

## A NOTE ON THE GAPS BETWEEN CONSECUTIVE ZEROS OF THE RIEMANN ZETA-FUNCTION

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ABSTRACT. Assuming the Riemann Hypothesis, we show that infinitely often consecutive non-trivial zeros of the Riemann zeta-function differ by at most 0.5155 times the average spacing and that infinitely often they differ by at least 2.6950 times the average spacing.

### 1. INTRODUCTION

Let  $\zeta(s)$  denote the Riemann zeta-function. Assuming the Riemann Hypothesis, the non-trivial zeros of  $\zeta(s)$  can be written as  $\rho = \frac{1}{2} \pm i\gamma$ , where  $\gamma$  is a positive real number. It is well known that, for  $T \geq 10$ ,

$$N(T) := \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Hence, if we let  $0 < \gamma \leq \gamma'$  denote consecutive ordinates of non-trivial zeros of  $\zeta(s)$ , we see that the average size of  $\gamma' - \gamma$  is  $2\pi/\log \gamma$ . Normalizing, we let

$$\lambda := \limsup_{\gamma > 0} \frac{(\gamma' - \gamma) \log \gamma}{2\pi}$$

and

$$\mu := \liminf_{\gamma > 0} \frac{(\gamma' - \gamma) \log \gamma}{2\pi},$$

and we observe that  $\mu \leq 1 \leq \lambda$ . It is expected that there are arbitrarily large and arbitrarily small (normalized) gaps between consecutive zeros of the Riemann zeta-function on the critical line; in other words, that  $\mu = 0$  and  $\lambda = +\infty$ . In this paper, we prove the following theorem.

**Theorem 1.1.** *Assume the Riemann Hypothesis. Then  $\lambda > 2.6950$  and  $\mu < 0.5155$ .*

We briefly describe the history of the problem. Very little is known unconditionally; however, Selberg (unpublished, but announced in [12]) has shown that  $\mu < 1 < \lambda$ . Assuming the Riemann Hypothesis, numerous authors [2, 5, 7, 8, 10]

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have obtained explicit bounds for  $\mu$  and  $\lambda$ . Theorem 1.1 improves the previously best known results under this assumption which were  $\mu < 0.5172$  due to Conrey, Ghosh, and Gonek [2] and  $\lambda > 2.6306$  due to R. R. Hall [5]. The results in Hall's paper are actually unconditional, but a lower bound for  $\lambda$  can only be obtained if the Riemann Hypothesis is assumed. Assuming the generalized Riemann Hypothesis for the zeros of Dirichlet  $L$ -functions, Conrey, Ghosh, and Gonek [3] have shown that  $\lambda > 2.68$ . Their method can be modified (see [11] and [1]) to show that  $\lambda > 3$ .

Understanding the distribution of the zeros of the zeta-function is important for a number of reasons. One reason, in particular, is the connection between the spacing of the zeros of  $\zeta(s)$  and the class number problem for imaginary quadratic fields. This is described by Conrey and Iwaniec in [4]; see also Montgomery and Weinberger [9]. Studying this connection led Montgomery [7] to investigate the pair correlation of the ordinates of the zeros of the zeta-function. He conjectured that, for any fixed  $0 < \alpha < \beta$ ,

$$\sum_{\substack{0 < \gamma, \tilde{\gamma} \leq T \\ \frac{2\pi\alpha}{\log T} \leq \tilde{\gamma} - \gamma \leq \frac{2\pi\beta}{\log T}}} 1 \sim N(T) \int_{\alpha}^{\beta} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right) du.$$

Here  $\gamma$  and  $\tilde{\gamma}$  run over two distinct sets of ordinates of the non-trivial zeros of  $\zeta(s)$ . Clearly, Montgomery's conjecture implies that  $\mu = 0$ . Moreover, F. J. Dyson observed that the eigenvalues of large, random complex Hermitian or unitary matrices have the same pair correlation function. This observation (among other things) has led to a stronger conjecture that the zeros of the zeta-function should behave, asymptotically, like the eigenvalues of large random matrices from the Gaussian Unitary Ensemble. These ideas led to the conjecture that  $\lambda = +\infty$ .

## 2. MONTGOMERY AND ODLYZKO'S METHOD FOR EXHIBITING IRREGULARITY IN THE GAPS BETWEEN CONSECUTIVE ZEROS OF $\zeta(s)$

Throughout the remainder of this paper, we assume the truth of the Riemann Hypothesis.

