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A MIXED MULTIPLICITY FORMULA FOR COMPLETE IDEALS IN 2-DIMENSIONAL RATIONAL SINGULARITIES

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Abstract. Let (R, m) be a 2-dimensional rational singularity with algebraically closed residue field and for which the associated graded ring is an integrally closed domain. We use the work of Göhner on condition (N) and the theory of degree functions developed by Rees and Sharp to give a very short and elementary proof of a formula for the (mixed) multiplicity of complete m-primary ideals in (R, m).

1. Introduction

The purpose of this paper is to prove a mixed multiplicity formula for complete m-primary ideals in a 2-dimensional rational singularity (R, m) with algebraically closed residue field and for which the associated graded ring is an integrally closed domain. Let I and J be complete m-primary ideals of (R, m). An ideal I is called integrally closed or complete if $\overline{I} = I$, where $\overline{I}$ denotes the integral closure of the ideal I. Let $R_1, \ldots, R_n$ be the common immediate base points of I and J. The rings $R_1, \ldots, R_n$ are 2-dimensional regular local rings. In this paper we will show that the following formula holds for the mixed multiplicity $e_1(I|J)$ of I and J:

$$e_1(I|J) = e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J) + \sum_{i=1}^{n} e_1(I^{R_i}|J^{R_i})$$

where I^{R_i}, respectively J^{R_i}, denotes the transform of I, respectively J, in R_i. The mixed multiplicity $e_1(I|J)$ of I and J is defined by $e(IJ) = e(I) + 2e_1(I|J) + e(J)$ [14, p. 1037].

The foundation of the theory of complete ideals in 2-dimensional regular local rings was laid by O. Zariski in 1938 [16]. The theory was developed further in Appendix 5 of [17] by Zariski and Samuel. The work of Zariski was continued by Lipman, who proved several results concerning multiplicities of complete ideals and values of complete ideals with respect to prime divisors of 2-dimensional regular local rings. Chapter 14 in [12] gives a concise overview of the theory of integrally closed ideals in 2-dimensional regular local rings, following the approach of Huneke in [5]. The following two results are of particular interest to us in this paper. One has the following formula for the mixed multiplicity $e_1(I|J)$ of complete m-primary

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ideals I and J in a 2-dimensional regular local ring (R, m) with algebraically closed residue field

$$e_1(I|J) = \sum_{S>R} \mathrm{ord}_S(I^S)\mathrm{ord}_S(J^S),$$

where the sum is over all 2-dimensional regular local rings (S, m_S) birationally dominating (R, m). This formula follows from [15, Theorems 3.2, 2.1] and from [9, Corollary 3.7]. Taking $I = J$, one has for a complete m-primary ideal I of a 2-dimensional regular local ring (R, m) with algebraically closed residue field that

$$e(I) = \sum_{S>R} \mathrm{ord}_S(I^S)^2.$$

Several aspects of the theory of complete ideals in 2-dimensional regular local rings have been generalized to other classes of normal local rings by different authors, including [1, 2], [4], [6], [7], and [8, 9]. In this paper we will work in 2-dimensional rational singularities, which form a class of rings that contains the 2-dimensional regular local rings.

Let (R, m) be a 2-dimensional analytically normal local domain with infinite residue field. (R, m) is called a rational singularity if $\overline{e}_2(I) = 0$ for any m-primary ideal I of R. For an m-primary ideal I of R, $\overline{e}_2(I)$ is defined as follows. For all $n \gg 0$, one has

$$\overline{H}_I(n) := \ell\left(\frac{R}{\overline{I^n}}\right) = \overline{e}_0(I)\begin{pmatrix} n+1 \\ 2 \end{pmatrix} - \overline{e}_1(I)\begin{pmatrix} n \\ 1 \end{pmatrix} + \overline{e}_2(I),$$

where the coefficients $\overline{e}_i(I)$ are integers. The polynomial $\overline{P}_I(n) := \overline{e}_0(I)\begin{pmatrix} n+1 \\ 2 \end{pmatrix} - \overline{e}_1(I)\begin{pmatrix} n \\ 1 \end{pmatrix} + \overline{e}_2(I)$ is called the normal Hilbert polynomial of I.

