CONTINUITY OF SPECTRAL AVERAGING

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Abstract. We consider averages $\kappa$ of spectral measures of rank one perturbations with respect to a $\sigma$-finite measure $\nu$. It is examined how various degrees of continuity of $\nu$ with respect to $\alpha$-dimensional Hausdorff measures ($0 \leq \alpha \leq 1$) are inherited by $\kappa$. This extends Kotani’s trick where $\nu$ is simply the Lebesgue measure.

1. Introduction

Let $A$ be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Fix a normalized vector $\phi \in \mathcal{H}$. Consider the family of rank one perturbations

$$A_\lambda := A + \lambda \langle \phi, . \rangle \phi,$$

indexed by the real parameter $\lambda$. Despite its simple form, the family in (1.1) proves to be a very useful tool in the study of discrete random Schrödinger operators. There, rank one perturbations correspond to fluctuations of the potential at a lattice site. Reference [1] summarizes several of these applications, among them the Simon-Wolf criterion for spectral localization, the theory of Aizenman-Molchanov for the Anderson model, and Wegner’s estimate.

Crucial to most of these applications is a result known as spectral averaging or Kotani’s trick. It allows one to relate the spectral behavior for fixed values of $\lambda$ to the spectral properties inherent to the entire family $\{A_\lambda\}$, i.e. upon a variation of $\lambda$.

Denote by $d\mu(x)$ and $d\mu_\lambda(x)$ the spectral measure with respect to $\phi$ for the operator $A$ and $A_\lambda$, respectively. Kotani’s trick is the following result:

**Theorem 1.1 (Kotani’s trick).** Let $B$ be a Borel set on the real line. Then

$$|B| = \int \mu_\lambda(B) d\lambda.$$

Here, $|.|$ denotes the Lebesgue measure.

Different proofs and applications of this result were given in [2] [3] [4] [5] [6] [7] [8] [9]. We note that for some purposes, among them the Simon-Wolf criterion, a weaker formulation is sufficient. This weaker result states that the Borel measure on the right hand side of (1.2) is absolutely continuous with respect to Lebesgue. In fact,
in the original proof of the Simon-Wolff criterion (see [9], Theorem 5) the authors
show that the measure
\[(1.3) \quad \kappa(.) = \int \mu_\lambda(.) \frac{1}{1 + \lambda^2} d\lambda \]
is mutually equivalent to the Lebesgue measure.

Equation (1.3) suggests the following generalization: For \(\nu\) a \(\sigma\)-finite Borel measure on \(\mathbb{R}\), define a measure \(\kappa\) by
\[(1.4) \quad \kappa(.) = \int \mu_\lambda(.) d\nu(\lambda) \,.
\]
Such averages were first considered in [10] for a finite measure \(\nu\). There, relation (3.4) was discovered for a finite measure \(\nu\) and was used to estimate the Hausdorff dimension of set \(\{\lambda : A_\lambda \text{ has some continuous spectrum}\}\) (see Theorem 5.2 therein).

In view of the measure defined in (1.4), Kotani’s trick (\(d\nu(\lambda) = d\lambda\)) and the result for \(d\nu(\lambda) = \frac{1}{1 + \lambda^2} d\lambda\) in (1.3) become statements about continuity properties of the measure \(\nu\) being inherited by \(\kappa\).

In this article we pursue this continuity-based approach to spectral averaging. We will show how various degrees of continuity of \(\nu\) with respect to \(\alpha\)-dimensional Hausdorff measures \((\alpha \leq 1)\) are inherited by \(\kappa\). For a definition of \(\alpha\) continuity see Definition 4.1. Our main result is the following theorem:

**Theorem 1.2.** If \(\nu\) is absolutely continuous with respect to Lebesgue, so is \(\kappa\). Additionally, if \(\nu\) is \(\alpha\)c, \(0 < \alpha < 1\), then \(\kappa\) is \(\delta\)c for all \(\delta < \alpha\).

Kotani’s trick then arises as a special case, where the density of \(d\kappa(x)\) can be calculated explicitly.