Let  $T$  be large and put  $K = T(\log T)^{-2}$ . Further, let

$$h(c) := c - \frac{\operatorname{Re} \left( \sum_{nk \leq K} a_k \overline{a_{nk}} g_c(n) \Lambda(n) n^{-1/2} \right)}{\sum_{k \leq K} |a_k|^2},$$

where

$$(2.1) \quad g_c(n) = \frac{2 \sin \left( \pi c \frac{\log n}{\log T} \right)}{\pi \log n}$$

and  $\Lambda(\cdot)$  is von Mangoldt's function defined by  $\Lambda(n) = \log p$  if  $n = p^k$  for a prime  $p$  and  $k \in \mathbb{N}$  and by  $\Lambda(n) = 0$  otherwise. In [8], by an argument using the Guinand-Weil explicit formula for the zeros of  $\zeta(s)$ , Montgomery and Odlyzko show that if  $h(c) < 1$  for some choice of  $c > 0$  and a sequence  $\{a_n\}$ , then  $\lambda \geq c$  and that if  $h(c) > 1$  for a choice of  $c > 0$  and a sequence  $\{a_n\}$ , then  $\mu \leq c$ . In particular, for any such choices of  $c$  and  $\{a_n\}$ , their method proves the existence of a pair of consecutive zeros of  $\zeta(s)$  with ordinates  $\gamma \leq \gamma'$  in the interval  $[T/2, 2T]$  which satisfy  $\gamma' - \gamma \geq \frac{2\pi c}{\log T}$  and  $\gamma' - \gamma \leq \frac{2\pi c}{\log T}$ , respectively.

Conrey, Ghosh, and Gonek [2], expanding on an idea of Mueller [10], have given an alternate and much simpler way of viewing this problem. Let

$$A(t) = \sum_{k \leq K} a_k k^{-it}$$

be a Dirichlet polynomial and set

$$M_1 = \int_{T/2}^{2T} |A(t)|^2 dt \quad \text{and} \quad M_2(c) = \int_{-\pi c/\log T}^{\pi c/\log T} \sum_{T/2 \leq \gamma \leq 2T} |A(\gamma + \alpha)|^2 d\alpha.$$

Then, clearly,  $M_2(c)$  is monotonically increasing and  $M_2(\mu) \leq M_1 \leq M_2(\lambda)$  when  $T$  is sufficiently large. Therefore, if it can be shown that  $M_2(c) < M_1$  for some choice of  $A(t)$  and  $c$ , then  $\lambda > c$ . Similarly, if  $M_2(c) > M_1$  for some choice of  $A(t)$  and  $c$ , then  $\mu < c$ . Using standard techniques to estimate  $M_1$  and  $M_2(c)$ , it can be shown that

$$M_2(c)/M_1 = h(c) + o(1).$$

Hence, this argument is seen to be equivalent to Montgomery and Odlyzko's method, described above. Moreover, we note that this formulation of the method suggests that we should choose a test function  $A(t)$  which is small near the zeros of  $\zeta(s)$  to exhibit large gaps between the zeros of the zeta-function and a test function  $A(t)$  which is large near the zeros of  $\zeta(s)$  to exhibit small gaps.

In [8], Montgomery and Odlyzko make the choices of

$$a_k^+ = \frac{1}{\sqrt{k}} f\left(\frac{\log k}{\log K}\right) \quad \text{and} \quad a_k^- = \frac{\lambda(k)}{\sqrt{k}} f\left(\frac{\log k}{\log K}\right)$$

(using the coefficients  $a_k^+$  to exhibit large gaps and  $a_k^-$  to exhibit small gaps), where  $f$  is a continuous function of bounded variation on  $[0, 1]$  normalized so that  $\int_0^1 |f|^2 = 1$  and  $\lambda(k)$ , the Liouville function, equals  $(-1)^{\Omega(k)}$ . Here,  $\Omega(k)$  denotes the total number of primes dividing  $k$ . By choosing  $f$  to be a certain modified Bessel function, the values  $\mu < 0.5179$  and  $\lambda > 1.9799$  are obtained. They mention that this choice of  $f$  is nearly optimal for their choice of coefficients  $\{a_k^\pm\}$ .

