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LOWER VOLUME ESTIMATES AND SOBOLEV INEQUALITIES

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Abstract. We consider complete manifolds with asymptotically non-negative curvature which enjoy a Euclidean-type Sobolev inequality and we get an explicit lower control on the volume of geodesic balls. In case the amount of negative curvature is small and the Sobolev constant is almost optimal, we deduce that the manifold is diffeomorphic to Euclidean space. This extends previous results by M. Ledoux and C. Xia.

1. Introduction

A Riemannian manifold $(M, \langle , \rangle)$ of dimension $\dim M = m > p \geq 1$ is said to support a Euclidean-type Sobolev inequality if there exists a constant $C_M > 0$ such that, for every $u \in C_c^\infty(M)$,

$$\left(\int_M |u|^{p^*} \, d\mathrm{vol}\right)^{\frac{1}{p^*}} \leq C_M \left(\int_M |\nabla u|^p \, d\mathrm{vol}\right)^{\frac{1}{p}}, \tag{1}$$

where

$$p^* = \frac{mp}{m-p}$$

and $d\mathrm{vol}$ denotes the Riemannian measure of M. Clearly, (1) implies that there exists a continuous imbedding $W^{1,p}(M) \hookrightarrow L^{p^*}(M)$ and can be expressed in the equivalent form

$$C_M^{-p} \leq \inf_{u \in \Lambda} \int_M |\nabla u|^p \, d\mathrm{vol},$$

where

$$\Lambda = \left\{ u \in L^{p^*}(M) : |\nabla u| \in L^p \text{ and } \int_M |u|^{p^*} \, d\mathrm{vol}_M = 1 \right\}.$$

The validity of (1), as well as the best value of the Sobolev constant C_M, have intriguing and deep connections with the geometry of the underlying manifold, many of which are discussed in the excellent lecture notes [5]. See also [8] for a survey in the more abstract perspective of Markov diffusion processes, and [9] for the relevance of (1) in the $L^{p,q}$-cohomology theory. For instance, we note that a complete manifold with non-negative Ricci curvature (but, in fact, a certain amount of negative curvature is allowed) and supporting a Euclidean-type Sobolev inequality is necessarily connected at infinity. This fact can be proved using (non-linear) potential-theoretic arguments; see [10], [11].

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It is known (see e.g. Proposition 4.2 in [5]) that

$$C_M \geq K(m,p), \tag{2}$$

where $K(m,p)$ is the best constant in the corresponding Sobolev inequality of $\mathbb{R}^m$. It was discovered by M. Ledoux, [7], that for complete manifolds of non-negative Ricci curvature, the equality in (2) forces M to be isometric to $\mathbb{R}^m$. This important rigidity result has been generalized by C. Xia, [13], by showing that, in case C_M is sufficiently close to $K(m,p)$, then M is diffeomorphic to $\mathbb{R}^m$. The key ingredient in the Ledoux-Xia argument is a sharp lower estimate for the growth of geodesic balls which depends explicitly on the Sobolev constant. Actually, it is known, [2], that the validity of (1) implies that there exists a (small) constant $\gamma = \gamma(m,p,C_M) > 0$ (depending continuously on C_M) such that

$$V(B_t) \geq \gamma V(\mathbb{B}_t). \tag{3}$$

However, to obtain the desired rigidity, one needs to get a sharp value for γ. In this paper, using a somewhat more geometric approach, we are able to extend this kind of estimate to manifolds with asymptotically non-negative curvature; see Theorem 1. As a consequence, using a theorem by S.-H. Zhu, [14], we will deduce rigidity even in this more general context.

Notation. In what follows, having fixed a reference origin $o \in M$, we set $r(x) = \mathrm{dist}_M(x,o)$, and we denote by B_t and ∂B_t the geodesic ball and sphere of radius $t > 0$ centered at o. The corresponding balls and spheres in the m-dimensional Euclidean space are denoted by $\mathbb{B}_t$ and $\partial\mathbb{B}_t$. Finally, the symbols $V(B_t)$ and $A(\partial B_t)$ stand, respectively, for the Riemannian volume of B_t and the $(m-1)$-dimensional Hausdorff measure of ∂B_t.