In a 2-dimensional rational singularity (R, m), the product of complete ideals is complete again. S.D. Cutkosky has shown in [3] that the converse also holds if (R, m) is a 2-dimensional analytically normal local domain with algebraically closed residue field.

Göhner [4, Corollary 3.11] has shown that a 2-dimensional rational singularity satisfies condition (N). Given a prime divisor v, there exists a unique complete m-primary ideal A_v in R with $T(A_v) = \{v\}$ and such that any complete m-primary ideal with unique Rees valuation v is a power of A_v. Also, there exists a positive integer N such that for every complete m-primary ideal I in R, there is a unique decomposition $I^N = \prod_{v \in T(I)} A_v^{e_v}$. Here $T(I)$ denotes the set of all Rees valuations of I.

The techniques we use to prove the (mixed) multiplicity formula are different from the methods used in the generalizations cited earlier, in the sense that our approach uses the theory of degree functions. The proof of our main result (Theorem 3.2) uses a result on the behavior of the degree coefficients $d(I, v)$ under a quadratic transformation, obtained in [13, Theorem 3.3]. These degree coefficients were introduced by D. Rees in [10].

Let (R, m) be a local domain with quotient field K. With an m-primary ideal I of R, Rees associated an integer-valued function d_I on $m \setminus \{0\}$ as follows:

$$d_I(x) = e\left(\frac{I + xR}{xR}\right),$$

where $e(\frac{I+xR}{xR})$ denotes the multiplicity of $\frac{I+xR}{xR}$. The function d_I is called the degree function defined by I.

For every prime divisor v of R, there is an associated non-negative integer $d(I,v)$, with $d(I,v)=0$ for all except finitely many v, such that

$$d_I(x) = \sum_v d(I,v)v(x) \qquad \forall 0 \neq x \in m,$$

where the sum is over all prime divisors v of R ([10], Theorem 3.2). By a prime divisor v of R we mean a discrete valuation v of K which is non-negative on R and has center m on R and whose residual transcendence degree is $\dim R - 1$. The set of all prime divisors of R will be denoted by $P(R)$. In case (R,m) is analytically unramified, $d(I,v) \neq 0$ for all $v \in P(R)$ that are Rees valuations of I as defined by Rees in [10], whereas $d(I,v')=0$ for all other prime divisors v' of R.

We will give more background information on degree functions and on quadratic transformations in Section 2.

We will assume for the remainder of this section that (R,m) is a 2-dimensional rational singularity with algebraically closed residue field R/m. We will also assume that the associated graded ring $\mathrm{gr}_m R$ is an integrally closed domain. This implies that ord_R is a valuation and that $B\ell_m R$ is a desingularization of R [4, 8]. Here $B\ell_m R$ denotes the scheme $\mathrm{Proj}(\bigoplus_{n\geq 0} m^n)$ obtained by blowing up m.

In Section 3 we prove a key lemma (Lemma 3.1). Two formulas follow from this key lemma (Theorem 3.2 and Corollary 3.3):

(1) If I and J are complete m-primary ideals of R, then

$$e_1(I|J) = e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J) + e_1(I^{R_1}|J^{R_1}) + \ldots + e_1(I^{R_n}|J^{R_n}),$$

where $R_1,\ldots,R_n$ are the common immediate base points of I and J.

(2) If I is a complete m-primary ideal of R, then

$$e(I) = e(m)\mathrm{ord}_R(I)^2 + e(I^{R_1}) + \ldots + e(I^{R_n}),$$

where $R_1,\ldots,R_n$ are the immediate base points of I.

