A PROPERTY OF LOCAL COHOMOLOGY MODULES
OF POLYNOMIAL RINGS

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Abstract. Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over a field $k$ of characteristic $p > 0$, and let $I = (f_1, \cdots, f_s)$ be an ideal of $R$. We prove that every associated prime $P$ of $H^i_I(R)$ satisfies $\dim R/P \geq n - \sum \deg f_i$. In characteristic 0 the question is open.

Throughout this paper, $R = k[x_1, \cdots, x_n]$ is the ring of polynomials in $n$ variables over a field $k$ of characteristic $p > 0$. The main result of this paper is the following:

Theorem 1. Assume $I = (f_1, \cdots, f_s)$ is an ideal of $R$ such that $\sum \deg f_i < n$. If $P$ is an associated prime of $H^i_I(R)$, then $\dim R/P \geq n - \sum \deg f_i$.

Whether the same result holds in characteristic 0 is still an open question. Before we give the proof of Theorem 1, we fix some notation. Denote the multi-index $(i_1, \cdots, i_n)$ by $\bar{i}$, especially $p^l - 1 = (p^l_1 - 1, \cdots, p^l_n - 1)$, where $l$ is a positive integer. Since all the results in this paper concern the vanishing of local cohomology modules and since extending the field is faithfully flat in the sequel, we can always enlarge the field to make it perfect and infinite. Then $R$ is a free $R^p$-module on the $p^{ln}$ monomials $e_{\bar{i}} = x_1^{i_1} \cdots x_n^{i_n}$, where $0 \leq i_j < p^l$ for every $j$. Let $F : R \longrightarrow R^p$ be the Frobenius homomorphism and denote the source and target of $F$ by $R_s$ and $R_t$ respectively; that is, $F : R_s \to R_t$. There are two associated functors

$$F^* : R_s\text{-mod} \to R_t\text{-mod},$$

such that $F^*(-) = R_t \otimes_{R_s} -$, and

$$F_* : R_t\text{-mod} \to R_s\text{-mod},$$

which is the restriction of scalars. For each $R_s$-module $N$, we have

$$F^*(N) = R_t \otimes_{R_s} N = \bigoplus \epsilon_{\bar{i}} \otimes_{R_s} N.$$

For each $f \in \text{Hom}_{R_s}(M, F^*(N))$, define $\tilde{f}_i = p^l_1 \circ f : F_*(M) \to N$, where

$$F^*(N) \xymatrix{y \mapsto \epsilon_{\bar{i}} \otimes_{R_s} N} \ar[r] & \epsilon_{\bar{i}} \otimes_{R_s} N$$

is the natural projection to the $\bar{i}$-component. There is a duality theorem in [3].

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Theorem 2 (Theorem 3.3 in [3]). For every $R_t$-module $M$ and every $R_s$-module $N$ there is an $R_t$-linear isomorphism
\[
\operatorname{Hom}_{R_s}(F_*(M), N) \cong \operatorname{Hom}_{R_t}(M, F^*(N))
\]
\[
g_{p-1}(-) \leftrightarrow (g = \bigoplus_i (\varepsilon_i \otimes_{R_s} g_i(-)))
\]
\[
g \leftrightarrow \bigoplus_i (\varepsilon_i \otimes_{R_s} g(e_{p-1-i}(-))).
\]

\[\text{Proposition 3. Assume } I = (f_1, \cdots, f_s) \text{ is an ideal of } R \text{ such that } \sum \deg f_i < n. \text{ Then } H^0_{R_t}(H^1_t(R)) = 0 \text{ for every maximal ideal } m.\]

\[\text{Proof. Let } K^t(f, R) = \text{the Koszul cocomplex of } R \text{ on } f_1', \cdots, f_s'; \text{ that is,}
\]
0 → $R \xrightarrow{d^0} \bigoplus_{1 \leq \alpha \leq s} R_{\alpha} \xrightarrow{d^1} \bigoplus_{1 \leq \alpha \lessdot \beta \leq s} R_{\alpha_1, \alpha_2} \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} R_{1, \ldots, s} → 0$,
where each $R_{\alpha_1, \ldots, \alpha_j}$ is just a copy of $R$ indexed by the tuple $(\alpha_1, \ldots, \alpha_j)$ and the differentials
\[d^j : \bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_j \leq s} R_{\alpha_1, \ldots, \alpha_j} \rightarrow \bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_{j+1} \leq s} R_{\alpha_1, \ldots, \alpha_{j+1}}\]
are given by
\[(d^j(r))_{\alpha_1, \ldots, \alpha_{j+1}} = \sum_{v=1}^{j+1} (-1)^v f_{\alpha_1, \ldots, \alpha_v} \cdots R_{\alpha_{j+1}}.
\]