In [2], Conrey, Ghosh, and Gonek choose the coefficients

$$a_k^+ = \frac{d_r(k)}{\sqrt{k}} \quad \text{and} \quad a_k^- = \frac{\lambda(k)d_r(k)}{\sqrt{k}},$$

where  $d_r(k)$  is a multiplicative function defined on integral powers of a prime  $p$  by

$$d_r(p^k) = \frac{\Gamma(k+r)}{\Gamma(r)k!}.$$

In this context, exhibiting large and small (normalized) gaps between consecutive zeros of the zeta-function becomes an optimization problem in the variable  $r$ . The choice  $r = 1.1$  yields  $\mu < 0.5172$ , and the choice  $r = 2.2$  yields  $\lambda > 2.3378$ .

In order to prove Theorem 1.1, we combine the approaches of [8] and [2]. We choose the coefficients

$$(2.2) \quad a_k^+ = \frac{d_r(k)}{\sqrt{k}} f\left(\frac{\log K/k}{\log K}\right) \quad \text{and} \quad a_k^- = \frac{\lambda(k)d_r(k)}{\sqrt{k}} f\left(\frac{\log K/k}{\log K}\right)$$

for sufficiently smooth functions  $f$ . This variant allows us to optimize over both  $r$  and  $f$  rather than over just  $r$  or just  $f$ .

We now provide further insight into the choice of these coefficients. For simplicity, suppose  $f$  is a polynomial. Since, for  $\text{Re } s > 1$ ,

$$\sum_{k \geq 1} \frac{d_r(k)}{k^s} = \zeta(s)^r \quad \text{and} \quad \sum_{k \geq 1} \frac{\lambda(k)d_r(k)}{k^s} = \left( \frac{\zeta(2s)}{\zeta(s)} \right)^r,$$

with our choice of coefficients  $\{a_k^+\}$  and  $\{a_k^-\}$  we see that the test function  $A(t)$  approximates  $\zeta(\frac{1}{2} + it)^r$  and  $\zeta(1 + 2it)^r / \zeta(\frac{1}{2} + it)^r$ , respectively, and should have the desired effect of making  $A(t)$  small (respectively large) near the zeros of  $\zeta(s)$ . Moreover, when we multiply  $d_r(k)$  by  $f(\frac{\log K/k}{\log K})$ , then  $A(t)$  behaves like a linear combination of  $\zeta(\frac{1}{2} + it)^r$  and its derivatives and an analogous comment applies to the other case. The presence of the function  $f$  leads to improved numerical results for bounds for  $\mu$  and  $\lambda$ .

With the coefficients  $\{a_k^\pm\}$  in (2.2), we define

$$h^\pm(c) := c - \frac{\text{Re} \left( \sum_{nk \leq K} a_k^\pm \overline{a_{nk}^\pm} g_c(n) \Lambda(n) n^{-1/2} \right)}{\sum_{k \leq K} |a_k^\pm|^2},$$

where  $g_c(n)$  is the arithmetic function defined in (2.1). In order to establish the bounds for  $\lambda$  and  $\mu$  in Theorem 1.1, we require the following lemma.

**Lemma 2.1.** *Let  $T$  be large,  $K = T(\log T)^{-2}$ , and  $r \geq 1$ . Then we have*

$$(2.3) \quad h^\pm(c) = c \mp \frac{2r \int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi cv)}{v} f(u-v) dv du}{\int_0^1 (1-u)^{r^2-1} f(u)^2 du} + O_{f,r,\varepsilon}((\log T)^{-1+\varepsilon}),$$

where  $f$  is a continuous, real-valued function of bounded variation on  $L^2[0, 1]$  and  $\varepsilon > 0$  is arbitrary.

We are now able to deduce Theorem 1.1 from Lemma 2.1.

*Proof of Theorem 1.1.* We begin with the lower bound for  $\lambda$ . Choosing  $r = 3.00$  and

$$f(x) = 1 + 11x + 42x^2 + 26x^3 - 75x^4$$

in (2.3), a numerical calculation shows that  $h^+(2.6950) < 1$  when  $T$  is sufficiently large. This provides the lower bound for  $\lambda$  in Theorem 1.1.