Theorem 1. *Let $(M, \langle,\rangle)$ be a complete, m-dimensional Riemannian manifold, $m \geq 3$, with*

$$^M\mathrm{Ric}(x) \geq -(m-1)G(r(x)) \text{ on } M \tag{4}$$

for some non-negative function $G \in C^0([0,+\infty))$. Assume that G satisfies the integrability condition

$$\int_0^\infty tG(t)dt = b < +\infty \tag{5}$$

and that the Euclidean-type Sobolev inequality (1) *holds on M, for some $1 < p < m$. Then*

$$e^{(m-1)b}V(\mathbb{B}_t) \geq V(B_t) \geq \hat{C}(m,p,C_M,b)V(\mathbb{B}_t), \tag{6}$$

where

$$\hat{C}(m,p,C_M,b) = e^{-b(m-1)}\left[\left(\frac{C_M}{K(m,p)}\right)^p + C_M^p C_2\right]^{-\frac{m}{p}} \tag{7}$$

and

$$0 < C_2 = C_2(m,p,b) \to 0, \text{ as } b \to 0.$$

Using Theorem 1 we shall deduce the announced topological rigidity result.

Corollary 2. *Given $m \geq 3$, $m > p$, there exist constants $b_0 = b_0(m,p) > 0$ and $\varepsilon_0 = \varepsilon_0(m,p) > 0$ such that the following holds:*

Let M be an m-dimensional complete manifold supporting the Sobolev inequality (1) *with $C_M \leq K(m,p)+\varepsilon$, for some $0<\varepsilon\leq\varepsilon_0$, and such that*

$$^{M}\operatorname{Sect} \geq -G(r(x)) \text{ on } M,$$

where G satisfies (5) *for some $0<b\leq b_0$. Then M is diffeomorphic to $\mathbb{R}^m$.*

2. Proof of the lower volume estimate

Recall that, in $\mathbb{R}^m$, the equality in (1) with the best constant $C_{\mathbb{R}^m}=K(m,p)$ is realized by the (radial) Bliss-Aubin-Talenti functions $\phi_\lambda(x)=\varphi_\lambda(|x|)$ for every $\lambda>0$, where $|x|$ is the Euclidean norm of x and $\varphi_\lambda(t)$ are the real-valued functions defined as

$$\varphi_\lambda(t)=\frac{\beta(m,p)\,\lambda^{\frac{m-p}{p^2}}}{\left(\lambda+t^{\frac{p}{p-1}}\right)^{\frac{m}{p}-1}}.$$

If we choose $\beta(m,p)>0$ such that

$$\int_{\mathbb{R}^m}\phi_\lambda^{p^*}(x)dx=1, \tag{8}$$

then

$$K(m,p)^{-p}=\int_{\mathbb{R}^m}|\varphi_\lambda'(|x|)|^p\,dx$$

and, by the standard calculus of variations, the extremal functions ϕ_λ obey the (non-linear) Yamabe-type equation

$$^{\mathbb{R}^m}\Delta_p\phi_\lambda=-K(m,p)^{-p}\,\phi_\lambda^{p^*-1}, \tag{9}$$

where

$$\Delta_p u=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \tag{10}$$

stands for the p-Laplacian of a given function u.