The paper is organized as follows: Section 2 summarizes some results of the theory of Borel transforms and rank one perturbations as needed for further development. After showing that mere continuity of \(\nu\) is inherited by \(\kappa\) (Theorem 3.1), we examine the situation for \(\nu\) being uniformly-\(\alpha\)-Hölder continuous (see Definition 3.3). In particular, we shall show that uniform 1-Hölder continuity of \(\nu\) is inherited by \(\kappa\). Kotani’s trick follows as a special case if \(d\nu(x) = dx\). Finally, in Sections 4 and 5 we employ the Rogers-Taylor decomposition of measures with respect to Hausdorff measures to prove Theorem 1.2.

2. Borel transforms and rank one perturbations

The key quantity to understanding the spectral properties of the family (1.1) is the Borel transform of the spectral measure \(d\mu\) associated with the unperturbed operator \(A\) and the vector \(\phi\). In general, if \(\eta\) is a Borel measure with \(\int \frac{1}{1+|y|} d\eta(y) < \infty\), for \(z \in \mathbb{H}^+\) we define
\[(2.1) \quad F_\eta(z) := \int \frac{d\eta(y)}{y - z} ,\]
the Borel transform of the measure \(\eta\). Letting \(z = x + i\epsilon, \epsilon > 0\), we may split \(F_\eta(x+i\epsilon)\) into its real and imaginary parts, i.e. \(F_\eta(x+i\epsilon) = Q_\eta(x+i\epsilon) + iP_\eta(x+i\epsilon)\), where
\[
Q_\eta(x + i\epsilon) = \int \frac{y - x}{(y - x)^2 + \epsilon^2} d\eta(y) ,
\]
\[
P_\eta(x + i\epsilon) = \int \frac{\epsilon}{(y - x)^2 + \epsilon^2} d\eta(y) .
\]
We shall refer to \( P_\eta \) and \( Q_\eta \) as Poisson and conjugate Poisson transforms of the measure \( \eta \), respectively. Whereas \( Q_\eta(x + i\epsilon) \) depends on the “symmetry” of \( \eta \) around \( x \), \( P_\eta(x + i\epsilon) \) carries information about the growth of the measure \( \eta \) at \( x \).

A detailed analysis about the asymptotic behavior of \( P_\eta \) and \( Q_\eta \) is given in [11].

The relation between the local growth of a measure and its Poisson transform follows from the following simple estimate: Given a measurable for any Borel set \( \mu \), the Poisson transform of a measure even if its Borel transform does not exist. A necessary and sufficient condition for \( P_\eta(x + i\epsilon) < \infty \), \( \epsilon > 0 \)

\[
(\text{2.2) } \epsilon^{1-\alpha} P_\eta(x + i\epsilon) \geq \epsilon^{1-\alpha} \int_{(\epsilon - \epsilon,x + \epsilon)} \frac{\epsilon}{(x - y)^2 + \epsilon^2} \, d\eta(y) \geq \frac{1}{2\epsilon^\alpha} M_\eta(x; \epsilon)
\]

where \( M_\eta(x; \epsilon) := \eta(x - \epsilon, x + \epsilon) \) denotes the growth function of \( \eta \) at \( x \).

**Remark 2.1.** As will be seen below (see Theorem 2.2), it is useful to consider the Poisson transform of a measure even if its Borel transform does not exist. A necessary and sufficient condition for \( P_\eta(x + i\epsilon) < \infty \), \( \epsilon > 0 \), is \( \int \frac{1}{1 + x^2} d\eta(x) < \infty \).

For \( \alpha \geq 0 \) we define

\[
(\text{2.3) } D^\alpha_\eta(x) := \limsup_{\epsilon \to 0^+} \frac{\eta(x - \epsilon, x + \epsilon)}{\epsilon^\alpha},
\]

the upper-\( \alpha \)-derivative of a measure \( \eta \) at a point \( x \in \mathbb{R} \).

The above estimate (2.2) leads to the following result proven e.g. in [10]:

**Proposition 2.1.** Let \( \alpha \in [0,1] \) and \( x \in \mathbb{R} \) be fixed. Then \( D^\alpha_\eta(x) \) and \( \limsup_{\epsilon \to 0^+} \epsilon^{1-\alpha} P_\eta(x + i\epsilon) \) are either both infinite, zero, or in \((0, +\infty)\).

