PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 1, January 2011, Pages 125–128
S 0002-9939(2010)10530-0
Article electronically published on August 3, 2010

A PROPERTY OF LOCAL COHOMOLOGY MODULES OF POLYNOMIAL RINGS

YI ZHANG

(Communicated by Irena Peeva)

Abstract. Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over a field k of characteristic $p > 0$, and let $I = (f_1, \cdots, f_s)$ be an ideal of R. We prove that every associated prime P of $H_I^i(R)$ satisfies $\dim R/P \geqslant n - \sum \deg f_i$. In characteristic 0 the question is open.

Throughout this paper, $R = k[x_1, \cdots, x_n]$ is the ring of polynomials in n variables over a field k of characteristic $p > 0$. The main result of this paper is the following:

Theorem 1. *Assume $I = (f_1, \cdots, f_s)$ is an ideal of R such that $\sum deg f_i < n$. If P is an associated prime of $H_I^i(R)$, then $dim R/P \geqslant n - \sum deg f_i$.*

Whether the same result holds in characteristic 0 is still an open question.

Before we give the proof of Theorem 1, we fix some notation. Denote the multi-index $(i_1, \cdots, i_n)$ by $\bar{i}$, especially $\overline{p^l - 1} = (p^l - 1, \cdots, p^l - 1)$, where l is a positive integer. Since all the results in this paper concern the vanishing of local cohomology modules and since extending the field is faithfully flat in the sequel, we can always enlarge the field to make it perfect and infinite. Then R is a free R^{p^l}-module on the p^{ln} monomials $e_{\bar{i}} = x_1^{i_1} \cdots x_n^{i_n}$, where $0 \leqslant i_j < p^l$ for every j. Let $F : R \xrightarrow{r \mapsto r^{p^l}} R$ be the Frobenius homomorphism and denote the source and target of F by R_s and R_t respectively; that is, $F : R_s \to R_t$. There are two associated functors

$$F^* : R_s\text{-mod} \to R_t\text{-mod},$$

such that $F^*(-) = R_t \otimes_{R_s} -$, and

$$F_* : R_t\text{-mod} \to R_s\text{-mod},$$

which is the restriction of scalars. For each R_s-module N, we have

$$F^*(N) = R_t \otimes_{R_s} N = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} N).$$

For each $f \in \mathrm{Hom}_{R_t}(M, F^*(N))$, define $f_{\bar{i}} = p_{\bar{i}} \circ f : F_*(M) \to N$, where

$$F^*(N) \xrightarrow{y \mapsto e_{\bar{i}} \otimes p_{\bar{i}}(y)} e_{\bar{i}} \otimes_{R_s} N$$

is the natural projection to the $\bar{i}$-component. There is a duality theorem in [3]:

Received by the editors March 29, 2010.
2010 *Mathematics Subject Classification.* Primary 13D45.
NSF support through grant DMS-0701127 is gratefully acknowledged.

Theorem 2 (Theorem 3.3 in [3]). *For every R_t-module M and every R_s-module N there is an R_t-linear isomorphism*

$$Hom_{R_s}(F_*(M), N) \cong Hom_{R_t}(M, F^*(N))$$
$$g_{\overline{p^l-1}}(-) \leftarrow (g = \bigoplus_{\bar{i}}(e_{\bar{i}} \otimes_{R_s} g_{\bar{i}}(-)))$$
$$g \mapsto \bigoplus_{\bar{i}}(e_{\bar{i}} \otimes_{R_s} g(e_{\overline{p^l-1}-\bar{i}}(-))).$$

Proposition 3. *Assume $I = (f_1, \cdots, f_s)$ is an ideal of R such that $\sum degf_i < n$. Then $H^0_m(H^i_I(R)) = 0$ for every maximal ideal m.*

Proof. Let $K^{\cdot}(\underline{f}^t, R)$ be the Koszul cocomplex of R on $f_1^t, \cdots, f_s^t$; that is,

$$0 \to R \xrightarrow{d^0} \bigoplus_{1\leqslant\alpha\leqslant s} R_\alpha \xrightarrow{d^1} \bigoplus_{1\leqslant\alpha_1<\alpha_2\leqslant s} R_{\alpha_1,\alpha_2} \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} R_{1,\cdots,s} \to 0,$$

where each $R_{\alpha_1,\cdots,\alpha_j}$ is just a copy of R indexed by the tuple $(\alpha_1, \cdots, \alpha_j)$ and the differentials

$$d^j : \bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_j\leqslant s} R_{\alpha_1,\cdots,\alpha_j} \to \bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_{j+1}\leqslant s} R_{\alpha_1,\cdots,\alpha_{j+1}}$$

are given by

$$(d^j(r))_{\alpha_1,\cdots,\alpha_{j+1}} = \sum_{v=1}^{v=j+1} (-1)^v f_v^t r_{\alpha_1,\cdots,\hat{\alpha}_v,\cdots,\alpha_{j+1}}.$$