Define

$$\hat{\phi}_\lambda : M\to\mathbb{R} \text{ as } \hat{\phi}_\lambda(x):=\varphi_\lambda(r(x)).$$

If we were in the assumptions of Ledoux and Xia, namely, if the Ricci curvature was non-negative, the idea of our proof would be simply to apply the Karp version of Stokes' theorem, [6], to the vector field $X_\lambda:=\hat{\phi}_\lambda|\nabla\phi_\lambda|^{p-2}\nabla\hat{\phi}_\lambda$, once we have observed that, by (9) and the Laplacian comparison theorem, each function $\hat{\phi}_\lambda$ on M satisfies

$$\Delta_p\hat{\phi}_\lambda\geq -K(m,p)^{-p}\,\hat{\phi}_\lambda^{p^*-1}.$$

This would lead directly to inequality (2.2) in [13], and the desired volume conclusion can be deduced. The strategy for the proof of the general case stated in Theorem 1 is completely similar. Clearly, this time we have to take into account the (small) perturbations of (10) introduced by the negative curvature. Note also that, due to the possible presence of cut-points, all the computations have to be performed in the sense of distributions.

Proof of Theorem 1. Let $h\in C^2([0,+\infty))$ be the solution of the problem

$$\begin{cases} h''(t)-G(t)h(t)=0,\\ h(0)=0,\ h'(0)=1,\end{cases}$$

and consider the m-dimensional model manifold M_h defined as $M_h := (\mathbb{R} \times \mathbb{S}^{m-1}, ds^2 + h^2(s)d\theta^2)$, where $d\theta^2$ is the standard metric on $\mathbb{S}^{m-1}$. We shall use an index h to denote objects and quantities referred to M_h. Thus, we denote by $\mathbb{B}_t^h$ and $\partial\mathbb{B}_t^h$ the geodesic ball and sphere of radius $t > 0$ in M_h. Moreover, we introduce the family of functions $\phi_{\lambda,h} : M_h \to \mathbb{R}$ defined by $\phi_{\lambda,h}((s,\theta)) := \varphi_\lambda(s)$. For later purposes, we recall that ([14], [10])

$$\mathrm{V}(\mathbb{B}_t^h) \geq \mathrm{V}\left(\mathbb{B}_t\right), \, t \geq 0. \tag{11}$$

Furthermore, we observe that, according to the Bishop-Gromov comparison theorem and its generalizations, [3], [10], $A\left(\partial B_t\right)/A\left(\partial\mathbb{B}_t^h\right)$ is a decreasing function of $t > 0$ and the following relations hold:

$$\begin{aligned} &A\left(\partial B_t\right) \leq A(\partial\mathbb{B}_t^h) \leq e^{b(m-1)} A\left(\partial\mathbb{B}_t\right), \\ &V\left(B_t\right) \leq V(\mathbb{B}_t^h) \leq e^{b(m-1)} V\left(\mathbb{B}_t\right). \end{aligned} \tag{12}$$

By the co-area formula, these imply that

$$\begin{cases} \text{(i)} & \int_M \hat{\phi}_\lambda^{p^*} d\mathrm{vol} \leq \int_{M_h} \phi_{\lambda,h}^{p^*} d\mathrm{vol}_h \leq e^{b(m-1)}, \\ \text{(ii)} & |\nabla\hat{\phi}_\lambda| \in L^p\left(M\right), \\ \text{(iii)} & r(x)^{-1}\widehat{\phi}_\lambda|\nabla\widehat{\phi}_\lambda|^{p-1} \in L^1(M), \\ \text{(iv)} & \int_{B_R} \widehat{\phi}_\lambda|\nabla\widehat{\phi}_\lambda|^{p-1} d\mathrm{vol} = o\left(R\right), \text{ as } R \to +\infty. \end{cases} \tag{13}$$

Here $d\mathrm{vol}_h$ stands for the Riemannian measure on M_h.