Proposition 2.1 will be used to analyze continuity with respect to Hausdorff measures of the measure \( \kappa \) defined in (1.4).

The following theorem is the key to spectral analysis of rank one perturbations. It provides a characterization of the components of \( \eta \) in a Lebesgue decomposition. Proof can be found e.g. in [11, 12].

**Theorem 2.2.** Let \( \eta \) be a Borel measure on the real line such that \( \int \frac{1}{1 + x^2} d\eta(y) < \infty \). The following statements characterize the components in the Lebesgue decomposition of \( \eta = \eta_{\text{sing}} + \eta_{\text{ac}} \):

(i) \( \eta_{\text{ac}}(x) = \frac{1}{\pi} P_\eta(x + i0) dx \),
(ii) \( \eta_{\text{sing}} \) is supported on \( \{ x : P_\eta(x + i0) = +\infty \} \).

Theorem 2.2 implies a characterization of the spectral properties of the family \( \{A_\lambda\} \). Out of this we shall only need the following statement related to the singular (pp+sc)-spectrum of \( \{A_\lambda\} \) (see [1], Theorem 12.2).

**Proposition 2.2.**

(i) \( \mu_{\lambda, \text{sing}} \) is supported on the set \( \{ x : F_\mu(x + i0) = -\frac{1}{\lambda} \} \).
(ii) The family of measures \( \{d\mu_{\lambda, \text{sing}}\} \) are mutually singular. In particular, a point \( x \in \mathbb{R} \) can be an atom for at most one value of \( \lambda \).

3. Spectral averaging

For a fixed \( \sigma \)-finite Borel measure \( \nu \), consider the measure \( \kappa \) introduced in (1.4). \( \kappa \) is well defined since for any polynomial \( p(x) \), \( \langle \phi, p(A_\lambda) \phi \rangle \) is a polynomial in \( \lambda \). Stone-Weierstraß and functional calculus then imply that \( \lambda \mapsto \mu_\lambda(B) \) is Borel measurable for any Borel set \( B \subseteq \mathbb{R} \).

We start our analysis of the continuity of \( \kappa \) in relation to the continuity of \( \nu \) with the following simple observation:
Theorem 3.1. If \( \nu \) is continuous, so is \( \kappa \).

Proof. Apply part (ii) of Proposition 2.2 to \( \kappa(\{x\}) = \int \mu_\lambda(\{x\}) d\nu(\lambda), x \in \mathbb{R} \). \( \square \)

The following simple relation between the Poisson transforms of \( \kappa \) and \( \nu \) is crucial to further analyze the continuity properties of \( \kappa \).

Proposition 3.1. Assume \( \int 1 + y^2 d\nu(y) < \infty \). Then

\[
P_\kappa(z) = P_\nu \left( -\frac{1}{F_\mu(z)} \right)
\]

for \( z \in \mathbb{H}^+ \).

Proof. Using the definition of \( \kappa \) in (1.4), the monotone convergence theorem implies

\[
\int f(x) d\kappa(x) = \int \left\{ \int f(x) d\mu_\lambda(x) \right\} d\nu(\lambda)
\]

for any measurable \( 0 \leq f \).

In particular, for \( z \in \mathbb{H}^+ \),

\[
P_\kappa(z) = \int P_{\mu_\lambda}(z) d\nu(\lambda)
\]

\[
= \int \frac{P_\mu(z)}{1 + \lambda F_\mu(z)} d\nu(\lambda) = P_\nu \left( -\frac{1}{F_\mu(z)} \right).
\]

Here, the second equality follows from the Aronszajn-Krein formula [1],

\[
F_{\mu_\lambda}(z) = \frac{F_\mu(z)}{1 + \lambda F_\mu(z)},
\]

which relates the Borel transforms of the spectral measures \( \mu_\lambda \) and \( \mu \). \( \square \)

Remark 3.2. If \( \nu \) is a finite measure an analogous result between the respective Borel transforms was first obtained in [10]:

\[
F_\kappa(z) = F_\nu \left( -\frac{1}{F_\mu(z)} \right).
\]

Note that for non-finite \( \nu \) the Borel transform will in general not exist (e.g. take \( \nu \) to be the Lebesgue measure). In fact for \( \sigma \)-finite \( \nu \), often the Poisson transform exists, whereas its Borel transform does not. In these cases we still have a relation between the Poisson transforms of \( \nu \) and \( \kappa \) as established in Proposition 3.1.