We now establish the upper bound for  $\mu$ . Choosing  $r = 1.23$  and

$$f(x) = 1 + 0.99x - 0.42x^2$$

in (2.3), a numerical calculation implies that  $h^-(0.5155) > 1$  for sufficiently large  $T$ . This provides the upper bound for  $\mu$  stated in Theorem 1.1. (See Table 1 and Table 2 in §3 for some other numerically optimal choices of  $f$ .)  $\square$

Our choices of  $r$  and  $f$  shall be explained in more detail in the next section. We conclude this section with the proof of Lemma 2.1.

*Proof of Lemma 2.1.* We begin by establishing the formula for  $h^+(c)$  in (2.3). We assume that  $r \geq 1$  so that  $d_r(mn) \leq d_r(m)d_r(n)$  for  $m, n \in \mathbb{N}$ . It is well known that, for fixed  $r \geq 1$ ,

$$(2.4) \quad \sum_{k \leq x} \frac{d_r(k)^2}{k} = A_r(\log x)^{r^2} + O((\log T)^{r^2-1})$$

uniformly for  $x \leq T$ ; here  $A_r$  is a certain arithmetical constant (the exact value is not important in our argument). By partial summation, we find that the denominator in the ratio of sums in the definition of  $h^+(c)$  is

$$\begin{aligned} \sum_{k \leq K} |a_k^+|^2 &= \int_{1^-}^K f\left(\frac{\log K/x}{\log K}\right)^2 d\left(\sum_{k \leq x} \frac{d_r(k)^2}{k}\right) \\ &= A_r r^2 \int_1^{K/p} f\left(\frac{\log K/x}{\log K}\right)^2 (\log x)^{r^2-1} \frac{dx}{x} + O_{f,r}((\log T)^{r^2-1}) \end{aligned}$$

by (2.4). By the variable change  $u = 1 - \frac{\log x}{\log K}$ , we have

$$\sum_{k \leq K} |a_k^+|^2 = A_r r^2 (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f(u)^2 du + O_{f,r}((\log T)^{r^2-1}),$$

where  $\varepsilon > 0$  is arbitrary.

We now evaluate the numerator in the ratio of sums in the definition of  $h^+(c)$ . If we let

$$N^+(c) := \sum_{nk \leq K} a_k^+ \overline{a_{nk}^+} g_c(n) \Lambda(n) n^{-1/2},$$

then

$$\begin{aligned} N^+(c) &= \frac{2}{\pi} \sum_{nk \leq K} \frac{d_r(k) d_r(kn) \Lambda(n)}{kn \log n} f\left(\frac{\log K/k}{\log K}\right) f\left(\frac{\log K/nk}{\log K}\right) \sin\left(\pi c \frac{\log n}{\log T}\right) \\ &= \frac{2}{\pi} \sum_{pk \leq K} \frac{d_r(k) d_r(kp)}{kp} f\left(\frac{\log K/k}{\log K}\right) f\left(\frac{\log K/pk}{\log K}\right) \sin\left(\pi c \frac{\log p}{\log T}\right) \\ &\quad + O_{f,r}((\log T)^{r^2-1}) \\ &= \frac{2r}{\pi} \sum_{p \leq K} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right)}{p} \sum_{k \leq K/p} \frac{d_r(k)^2}{k} f\left(\frac{\log K/k}{\log K}\right) f\left(\frac{\log K/pk}{\log K}\right) \\ &\quad + O_{f,r}((\log T)^{r^2-1}), \end{aligned}$$

where the sum over  $p$  runs over the primes. By Stieltjes' integration and a variable change, the inner sum in the main term of the last expression for  $N^+(c)$  is

$$\begin{aligned} &\int_{1^-}^{K/p} f\left(\frac{\log K/x}{\log K}\right) f\left(\frac{\log K/px}{\log K}\right) d\left(\sum_{k \leq x} \frac{d_r(k)^2}{k}\right) \\ &= A_r r^2 \int_1^{K/p} f\left(\frac{\log K/x}{\log K}\right) f\left(\frac{\log K/px}{\log K}\right) (\log x)^{r^2-1} \frac{dx}{x} + O_{f,r}((\log T)^{r^2-1}) \\ &= A_r r^2 (\log K)^{r^2} \int_{\frac{\log p}{\log K}}^1 (1-u)^{r^2-1} f(u) f\left(u - \frac{\log p}{\log K}\right) du + O_{f,r}((\log T)^{r^2-1}). \end{aligned}$$