Now, by Laplacian comparison, [10], assumption (4) yields

$$\Delta r \leq \frac{(m-1)e^b}{r}$$

pointwise on $M \setminus \mathrm{cut}(o)$ and weakly on all of M. This means that

$$-\int_M \langle \nabla r, \nabla\eta \rangle \, d\mathrm{vol} \leq \int_M \eta \frac{(m-1)e^b}{r} d\mathrm{vol}, \tag{14}$$

for all $0 \leq \eta \in W_c^{1,2}(M)$. Let $0 \leq \xi \in C_c^\infty(M)$ to be chosen later and apply (14) with

$$\eta = -\left(\xi\hat{\phi}_\lambda\right)|\varphi_\lambda'(r)|^{p-2}\varphi_\lambda'(r),$$

thus obtaining

$$\begin{aligned} &\int_M \varphi_\lambda'(r)|\varphi_\lambda'(r)|^{p-2}\left\langle \nabla r, \nabla\left(\xi\hat{\phi}_\lambda\right)\right\rangle d\mathrm{vol} \\ &\leq -\int_M |\varphi_\lambda'(r)|^{p-2}\left[(p-1)\varphi_\lambda''(r) + \frac{(m-1)e^b}{r}\varphi_\lambda'(r)\right]\left(\xi\hat{\phi}_\lambda\right) d\mathrm{vol}. \end{aligned} \tag{15}$$

On the other hand, according to (9),

$$|\varphi_\lambda'(t)|^{p-2}\left\{(p-1)\varphi_\lambda''(t) + \frac{m-1}{t}\varphi_\lambda'(t)\right\} = -K(m,p)^{-p}\varphi_\lambda^{p^*-1}(t)$$

for all $t > 0$, and inserting into (15) gives

$$
\begin{aligned}
(16)\qquad & -\int_M K(m,p)^{-p}\xi\hat{\phi}_\lambda^{p^*}\,d\mathrm{vol} + \int_M \hat{\phi}_\lambda\varphi_\lambda'(r)|\varphi_\lambda'(r)|^{p-2}\frac{(m-1)(e^b-1)}{r}\xi d\mathrm{vol} \\
&\le -\int_M \xi\left|\nabla\hat{\phi}_\lambda\right|^p d\mathrm{vol} - \int_M \hat{\phi}_\lambda\varphi_\lambda'(r)|\varphi_\lambda'(r)|^{p-2}\left\langle\nabla r,\nabla\xi\right\rangle d\mathrm{vol} \\
&\le -\int_M \xi\left|\nabla\hat{\phi}_\lambda\right|^p d\mathrm{vol} - \int_M \hat{\phi}_\lambda\varphi_\lambda'(r)|\varphi_\lambda'(r)|^{p-2}|\nabla\xi| d\mathrm{vol}.
\end{aligned}
$$

Now choose $\xi = \xi_R$ such that $\xi \equiv 1$ on B_R, $\xi \equiv 0$ on $M \setminus B_{2R}$ and $|\nabla\xi| < 2/R$. Then, taking the limits as $R \to +\infty$ in (16) and recalling (13), we obtain

$$
\begin{aligned}
&\int_M \left|\nabla\hat{\phi}_\lambda\right|^p d\mathrm{vol} - K(m,p)^{-p}\int_M \hat{\phi}_\lambda^{p^*}\,d\mathrm{vol} \\
&\le (m-1)\left(\tfrac{m-p}{p-1}\right)^{p-1}\beta^p(e^b-1)\lambda^{\frac{m-p}{p}}\int_M(\lambda + r^{\frac{p}{p-1}})^{-(m-1)}d\mathrm{vol},
\end{aligned}
$$

proving that

$$
(17)\qquad \frac{\int_M \left|\nabla\hat{\phi}_\lambda\right|^p}{\int_M \hat{\phi}_\lambda^{p^*}} \le K\,(m,p)^{-p} + C_1,
$$

where we have set

$$
(18)\quad C_1(m,p,\lambda,b) := (m-1)\left(\tfrac{m-p}{p-1}\right)^{p-1}\beta^{-\frac{p^2}{m-p}}(e^b-1)\frac{\int_M(\lambda + r^{\frac{p}{p-1}})^{-(m-1)}d\mathrm{vol}}{\lambda\int_M(\lambda + r^{\frac{p}{p-1}})^{-m}d\mathrm{vol}}.
$$