In order to prove finer statements on the continuity of \( \kappa \), we first establish some results for uniformly Hölder continuous \( \nu \). Recall the following definition:

Definition 3.3. Let \( \eta \) be a \( \sigma \)-finite Borel measure on the real line and \( \alpha \geq 0 \). \( \eta \) is uniformly \( \alpha \) Hölder continuous (U\( \alpha \)H) if for some constant \( K \), \( \eta(I) \leq K |I|^\alpha \) for any interval \( I \).

Remark 3.4. (i) U1H implies absolute continuity.

(ii) Using the Rogers-Taylor decomposition theorem (see Theorem 4.2), there are no non-trivial U\( \alpha \)H measures for \( \alpha > 1 \).

For \( \nu \) U\( \alpha \)H, Proposition 3.1 implies the following key estimate for the Poisson transform of \( \kappa \):
Proposition 3.2. If $\nu$ is $U\alpha H$, $0 \leq \alpha \leq 1$, then for some constant $C_\alpha$ and all $z \in \mathbb{H}^+$

(3.5) \[ P_\kappa(z) \leq C_\alpha \left( \frac{|F_\mu(z)|^2}{P_\mu(z)} \right)^{1-\alpha}. \]

In particular, $\int \frac{1}{1+x^2} \, d\kappa(x) < \infty$, whence $\kappa$ is a locally finite Borel measure (i.e. finite on compact sets).

Proof. Let $z \in \mathbb{H}^+$. Recasting $P_\nu(z)$ in terms of the Lebesgue-Stieltjes measure induced by $M_\nu(\text{Re}\{z\}; \delta)$, we get

(3.6) \[ P_\nu(z) = \text{Im} \{ z \} \int_0^{+\infty} \frac{dM_\nu(\text{Re}\{z\}; \delta)}{\delta^2 + \text{Im} \{ z \}^2} \leq \text{Im} \{ z \} 2K \int_0^{+\infty} \frac{\delta^{\alpha+1}}{\left( \delta^2 + \text{Im} \{ z \}^2 \right)^2} \, d\delta \]

\[ = \frac{\pi \alpha K}{2 \sin \left( \frac{\pi \alpha}{2} \right)} (\text{Im} \{ z \})^{\alpha-1}. \]

Here, the second equality follows, using integration by parts; the last equality is obtained by contour integration. For $\alpha = 0$, the last equality in (3.6) is to be interpreted in the limit $\alpha \to 0$, i.e. $P_\nu(z) \leq K \text{Im} \{ z \}^{-1}$.

In particular, for $\nu \sim U1H$, (3.6) establishes $\int \frac{1}{1+x^2} \, d\nu(x) < \infty$. Application of Proposition 3.1 hence yields the desired estimate.

Theorem 3.5. If $\nu$ is $U1H$, so is $\kappa$.

Proof. Let $0 \leq f$ be continuous of compact support. Using Proposition 3.2

\[ \int f(x) \, dx \geq \limsup_{\epsilon \to 0^+} \frac{1}{C_1} \int f(x) P_\kappa(x + i\epsilon) \, dx \]

\[ = \limsup_{\epsilon \to 0^+} \frac{1}{C_1} \int \left( \int f(x) \frac{\epsilon}{(x-y)^2 + \epsilon^2} \, dx \right) \, d\kappa(y) \]

\[ \geq \frac{1}{C_1} \int \left( \lim_{\epsilon \to 0^+} \int f(x) \frac{\epsilon}{(x-y)^2 + \epsilon^2} \, dx \right) \, d\kappa(y) \]

(3.7) \[ = \frac{\pi}{C_1} \int f(y) \, d\kappa(y). \]

Here, the second equality follows from Tonelli, whereas the second inequality uses Fatou’s Lemma. Note that $\sigma$-finiteness of $\kappa$ is implied by Proposition 3.2.

