \Delta\varphi_r(z) - (1-\delta)\frac{1}{\pi r^2} \int_{B_r(z)} \Delta\varphi(\xi) dA_\xi \ge m\delta > 0.$$

We fix such an r and δ for the remainder of the proof. From the fact that $|\varphi-\varphi_r|\leq C_r$, it follows that the estimates for $\hat{F}$ given above hold for φ replaced with φ_r. Observe that $\bar{\partial}\hat{F}$ is supported on $\bigcup_{\gamma\in\Gamma}T_\gamma$ and so we have that

$$\begin{aligned}
\int_{\mathbb{C}}|\bar{\partial}\hat{F}(z)|^2e^{-\psi_r(z)}dA_z &= \sum_\gamma\int_{T_\gamma}|\bar{\partial}\eta_\gamma(z)|^2|F_\gamma(z)|^2e^{-\psi_r(z)}dA_z\\
&\lesssim \sum_\gamma\frac{1}{\rho_\gamma^2}\int_{T_\gamma}|F_\gamma(z)|^2e^{-\psi_r(z)}dA_z\\
&\lesssim_r \sum_\gamma\frac{1}{\rho_\gamma^{2+2n_\gamma}}\int_{T_\gamma}|F_\gamma(z)|^2e^{-\varphi(z)}dA_z\\
&\lesssim_r \sum_\gamma\frac{1}{\rho_\gamma^{2+2n_\gamma}}\int_{B_\gamma}|F_\gamma(z)|^2e^{-\varphi(z)}dA_z\\
&\lesssim \sum_\gamma|a_\gamma|^2\frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}}\\
&< +\infty.
\end{aligned}$$

In the second to last inequality we used (3.1).

Then by Hörmander's Theorem (Theorem 4.2) there exists a function U such that $\bar{\partial}U=\bar{\partial}\hat{F}$ and

$$\int_{\mathbb{C}}|U(z)|^2e^{-\psi_r(z)}dA_z\lesssim\int_{\mathbb{C}}|\bar{\partial}\hat{F}(z)|^2e^{-\psi_r(z)}dA_z<+\infty.$$

The fact that $e^{-\psi_r(z)}$ is not locally integrable at γ forces $U(\gamma)=0$ for all $\gamma\in\Gamma$. We also have that

$$\begin{aligned}
\int_{\mathbb{C}}|U(z)|^2e^{-\varphi(z)}dA_z\lesssim_r\int_{\mathbb{C}}|U(z)|^2e^{-\varphi_r(z)}dA_z &\leq \int_{\mathbb{C}}|U(z)|^2e^{-\psi_r(z)}dA_z\\
&\lesssim\sum_\gamma|a_\gamma|^2\frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}}<+\infty,
\end{aligned}$$

where the first inequality follows from our comment about the equivalence of the φ and φ_r norms and the second from Lemma 2.6. We now define the function $F:=\hat{F}-U$. We immediately see that $F(\gamma)=a_\gamma$ and that F is holomorphic. Finally we have that

$$\int_{\mathbb{C}}|F|^2e^{-\varphi}dA<+\infty$$

since $\hat{F}$ and U both have finite L^2-norms. The proof is complete. □

4. Appendix

We state and outline a proof of a version of Hörmander's Theorem, which we require since it is hard to find in the literature in the exact form we require. We claim absolutely no originality here. The following "smooth" version can be found in [3] as Theorem 4.4:

Theorem 4.1. *Let ψ be a $\mathscr{C}^2$-smooth real-valued function on $\mathbb{C}$ such that there exists a $\delta > 0$ so that $\Delta\psi \geq \delta$. Then given any function $f \in L^2_\psi(\mathbb{C})$ there exists a function $u \in L^2_\psi(\mathbb{C})$ such that $\bar{\partial}u = f$ in the sense of distributions and*

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA.$$

In many applications of Hörmander's theorem (including ours) it is important to be able to relax the regularity assumption on the weight function ψ. This can be done through a standard regularization procedure as follows.

Theorem 4.2. *Let ψ be an L^1_{loc} real-valued function on $\mathbb{C}$ such that there exists a $\delta > 0$ so that $\Delta\psi \geq \delta$ in the sense of positive distributions. Then given any function $f \in L^2_\psi(\mathbb{C})$ there exists a function $u \in L^2_\psi(\mathbb{C})$ such that $\bar{\partial}u = f$ in the sense of distributions and we have the following estimate on its norm:*

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA.$$

Proof. Given a ψ satisfying the assumptions of the theorem and any $\epsilon > 0$ there exists a smooth function ψ_ϵ such that $\Delta\psi_\epsilon \geq \delta$ and $\psi_\epsilon \searrow \psi$ as $\epsilon \searrow 0$. In fact, ψ_ϵ is a convolution of ψ with a positive radial bump function (see [3]). Then Theorem 4.1 asserts that there exists a family of functions $\{u_\epsilon\}$ such that $\bar{\partial}u_\epsilon = f$ and

$$\int_{\mathbb{C}} |u_\epsilon|^2 e^{-\psi_\epsilon} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi_\epsilon} dA \leq \int_{\mathbb{C}} |f|^2 e^{-\psi} dA. \tag{4.1}$$

If we let $U_\epsilon := u_\epsilon e^{-\frac{1}{2}\psi_\epsilon}$, then (4.1) says that $\{U_\epsilon\} \subset L^2(\mathbb{C})$ is a uniformly bounded family and so has a subsequence which converges weakly to some function U. It now follows by construction that $u = Ue^{\frac{1}{2}\psi}$ solves $\bar{\partial}u = f$ (in the sense of distributions) and

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA. \qquad \square$$

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