Let $l > 1$ be an integer and $F : R_s \xrightarrow{r\mapsto r^{p^l}} R_t$ be the Frobenius homomorphism. Let

$$\phi : H^i(K^{\cdot}(\underline{f}, R)) \to F^*(H^i(K^{\cdot}(\underline{f}, R))) = H^i(F^*(K^{\cdot}(\underline{f}, R)))$$

be the map induced by the chain map

$$\phi^{\cdot} : K^{\cdot}(\underline{f}, R) \to F^*(K^{\cdot}(\underline{f}, R)) = K^{\cdot}(\underline{f}^{p^l}, F^*(R))$$

that sends each $R_{\alpha_1,\cdots,\alpha_j}$ to $F^*(R_{\alpha_1,\cdots,\alpha_j}) \cong R_{\alpha_1,\cdots,\alpha_j}$ via multiplication by $f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_j}^{p^l-1}$. By Proposition 1.11 in [2], $H^i_I(R)$ is the direct limit of

$$H^i(K^{\cdot}(\underline{f}, R)) \xrightarrow{\phi} F^*(H^i(K^{\cdot}(\underline{f}, R))) \xrightarrow{F^*(\phi)} (F^*)^2(H^i(K^{\cdot}(\underline{f}, R))) \xrightarrow{(F^*)^2(\phi)} \cdots .$$

Since $H^i(K^{\cdot}(\underline{f}, R))$ is a subquotient of $\bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_i\leqslant s} R_{\alpha_1,\cdots,\alpha_i}$, assume that $H^i(K^{\cdot}(\underline{f}, R)) \subseteq \bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_i\leqslant s} R_{\alpha_1,\cdots,\alpha_i}/Q$ for a submodule Q. Notice that ϕ is the multiplication by $f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}$ in the $(\alpha_1, \cdots, \alpha_i)$ component. For each $((g_{\alpha_1,\cdots,\alpha_i}) + Q) \in \bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_i\leqslant s} R_{\alpha_1,\cdots,\alpha_i}/Q$, suppose $l > \sum \deg g_{\alpha_1,\cdots,\alpha_i}$. Then

$$\begin{aligned}
&\deg(g_{\alpha_1,\cdots,\alpha_i} \cdot f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}) \\
&\leqslant \deg g_{\alpha_1,\cdots,\alpha_i} + (p^l-1) \cdot \deg(f_1 \cdots f_s) \\
&< l + (p^l-1) \cdot (n-1) \leqslant n \cdot (p^l-1) \\
&= \deg x_1^{p^l-1} \cdots x_n^{p^l-1},
\end{aligned}$$

where the second inequality holds since $\deg g_{\alpha_1,\cdots,\alpha_i} < l$ and $\deg(f_1\cdots f_s) < n$. In other words, all $g_{\alpha_1,\cdots,\alpha_i}\cdot f_{\alpha_1}^{p^l-1}\cdots f_{\alpha_i}^{p^l-1}$ have zero $e_{\overline{p^l-1}} = x_1^{p^l-1}\cdots x_n^{p^l-1}$ components in $F^*(R_{\alpha_1,\cdots,\alpha_i})$. Since

$$F^*(\bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_i\leqslant s} R_{\alpha_1,\cdots,\alpha_i}/Q) \cong (\bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_i\leqslant s} F^*(R_{\alpha_1,\cdots,\alpha_i}))/F^*(Q),$$

$\phi((g_{\alpha_1,\cdots,\alpha_i})+Q) = ((g_{\alpha_1,\cdots,\alpha_i}\cdot f_{\alpha_1}^{p^l-1}\cdots f_{\alpha_i}^{p^l-1})+F^*(Q))$ has a zero $e_{\overline{p^l-1}}$ component in $F^*(H^i(K^\cdot(\underline{f},R)))$. Therefore

$$\phi_{\overline{p^l-1}}((g_{\alpha_1,\cdots,\alpha_i})+Q) = p_{\overline{p^l-1}}((g_{\alpha_1,\cdots,\alpha_i}\cdot f_{\alpha_1}^{p^l-1}\cdots f_{\alpha_i}^{p^l-1})+F^*(Q)) = 0.$$

Since the corresponding morphism

$$\psi : F_*(H^i(K^\cdot(\underline{f},R))) \to H^i(K^\cdot(\underline{f},R))$$

under Theorem 2 is exactly $\phi_{\overline{p^l-1}}$, we see that $\psi((g_{\alpha_1,\cdots,\alpha_i})+Q) = 0$.

Since $H_m^0(H^i(K^\cdot(\underline{f},R)))$ is of finite length, we can choose l large enough such that ψ sends a k-basis of $H_m^0(H^i(K^\cdot(\underline{f},R)))$ to 0. Hence $\phi=0$ on $H_m^0(H^i(K^\cdot(\underline{f},R)))$ by Theorem 2.