Let $l > 1$ be an integer and $F : R_t \xrightarrow{r^p} R_t$ be the Frobenius homomorphism. Let
\[
\phi : H^i(K^t(f, R)) \xrightarrow{\phi} F^*(H^i(K^t(f, R))) = F^*(H^i(K^t(f, R)))
\]
be the map induced by the chain map
\[
\phi : K^t(f, R) \rightarrow F^*(K^t(f, R)) = K^t(f, R)
\]
that sends each $R_{\alpha_1, \ldots, \alpha_j}$ to $F^*(R_{\alpha_1, \ldots, \alpha_j}) \cong R_{\alpha_1, \ldots, \alpha_j}$ via multiplication by $f_{\alpha_1}^{p-1} \cdots f_{\alpha_j}^{p-1}$. By Proposition 1.11 in [2], $H^1_t(R)$ is the direct limit of
\[
H^i(K^t(f, R)) \xrightarrow{\phi} H^i(K^t(f, R)) \xrightarrow{F^*(\phi)} F^*(K^t(f, R)) \xrightarrow{(F^*)^2(\phi)} \cdots.
\]

Since $H^i(K^t(f, R))$ is a subquotient of $\bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_j \leq s} R_{\alpha_1, \ldots, \alpha_j}$, assume that $H^i(K^t(f, R)) \subseteq \bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_j \leq s} R_{\alpha_1, \ldots, \alpha_j}/Q$ for a submodule $Q$. Notice that $\phi$ is the multiplication by $f_{\alpha_1}^{p-1} \cdots f_{\alpha_j}^{p-1}$ in the $(\alpha_1, \ldots, \alpha_j)$ component. For each $((g_{\alpha_1, \ldots, \alpha_i}) + Q) \in \bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_j \leq s} R_{\alpha_1, \ldots, \alpha_j}/Q$, suppose $l > \sum \deg g_{\alpha_1, \ldots, \alpha_i}$. Then
\[
\deg (g_{\alpha_1, \ldots, \alpha_i} \cdot f_{\alpha_1}^{p-1} \cdots f_{\alpha_i}^{p-1})
\]
\[
\leq \deg g_{\alpha_1, \ldots, \alpha_i} + (p^l - 1) \cdot \deg (f_1 \cdots f_s)
\]
\[
< l + (p^l - 1) \cdot (n - 1) \leq n \cdot (p^l - 1)
\]
\[
= \deg x_1^{p^l-1} \cdots x_n^{p^l-1}.
\]
where the second inequality holds since \( \deg g_{\alpha_1, \ldots, \alpha_s} < l \) and \( \deg (f_1 \cdots f_s) < n \). In other words, all \( g_{\alpha_1, \ldots, \alpha_s}, f_{\alpha_1}^{p^l - 1} \cdots f_{\alpha_s}^{p^l - 1} \) have zero \( e_{p^l - 1} \) component in \( F^*(R_{\alpha_1, \ldots, \alpha_s}) \). Since

\[
F^*(\bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_s \leq n} R_{\alpha_1, \ldots, \alpha_s}/Q) \cong \bigoplus_{1 \leq \alpha_1 < \cdots < \alpha_s \leq n} F^*(R_{\alpha_1, \ldots, \alpha_s})/F^*(Q),
\]

\( \phi((g_{\alpha_1, \ldots, \alpha_s}) + Q) = ((g_{\alpha_1, \ldots, \alpha_s}, f_{\alpha_1}^{p^l - 1} \cdots f_{\alpha_s}^{p^l - 1}) + F^*(Q)) \) has a zero \( e_{p^l - 1} \) component in \( F^*(H^i(K^t (f, R))) \). Therefore

\[
\phi_{p^l - 1}((g_{\alpha_1, \ldots, \alpha_s}) + Q) = p^l_{p^l - 1}((g_{\alpha_1, \ldots, \alpha_s}, f_{\alpha_1}^{p^l - 1} \cdots f_{\alpha_s}^{p^l - 1}) + F^*(Q)) = 0.
\]

Since the corresponding morphism

\( \psi : F_*(H^i(K^t (f, R))) \to H^i(K^t (f, R)) \)

under Theorem 2 is exactly \( \phi_{p^l - 1} \), we see that \( \psi((g_{\alpha_1, \ldots, \alpha_s}) + Q) = 0. \)

Since \( H^0_m(H^i(K^t (f, R))) \) is of finite length, we can choose \( l \) large enough such that \( \psi \) sends a \( k \)-basis of \( H^0_m(H^i(K^t (f, R))) \) to 0. Hence \( \phi = 0 \) on \( H^0_m(H^i(K^t (f, R))) \) by Theorem 2.