By combining the above estimates and interchanging the order of summation and integration, we conclude that  $N^+(c) = M^+(c) + O_{f,r}((\log T)^{r^2-1})$ , where

$$\begin{aligned} M^+(c) &= \frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_{\frac{\log 2}{\log K}}^1 (1-u)^{r^2-1} f(u) \sum_{2 \leq p \leq K^u} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right)}{p} f\left(u - \frac{\log p}{\log K}\right) du \\ &= \frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f(u) \sum_{2 \leq p \leq K^u} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right)}{p} f\left(u - \frac{\log p}{\log K}\right) du \\ &\quad + O_{f,r,\varepsilon}((\log T)^{r^2-1+\varepsilon}). \end{aligned}$$

By the prime number theorem with remainder term, it follows that

$$\sum_{2 \leq p \leq K^u} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right)}{p} f\left(u - \frac{\log p}{\log K}\right) = \int_2^{K^u} \frac{\sin\left(\pi c \frac{\log x}{\log T}\right)}{x \log x} f\left(u - \frac{\log x}{\log K}\right) dx + O_{f,r}\left(\frac{1}{\log T}\right).$$

By the variable change  $v = \frac{\log x}{\log K}$ , the integral is

$$\int_{\frac{\log 2}{\log K}}^u \frac{\sin\left(\pi c v \frac{\log K}{\log T}\right)}{v} f(u-v) dv = \int_0^u \frac{\sin(\pi c v)}{v} f(u-v) dv + O_{f,r,\varepsilon}((\log T)^{-1+\varepsilon}).$$

Hence,

$$\begin{aligned} N^+(c) &= \frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi c v)}{v} f(u-v) dv du \\ &\quad + O_{f,r,\varepsilon}((\log T)^{r^2-1+\varepsilon}). \end{aligned}$$

Combining our formulae for  $\sum_{k \leq K} |a_k^+|^2$  and  $N^+(c)$ , we find that

$$\begin{aligned} (2.5) \quad h^+(c) &= c - \frac{2r \int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi c v)}{v} f(u-v) dv du}{\int_0^1 (1-u)^{r^2-1} f(u)^2 du} \\ &\quad + O_{f,r,\varepsilon}((\log T)^{-1+\varepsilon}), \end{aligned}$$

as claimed.

Since the proof of the formula for  $h^-(c)$  is very similar to the proof of the formula for  $h^+(c)$ , we simply indicate the changes that need to be made in the above argument. In this case, we would consider the coefficients  $a_k^- = \frac{\lambda(k) d_r(k)}{\sqrt{k}} f\left(\frac{\log K/k}{\log K}\right)$ . Note that

$$(2.6) \quad \lambda(n)^2 = 1 \quad \text{and} \quad \lambda(pn) = -\lambda(n)$$

for every  $n \in \mathbb{N}$  and every prime  $p$ . The first identity in (2.6) implies that

$$\sum_{k \leq K} |a_k^-|^2 = \sum_{k \leq K} |a_k^+|^2 = \sum_{k \leq K} \frac{d_r(k)^2}{k} f\left(\frac{\log K/k}{\log K}\right)^2,$$

and using the second identity in (2.6) it is not hard to show that

$$\begin{aligned} &\sum_{nk \leq K} a_k^- \overline{a_{nk}^-} g_c(n) \Lambda(n) n^{-1/2} \\ &= -\frac{2}{\pi} \sum_{pk \leq K} \frac{d_r(k) d_r(kp)}{kp} f\left(\frac{\log K/k}{\log K}\right) f\left(\frac{\log K/pk}{\log K}\right) \sin\left(\pi c \frac{\log p}{\log T}\right) \\ &\quad + O_{f,r}((\log T)^{r^2-1}). \end{aligned}$$

Each of these expressions were dealt with in our evaluation of  $h^+(c)$ . The only difference is the  $-$  sign in the second identity. Thus, by the above calculations, we find that

$$(2.7) \quad h^-(c) = c + \frac{2r \int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi cv)}{v} f(u-v) dv du}{\pi \int_0^1 (1-u)^{r^2-1} f(u)^2 du} + O_{f,r,\varepsilon}((\log T)^{-1+\varepsilon}).$$