Since local cohomology commutes with direct limit, $H_m^0(H_I^i(R))$ has the "generating morphism" (see [2, Definition 1.9]),

$$\phi : H_m^0(H^i(K^\cdot(\underline{f},R))) \to F^*(H_m^0(H^i(K^\cdot(\underline{f},R)))),$$

which is zero as has just been shown. Therefore, $H_m^0(H_I^i(R)) = 0$ by Proposition 2.3 in [2]. □

Proof of Theorem 1. Let $h = \text{height}P$. Suppose on the contrary we have $\dim R/P < n - \sum \deg f_i$. By Noether's normalization lemma ([1, Exercise 5.16]), there are elements $y_1,\cdots,y_{n-h}\in R$ which are algebraically independent over k and such that $k[y_1,\cdots,y_{n-h}]\cap P = 0$ and R/P is integral over $k[y_1,\cdots,y_{n-h}]$. Moreover, since k is infinite, these $y_1,\cdots,y_{n-h}$ can be chosen to be linear combinations of $x_1,\cdots,x_n$. Therefore, we can assume without loss of generality that $x_1,\cdots,x_n$ are such that $k[x_1,\cdots,x_{n-h}]\cap P = 0$ and P is a maximal ideal in $R' = K[x_{n-h+1},\cdots,x_n]$, where K is the fraction field of $k[x_1,\cdots,x_{n-h}]$. By Proposition 3, we get $H_P^0(H_I^i(R')) = 0$, which contradicts the assumption that P is an associated prime of $H_I^i(R)$. □

When $i = s$, the local cohomology module $H_I^s(R)$ has a simple "generating morphism" [2, Definition 1.9], namely, $R/I \to R/I^{[p]}$. If $H_m^0(R/I) = 0$ in the first place, then we get $H_m^0(H_I^s(R)) = 0$ immediately. Therefore, Proposition 3 suggests the following:

Question 4. Let $R = k[x_1,\cdots,x_n]$ be the ring of polynomials such that k is any field, and let m be any maximal ideal. Assume $I = (f_1,\cdots,f_s)$ is an ideal of R such that $\deg f_1 + \cdots + \deg f_s < n$. Is $H_m^0(R/I) = 0$?

Proposition 5. *Assume the answer to Question* 4 *is positive. Then proj.dim.*$R/I \leqslant \sum \deg f_i$.

Proof. We claim that there is a linear combination of variables with coefficients in k, which is R/I-regular. Indeed, assume all linear combinations of variables are zero-divisors of R/I. Choose distinct elements $c_1,\cdots,c_n$ of the field k which are not roots of unity, and set $y_t = c_1^t x_1 + \cdots + c_n^t x_n$. Then any n of these are linearly

independent over k. Since each y_t is a zero-divisor of R/I, it is contained in some associated prime of R/I. Since the associated prime ideals of R/I are finitely many, one of them must contain more than n y_t's, and hence contains the maximal ideal $m = (x_1, \cdots, x_n)$. This contradicts the assumption that $H^0_m(R/I) = 0$.

By the Auslander-Buchsbaum theorem, it suffices to prove that

$$\mathrm{depth} R/I \geqslant n - \sum \deg f_i.$$

If $\sum \deg f_i = n-1$, then $\mathrm{depth} R/I \geqslant 1$ by the claim above. If $\sum \deg f_i < n-1$, after a linear change of variables, assume that x_n is R/I-regular. Modulo x_n, the new ideal $\bar{I} = (I+(x_n))/(x_n)$ satisfies $H^0_{\bar{m}}(\bar{R}/\bar{I}) = 0$ by a positive answer to Question 4. Since $\mathrm{depth} R/I = \mathrm{depth} \bar{R}/\bar{I} + 1$, we are done by induction. □

M. Stillman asked the following question:

Question 6 ([4, Problem 3.14])**.** Fix a sequence of natural numbers $d_1, \cdots, d_s$. Does there exist a number q such that proj.dim.$R/I \leqslant q$ when R is a polynomial ring (over any field k) and I is an ideal with homogeneous generators of degrees $d_l, \cdots, d_s$?

Thus a positive answer to Question 4 would also give a positive answer to M. Stillman's question and would even provide an upper bound on q.

Added in proof. Jason McCullough gave a negative answer to our Question 4 (J. McCullough, A family of ideals with few generators in low degree and large projective dimension. Preprint, arXiv:1005.3361).

Acknowledgments

The above result is from the author's thesis. He thanks his advisor, Professor Gennady Lyubeznik, for his guidance, support and patience. He also thanks Professor Craig Huneke for pointing out M. Stillman's Question 6 and its connection with the main theorem of this paper.

References

[1] M. F. Atiyah, I. G. Macdonald, Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR0242802 (39:4129)

[2] G. Lyubeznik, F-modules: Applications to local cohomology and D-modules in characteristic $p > 0$. J. Reine Angew. Math. 491 (1997), 65–130. MR1476089 (99c:13005)

[3] G. Lyubeznik, W. Zhang, Y. Zhang, A property of the Frobenius map of a polynomial ring. Preprint, arXiv: 1001.2949.

[4] I. Peeva, M. Stillman, Open problems on syzygies and Hilbert functions. J. Commut. Algebra 1 (2009), no. 1, 159–195. MR2462384 (2009i:13024)

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: zhang397@umn.edu