2. Background

Let (R,m) be a 2-dimensional Noetherian local domain with fraction field K. We will now briefly recall the definition of the Rees valuations and the Rees valuation rings of an m-primary ideal I of R. Let t be an indeterminate over R and let $R[It,t^{-1}]$ be the following subring of $R[t,t^{-1}]$: $R[It,t^{-1}] = \bigoplus_{n\in\mathbb{Z}} I^n t^n$, where $I^n = R$ if $n \leq 0$. Let $\{P_1,\ldots,P_n\}$ be the set of minimal primes of $(t^{-1})\overline{R[It,t^{-1}]}$, where $\overline{R[It,t^{-1}]}$ denotes the integral closure of $R[It,t^{-1}]$ in its fraction field $K(t)$. Then $(\overline{R[It,t^{-1}]})_{P_i}$ is a discrete valuation ring of $K(t)$ for $i=1,\ldots,n$ and

$$V_i := (\overline{R[It,t^{-1}]})_{P_i} \cap K \qquad (i=1,\ldots,n)$$

are the Rees valuation rings of I. The corresponding discrete valuations $v_1,\ldots,v_n$ are called the Rees valuations of I. The set $\{v_1,\ldots,v_n\}$ of these Rees valuations is denoted by $T(I)$. Using the Rees valuation rings of I, the integral closure $\overline{I}$ of I is given by

$$\overline{I} = \bigcap_{i=1}^{n} IV_i \cap R.$$

The degree function d_I of an m-primary ideal I in a Noetherian local domain (R, m) can be written as

$$d_I(x) = \sum_{v \in P(R)} d(I, v)v(x) \qquad \forall 0 \neq x \in m.$$

In [11, Lemma 5.1] Rees and Sharp have proved that for m-primary ideals I and J in a 2-dimensional Noetherian local domain (R, m), one has that

$$d(IJ, v) = d(I, v) + d(J, v)$$

for every prime divisor v of R . If R is analytically unramified and normal, then it follows that

$$T(IJ) = T(I) \cup T(J).$$

In [11, Theorem 4.3] it is shown that the multiplicity $e(I)$ of an m-primary ideal I in a 2-dimensional local domain (R, m) is given by

$$e(I) = \sum_{v \in P(R)} d(I, v)v(I).$$

Rees and Sharp define for I and J m-primary ideals in a 2-dimensional Cohen-Macaulay local domain (R, m):

$$d_I(J) = \min\{d_I(x) \mid 0 \neq x \in J\}.$$

One has [11, Theorem 5.2]

$$d_I(J) = \sum_{v \in P(R)} d(I, v)v(J)$$

and

$$d_I(J) = d_J(I) = e_1(I|J). \tag{2.1}$$

For technical reasons, we define $d_I(R) = 0$ for I an m-primary ideal of R.

Let I and J be m-primary ideals in a 2-dimensional Cohen-Macaulay local domain (R, m); then, the following three statements are equivalent [11, Corollary 5.3]:

(1) $\bar{I} = \bar{J}$,
(2) $d_I(x) = d_J(x) \quad \forall x \in m \setminus \{0\}$,
(3) $d(I, v) = d(J, v) \quad \forall v \in P(R)$.

Finally, we briefly recall the following notions: immediate quadratic transform R_1 of R, transform I_1 of an ideal I of R in R_1, immediate base point R_1 of an ideal I of R. Let (R, m) be a 2-dimensional rational singularity with infinite residue field and for which the associated graded ring is an integrally closed domain. If $x \in m \setminus m^2$ and if N is a maximal ideal in $R[\frac{m}{x}]$ lying over m (i.e. $N \cap R = m$), then

$$R_1 = R[\frac{m}{x}]_N$$

is called an immediate (or a first) quadratic transform of R. Let I be an m-primary ideal of R. If $\mathrm{ord}_R(I) = r$ (i.e. $I \subseteq m^r$ but $I \not\subseteq m^{r+1}$), then we have in R_1 that

$$IR_1 = x^r I_1,$$

where I_1 denotes an ideal in R_1 called the transform of I in R_1. In case $I_1 \neq R_1$, we say that (R_1, m_1) is an immediate base point of I. Here m_1 denotes the maximal

ideal of the local ring R_1. A given m-primary ideal I in R has only finitely many immediate base points.