This completes the proof of Lemma 2.1.  $\square$

### 3. NUMERICAL CALCULATIONS

In this section, we summarize the numerical calculations which led to Theorem 1.1. This theorem establishes the best known bounds for  $\lambda$  and  $\mu$  assuming the Riemann Hypothesis; however, we are still far from proving the conjectured values of  $\mu = 0$  and  $\lambda = \infty$ . In fact, it is known that this is not attainable using Montgomery and Odlyzko's method with Dirichlet polynomials of length  $\leq T$ . Specifically, in [2], it is shown that  $h(c) < 1$  if  $c < \frac{1}{2}$  and  $h(c) > 1$  if  $c \geq 6.2$ . Moreover, the authors note, without proof, that  $h(c) > 1$  if  $c \geq 3.74$ . It would be interesting to better understand the limitations of this method and, in particular, if it can be used to show that  $\mu \leq \frac{1}{2}$ .

We have not been able to prove that our bounds for  $\lambda$  and  $\mu$  in Theorem 1.1 are the optimal bounds for our choice of coefficients  $\{a_k^\pm\}$  in (2.2). In the special case of  $r = 1$ , this optimization problem has been solved (in terms of prolate spheroidal wave functions). See comments in [8] and the articles [13] and [6]. When  $r \neq 1$ , the analogous optimization problem seems considerably more difficult. Instead of trying to solve it explicitly, we have instead chosen  $f$  to be a polynomial of low degree ( $\leq 6$ ) having the form  $f(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_k x^k$ . Using Mathematica, we numerically evaluated (2.5) and (2.7) for each choice of  $c$  and  $r$  in terms of the coefficients  $\alpha_1, \alpha_2, \dots, \alpha_k$ . Then, using the Minimize/Maximize commands, we were able to find numerically optimal polynomials of each degree. Our results are summarized in the following tables. The coefficients of the polynomials in Table 1 are rounded to the nearest integer, and the coefficients in Table 2 are rounded to two significant figures.

TABLE 1. Using the coefficients  $\{a_k^+\}$  defined in (2.2), the following table displays some numerically optimal polynomials of low degree for which  $h^+(c) < 1$ .

Degree	Value of $c$	Value of $r$	Polynomial
0	2.3378	2.17	1
1	2.6779	2.87	$1 + 30x$
2	2.6938	3.02	$1 + 14x + 39x^2$
3	2.6949	3.00	$1 + 9x + 60x^2 - 45x^3$
4	2.6950	3.00	$1 + 11x + 42x^2 + 26x^3 - 75x^4$
5	2.6950	3.00	$1 + 12x + 35x^2 + 61x^3 - 155x^4 + 60x^5$
6	2.6950	3.00	$1 + 12x + 39x^2 + 37x^3 - 67x^4 - 77x^5 + 76x^6$

TABLE 2. Using the coefficients  $\{a_k^-\}$  defined in (2.2), the following table displays some numerically optimal polynomials of low degree for which  $h^-(c) > 1$ .

Degree	Value of $c$	Value of $r$	Polynomial
0	.5172	1.1	1
1	.5156	1.23	$1 + 0.59x$
2	.5155	1.23	$1 + 0.99x - 0.42x^2$
3	.5155	1.23	$1 + 0.9x - 0.19x^2 - 0.16x^3$

Our numerical calculations seem to suggest that polynomials of low degree work well; it does not seem like there is much to gain by taking  $f$  to be a polynomial of degree greater than 4. To demonstrate this phenomenon, we observe that one can recover the bounds for  $\lambda$  and  $\mu$ , in the case of  $r = 1$ , derived in [8] using polynomials of low degree in place of the modified Bessel functions. Letting  $f(x) = 1 + 6.47x + 15.36x^2 - 43.65x^3 + 21.83x^4$ , a numerical calculation shows that  $h^+(1.9799) < 1$ , and if we let  $f(x) = 1 + 0.465x - 0.465x^2$ , then it can be shown that  $h^-(0.5179) > 1$ . These are the nearly optimal values obtained by Montgomery and Odlyzko in [8] when  $r = 1$ .

From Table 1 it appears that the optimal value that can be obtained for  $\lambda$  occurs when  $r \approx 3$ . It should be noted that  $r = 3$  does not appear to give the optimal value, as we are able to show that  $r = 2.998$  gives a slightly better value for  $\lambda$  using polynomials of low degree. It would be interesting to determine, in the spirit of the articles [13] and [6], the choices of  $r$  and  $f$  which give the optimal values for  $\lambda$  and  $\mu$ .

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