By (12) and computing explicitly the integrals on $\mathbb{R}^m$, we get

$$
\begin{aligned}
\int_M(\lambda + r^{\frac{p}{p-1}})^{-(m-1)}d\mathrm{vol} &\le \int_0^\infty \frac{A(\partial\mathbb{B}_t)e^{b(m-1)}}{(\lambda + t^{\frac{p}{p-1}})^{m-1}} \\
&= e^{b(m-1)}A(\partial\mathbb{B}_1)\lambda^{-\frac{m-p}{p}}\frac{\Gamma(m-\frac{m}{p})\Gamma(\frac{m}{p}-1)}{\frac{p}{p-1}\Gamma(m-1)},
\end{aligned}
$$

where Γ denotes the Euler Gamma function. On the other hand,

$$
(19)\qquad \frac{V(B_t)}{\left(\lambda + t^{\frac{p}{p-1}}\right)^m} \le \frac{V(\mathbb{B}_t^h)}{\left(\lambda + t^{\frac{p}{p-1}}\right)^m} \le e^{b(m-1)}\frac{V(\mathbb{B}_t)}{\left(\lambda + t^{\frac{p}{p-1}}\right)^m} \to 0,
$$

as $t \to \infty$. Therefore, we can integrate by parts using the co-area formula, apply (3), integrate by parts again and compute explicitly the integrals on $\mathbb{R}^m$, thus obtaining

$$
\begin{aligned}
\int_M(\lambda + r^{\frac{p}{p-1}})^{-m}d\mathrm{vol} &= \int_0^\infty V(B_t)\left(-\frac{d}{dt}(\lambda + t^{\frac{p}{p-1}})^{-m}\right)dt \\
&\ge \gamma\int_0^\infty V(\mathbb{B}_t)\left(-\frac{d}{dt}(\lambda + t^{\frac{p}{p-1}})^{-m}\right)dt \\
&= \gamma\int_0^\infty A(\partial\mathbb{B}_t)\frac{d}{dt}(\lambda + t^{\frac{p}{p-1}})^{-m}dt \\
&= \gamma A(\partial\mathbb{B}_1)\lambda^{-\frac{m}{p}}\frac{\Gamma(m-\frac{m}{p})\Gamma(\frac{m}{p})}{\frac{p}{p-1}\Gamma(m)}.
\end{aligned}
$$

Inserting into (18) and (17), it follows that

$$\frac{\int_M \left|\nabla \hat{\phi}_\lambda\right|^p}{\int_M \hat{\phi}_\lambda^{p^*}} \leq K\,(m,p)^{-p} + C_2, \tag{20}$$

where

$$C_2(m,p,b) := \tfrac{(m-1)^2 p}{m-p}\left(\tfrac{m-p}{p-1}\right)^{p-1} \beta^{-\frac{p^2}{m-p}} (e^b - 1)\frac{e^{b(m-1)}}{\gamma}.$$

On the other hand, because of (13), we can use $\hat{\phi}_\lambda$ into (1) and get

$$\frac{\int_M \left|\nabla \hat{\phi}_\lambda\right|^p}{\left(\int_M \hat{\phi}_\lambda^{p^*}\right)^{\frac{p}{p^*}}} \geq C_M^{-p}. \tag{21}$$

Combining (20) and (21) we obtain

$$\int_M \hat{\phi}_\lambda^{p^*} \geq C_3^{-1},$$

with

$$C_3(m,p,b,C_M) := \left[\left(\frac{C_M}{K(m,p)}\right)^p + C_M^p C_2\right]^{m/p}.$$