Since local cohomology commutes with direct limit, \( H^0_m(H^i_j(R)) \) has the “generating morphism” (see [2, Definition 1.9]),

\[
\phi : H^0_m(H^i(K^t (f, R))) \to F^*(H^0_m(H^i(K^t (f, R)))),
\]

which is zero as has just been shown. Therefore, \( H^0_m(H^i_j(R)) = 0 \) by Proposition 2.3 in [2].

**Proof of Theorem 1.** Let \( h = \text{height} P \). Suppose on the contrary we have \( \dim R/P < n - \sum \deg f_i \). By Noether’s normalization lemma ([1 Exercise 5.16]), there are elements \( y_1, \ldots, y_{n-h} \in R \) which are algebraically independent over \( k \) and such that \( k[y_1, \ldots, y_{n-h}] \cap P = 0 \) and \( R/P \) is integral over \( k[y_1, \ldots, y_{n-h}] \). Moreover, since \( k \) is infinite, these \( y_1, \ldots, y_{n-h} \) can be chosen to be linear combinations of \( x_1, \ldots, x_n \). Therefore, we can assume without loss of generality that \( x_1, \ldots, x_n \) are such that \( k[x_1, \ldots, x_{n-h}] \cap P = 0 \) and \( P \) is a maximal ideal in \( R' = K[x_{n-h+1}, \ldots, x_n] \), where \( K \) is the fraction field of \( k[x_1, \ldots, x_{n-h}] \). By Proposition 3, we get \( H^0_{m_p}(H^s_j(R')) = 0 \), which contradicts the assumption that \( P \) is an associated prime of \( H^s_j(R) \). □

When \( i = s \), the local cohomology module \( H^s_j(R) \) has a simple “generating morphism” [2 Definition 1.9], namely, \( R/I \to R/\bar{I}^l \). If \( H^0_m(R/I) = 0 \) in the first place, then we get \( H^0_m(H^s_j(R)) = 0 \) immediately. Therefore, Proposition 3 suggests the following:

**Question 4.** Let \( R = k[x_1, \ldots, x_n] \) be the ring of polynomials such that \( k \) is any field, and let \( m \) be any maximal ideal. Assume \( I = (f_1, \ldots, f_s) \) is an ideal of \( R \) such that \( \deg f_1 + \cdots + \deg f_s < n \). Is \( H^0_m(R/I) = 0 \)?

**Proposition 5.** Assume the answer to Question 4 is positive. Then \( \text{proj. dim} R/I \leq \sum \deg f_i \).

**Proof.** We claim that there is a linear combination of variables with coefficients in \( k \), which is \( R/I \)-regular. Indeed, assume all linear combinations of variables are zero-divisors of \( R/I \). Choose distinct elements \( c_1, \ldots, c_n \) of the field \( k \) which are not roots of unity, and set \( y_i = c_1^i x_1 + \cdots + c_n^i x_n \). Then any \( n \) of these are linearly
independent over $k$. Since each $y_t$ is a zero-divisor of $R/I$, it is contained in some associated prime of $R/I$. Since the associated prime ideals of $R/I$ are finitely many, one of them must contain more than $n$ $y_t$’s, and hence contains the maximal ideal $m = (x_1, \cdots, x_n)$. This contradicts the assumption that $H^n_{m}(R/I) = 0$.

By the Auslander-Buchsbaum theorem, it suffices to prove that 
\[
\text{depth} R/I \geq n - \sum \deg f_i.
\]
If $\sum \deg f_i = n - 1$, then $\text{depth} R/I \geq 1$ by the claim above. If $\sum \deg f_i < n - 1$, after a linear change of variables, assume that $x_n$ is $R/I$-regular. Modulo $x_n$, the new ideal $\overline{I} = (I + (x_n))/(x_n)$ satisfies $H^n_{\overline{m}}(\overline{R}/\overline{I}) = 0$ by a positive answer to Question 4. Since $\text{depth} R/I = \text{depth} \overline{R}/\overline{I} + 1$, we are done by induction. 

M. Stillman asked the following question:

**Question 6** ([4, Problem 3.14]). Fix a sequence of natural numbers $d_1, \cdots, d_s$. Does there exist a number $q$ such that $\text{proj} \text{.dim.} R/I \leq q$ when $R$ is a polynomial ring (over any field $k$) and $I$ is an ideal with homogeneous generators of degrees $d_1, \cdots, d_s$?

Thus a positive answer to Question 6 would also give a positive answer to M. Stillman’s question and would even provide an upper bound on $q$.

**Added in proof.** Jason McCullough gave a negative answer to our Question 4 (J. McCullough, A family of ideals with few generators in low degree and large projective dimension. Preprint, arXiv:1005.3361).

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**References**


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