In [13, Theorem 3.3] the following result is obtained. Let $v \neq \mathrm{ord}_R$ be a prime divisor of R and let A_v be the unique complete m-primary ideal of R with $T(A_v) = \{v\}$ and such that any complete m-primary ideal of R with v as its unique Rees valuation is a power of A_v. Let R_1 denote the unique immediate base point of A_v and let $A_v^{R_1}$ denote the transform of A_v in R_1. Then

$$d(A_v, v) = d(A_v^{R_1}, v).$$

3. A mixed multiplicity formula

In this section we will prove a formula for the mixed multiplicity of complete m-primary ideals in a 2-dimensional rational singularity (R, m) with algebraically closed residue field and for which the associated graded ring is an integrally closed domain. This formula will be obtained as a corollary of the following key lemma.

Lemma 3.1. *Let (R, m) be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring is an integrally closed domain. Let $v \neq ord_R$ be a prime divisor of R and let A_v be the unique complete m-primary ideal of R with $T(A_v) = \{v\}$ and such that any complete m-primary ideal of R with v as its unique Rees valuation is a power of A_v. Let J be a complete m-primary ideal of R. Then*

$$d_{A_v}(J) = e(m)\mathrm{ord}_R(A_v)\mathrm{ord}_R(J) + d_{A_v^{R_1}}(J^{R_1}),$$

where R_1 is the unique immediate base point of A_v.

Proof. Let $R_1 = R[\frac{m}{x}]_N$ (with $x \in m \setminus m^2$ and N a maximal ideal in $R[\frac{m}{x}]$ lying over m) be the immediate base point of A_v. Put $J_1 = J^{R_1}$, $r = \mathrm{ord}_R(A_v)$ and $s = \mathrm{ord}_R(J)$. Then $x^s J_1 = JR_1$ and $v(J) = v(J_1) + sv(m)$.

From (2.1), it follows that

$$d_{A_v}(m) = d_m(A_v).$$

So $d(A_v, v)v(m) = d(m, \mathrm{ord}_R)r$. This implies that

$$d(A_v, v)v(J_1) + d(A_v, v)sv(m) = d(A_v, v)v(J_1) + d(m, \mathrm{ord}_R)rs.$$

From [13, Theorem 3.3], it follows that $d(A_v, v) = d(A_v^{R_1}, v)$. So

$$d(A_v, v)v(J) = d(A_v, v)\left(v(J_1) + sv(m)\right) = d(m, \mathrm{ord}_R)rs + d(A_v^{R_1}, v)v(J_1).$$

Since $T(A_v^{R_1}) = \{v\}$, this implies that $d_{A_v}(J) = e(m)rs + d_{A_v^{R_1}}(J^{R_1})$. □

A first consequence of this key lemma is the following mixed multiplicity formula.

Theorem 3.2. *Let (R, m) be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring is an integrally closed domain. Let I, J be complete m-primary ideals of R. Then*

$$e_1(I|J) = d_I(J) = d_J(I) = e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J) + \sum_{i=1}^{n} e_1(I^{R_i}|J^{R_i}), \tag{3.1}$$

where $R_1, \ldots, R_n$ are the common immediate base points of I and J.

Proof. Let $T(I) = \{v_1, \dots, v_n\}$. First assume that $\mathrm{ord}_R \notin T(I)$. Using [4, Corollary 3.11], we can write

$$I^N = A_{v_1}^{e_1} \dots A_{v_n}^{e_n}.$$

Suppose that R_i is the unique immediate base point of A_{v_i}, for $i = 1, \dots, n$. Then

$$d_{I^N}(J) = \sum_{i=1}^{n} e_i d_{A_{v_i}}(J).$$

From Lemma 3.1 and from the theory of Rees and Sharp, it follows that

$$\begin{aligned} d_{I^N}(J) &= \sum_{i=1}^{n} e_i \left(e(m)\mathrm{ord}_R(A_{v_i})\mathrm{ord}_R(J) + d_{A_{v_i}^{R_i}}(J^{R_i}) \right) \\ &= e(m)\mathrm{ord}_R(I^N)\mathrm{ord}_R(J) + \sum_{i=1}^{n} e_i d_{A_{v_i}^{R_i}}(J^{R_i}) \\ &= e(m)\mathrm{ord}_R(I^N)\mathrm{ord}_R(J) + \sum_{i=1}^{n} d_{(A_{v_i}^{e_i})^{R_i}}(J^{R_i}) \\ &= e(m)\mathrm{ord}_R(I^N)\mathrm{ord}_R(J) + \sum_{i=1}^{n} d_{(I^N)^{R_i}}(J^{R_i}) \\ &= N\left(e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J) + \sum_{i=1}^{n} d_{I^{R_i}}(J^{R_i}) \right). \end{aligned}$$