From this latter, using (13), (19), the co-area formula and integrating by parts, it follows that

$$\begin{aligned} 0 &\leq C_3 e^{b(m-1)} \int_M \hat{\phi}_\lambda^{p^*} d\mathrm{vol} - \int_{M_h} \phi_{\lambda,h}^{p^*} d\mathrm{vol}_h \\ &= \int_0^\infty v_{M,h}(t) V(\mathbb{B}_t^h) \frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right) dt, \end{aligned} \tag{22}$$

where, by Bishop-Gromov, the function

$$v_{M,h}(t) := \left[C_3 e^{b(m-1)} \frac{V(B_t)}{V(\mathbb{B}_t^h)} - 1\right]$$

is non-increasing. In view of (11), in order to prove (6), it is enough to show that $\lim_{t\to\infty} v_{M,h}(t) \geq 0$. By contradiction, suppose there exist positive constants ϵ and T such that $v_{M,h}(t) \leq -\epsilon$ for all $t \geq T$. In this assumption, $T_0 := \sup\{t < T : v_{M,h}(t) \geq 0\}$ is well defined and $0 < T_0 < T$. Then

$$\begin{aligned} &\int_0^\infty v_{M,h}(t) V(\mathbb{B}_t^h) \frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right) dt \\ &\leq v_{M,h}(0) \int_0^{T_0} V(\mathbb{B}_t^h) \frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right) dt \\ &- \epsilon \int_T^\infty V(\mathbb{B}_t^h) \frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right) dt. \end{aligned} \tag{23}$$

Observe that the 1-parameter family of functions

$$V(\mathbb{B}_t) \frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right) = \omega_m \frac{mp}{p-1} \beta\,(m,p)^{p^*}\, \lambda^{\frac{m}{p}} \frac{t^{\frac{1}{p-1}+m}}{\left(\lambda + t^{\frac{p}{p-1}}\right)^{m+1}}$$

is decreasing in λ, provided $\lambda \gg 1$. Then we can apply the dominated convergence theorem to deduce

$$\begin{aligned}(24)\qquad &\lim_{\lambda\to+\infty}\int_0^{T_0} V(\mathbb{B}_t^h)\frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right)dt\\ &\le e^{b(m-1)}\lim_{\lambda\to+\infty}\int_0^{T_0} V(\mathbb{B}_t)\frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right)dt = 0.\end{aligned}$$

On the other hand, using (11) and the co-area formula, we have

$$\begin{aligned}\int_0^\infty V(\mathbb{B}_t^h)\frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right)dt &\ge \int_0^\infty V(\mathbb{B}_t)\frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right)dt\\ &\ge \int_0^\infty A(\partial\mathbb{B}_t)\varphi_\lambda^{p^*}(t)dt\\ &= \int_{\mathbb{R}^m}\phi_\lambda^{p^*}(x)\,dx = 1,\end{aligned}$$

for all $\lambda > 0$. Therefore, by (24),

$$(25)\qquad \lim_{\lambda\to+\infty}\int_T^\infty V(\mathbb{B}_t^h)\frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right)dt \ge 1.$$

Inserting (24) and (25) into (23) we conclude that, up to choosing $\lambda > 0$ large enough,

$$\int_0^\infty v_{M,h}(t)V(\mathbb{B}_t^h)\frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right)dt < 0,$$

which contradicts (22). Setting $\hat{C}(m,p,C_M,b) := C_3^{-1}e^{-b(m-1)}$, we have thus proven the validity of (6). □

Corollary 2 now follows immediately from (6) recalling the explicit form of $\hat{C}$ given in (7) and using the next result due to S.-H. Zhu, [14], later quantified by M. Bazanfaré, [1].

Theorem 3. *Let $(M,\langle,\rangle)$ be a complete m-dimensional Riemannian manifold such that*

$$^M\mathrm{Sect} \ge -G(r(x)) \text{ on } M,$$

where G satisfies (5) *for some $b > 0$. Then there exists an explicit constant $0 < \omega_0 = \omega_0(b) \to 1/2$ as $b \to 0$ such that, if*

$$V(B_r(o)) \ge cV(\mathbb{B}_r),\ \forall r > 0,$$

and for some $c \ge \omega_0$, then M is diffeomorphic to $\mathbb{R}^m$.

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