Since $d_{I^N}(J) = N d_I(J)$, this implies that

$$e_1(I|J) = d_I(J) = d_J(I) = e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J) + \sum_{i=1}^{n} d_{I^{R_i}}(J^{R_i}).$$

Since $d_{I^{R_i}}(J^{R_i}) = 0$ if $J^{R_i} = R_i$, formula (3.1) follows.

Now assume that $\mathrm{ord}_R \in T(I)$. In case $T(I) = \{\mathrm{ord}_R\}$, we have $I = m^k$ with $k = \mathrm{ord}_R(I)$, so

$$d_I(J) = d_{m^k}(J) = k d_m(J) = k d(m, \mathrm{ord}_R)\mathrm{ord}_R(J) = e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J),$$

which implies formula (3.1) in this case. In case $T(I) = \{\mathrm{ord}_R, v_2, \dots, v_n\}$, we have $I^N = m^k I_1$, where I_1 is a complete m-primary ideal of R with $\mathrm{ord}_R \notin T(I_1)$: $T(I_1) = \{v_2, \dots, v_n\}$. We will now show how the mixed multiplicity formula (3.1) follows from the fact that this formula already holds for I_1. First note that I and I_1 have the same immediate base points $R_2, \dots, R_n$ and that $(I^N)^{R_i} = I_1^{R_i}$ for any immediate base point R_i of I. Using the theory of Rees and Sharp, we see that

$$d_{I^N}(J) = \sum_{v \in T(I^N)} d(I^N, v)v(J) = d(I^N, \mathrm{ord}_R)\mathrm{ord}_R(J) + \sum_{i=2}^{n} d(I^N, v_i)v_i(J).$$

Since $I^N = m^k I_1$, it follows that

$$d(I^N, v_i) = d(I_1, v_i) \qquad \forall i = 2, \dots, n$$

and

$$d(I^N, \mathrm{ord}_R) = k d(m, \mathrm{ord}_R) = k e(m).$$

Together with the fact that formula (3.1) holds for $e_1(I_1|J)$, this implies that

$$\begin{aligned}
d_{I^N}(J) &= ke(m)\mathrm{ord}_R(J) + \sum_{i=2}^{n} d(I_1, v_i)v_i(J) \\
&= ke(m)\mathrm{ord}_R(J) + d_{I_1}(J) \\
&= ke(m)\mathrm{ord}_R(J) + e(m)\mathrm{ord}_R(I_1)\mathrm{ord}_R(J) + \sum_{i=2}^{n} d_{I_1^{R_i}}(J^{R_i}) \\
&= e(m)\left(k + \mathrm{ord}_R(I_1)\right)\mathrm{ord}_R(J) + \sum_{i=2}^{n} d_{(I^N)^{R_i}}(J^{R_i}) \\
&= e(m)\mathrm{ord}_R(I^N)\mathrm{ord}_R(J) + \sum_{i=2}^{n} d_{(I^N)^{R_i}}(J^{R_i}) \\
&= N\left(e(m)\mathrm{ord}_R(I)\mathrm{ord}_R(J) + \sum_{i=2}^{n} d_{I^{R_i}}(J^{R_i})\right).
\end{aligned}$$

Since $d_{I^N}(J) = Nd_I(J)$, we can conclude that formula (3.1) holds. □

From the previous theorem, we immediately obtain the following result.

Corollary 3.3. *Let (R, m) be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring is an integrally closed domain. Let I be a complete m-primary ideal of R. Then*

$$e(I) = e(m)\mathrm{ord}_R(I)^2 + e(I^{R_1}) + \ldots + e(I^{R_n}),$$

where $R_1, \ldots, R_n$ are the immediate base points of I.

Proof. This formula follows immediately from Theorem 3.2 by taking $I = J$. □

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