Theorem 3.5 in particular implies $d\kappa(x) \ll dx$. Spectral averaging now arises as a special case where the density of $\kappa$ can be calculated explicitly.

Proof of Theorem 1.1. Since the Poisson transform of the Lebesgue measure $P_{\text{Leb}}(z) = \pi$, all $z \in \mathbb{H}^+$, Theorem 2.2(i) and Proposition 3.1 yield $d\kappa(x) = dx$. □
4. Continuity with respect to Hausdorff measures

In this section we analyze the degree of continuity of \( \kappa \) induced by measures \( \nu \) with lesser degree of continuity than considered in the previous section. To this end we make the following definitions:

**Definition 4.1.** For \( 0 \leq \alpha \) let \( h^\alpha \) denote the \( \alpha \)-dimensional Hausdorff measure on \( \mathbb{R} \). Let \( \eta \) be a Borel measure on the real line.

1. \( \eta \) is called \( \alpha \)-continuous (ac) if \( \eta(B) = 0 \) whenever \( h^\alpha(B) = 0 \).
2. \( \eta \) is called \( \alpha \)-singular if \( \eta \) is supported on a set of zero measure \( h^\alpha \).

The main tools for proving Theorem 1.2 are the following two results due to Rogers and Taylor \[13, 14, 15\]:

**Theorem 4.2 (Rogers and Taylor - 1 (see Theorem 67 in \[13\]).** Let \( \eta \) be a \( \sigma \)-finite Borel measure on \( \mathbb{R} \) and \( 0 \leq \alpha \leq 1 \). Consider the sets \( T^\alpha_{\eta;0+} := \{ x : 0 \leq D^\alpha_\eta(x) < \infty \} \)
and \( T^\alpha_{\eta;\infty} := \{ x : D^\alpha_\eta(x) = \infty \} \). Then \( T^\alpha_{\eta;0+} \) and \( T^\alpha_{\eta;\infty} \) are Borel measurable and

- (i) \( \eta \) is \( \alpha \)-continuous on \( T^\alpha_{\eta;0+} \),
- (ii) \( h^\alpha(T^\alpha_{\eta;\infty}) = 0 \) and \( \eta \) is \( \alpha \)-singular on \( T^\alpha_{\eta;\infty} \).

**Theorem 4.3 (Rogers and Taylor - 2 (see Theorem 68 in \[13\]).** Let \( \eta \) be \( \sigma \)-finite and \( \alpha \)-continuous, \( \alpha \geq 0 \). For \( \epsilon > 0 \), there exist mutually singular measures \( \eta_1 \) and \( \eta_2 \) with \( \eta = \eta_1 + \eta_2 \) such that

- (i) \( \eta_1 \) is UoH and
- (ii) \( \eta_2(\mathbb{R}) < \epsilon \).

**Remark 4.4.** Depending on \( D^\alpha_\eta \), Theorem 4.2 decomposes \( \eta \) into an \( \alpha \)-continuous and an \( \alpha \)-singular component. It thus generalizes the usual Lebesgue decomposition for \( \alpha = 1 \). The relevance of the Rogers-Taylor decomposition in spectral theory was pointed out by Last; see \[16\].

By Theorem 4.3 any \( \alpha \)-continuous measure is almost UoH. Hence, the proof of Theorem 1.2 boils down to establishing the statement for a UoH measure \( \nu \). To this end we shall use the following lemma, which quantifies the asymptotic growth of \( P_\eta \) and \( Q_\eta \) near the support of a probability measure \( \eta \).

**Lemma 4.5.** Let \( \eta \) be a probability (Borel) measure on \( \mathbb{R} \). Then for \( x \in \mathbb{R} \) and \( \epsilon > 0 \)

\[
\max\{P_\eta(x + i\epsilon), |Q_\eta(x + i\epsilon)|\} \leq \frac{2}{\epsilon} \sum_{n=0}^{\infty} 2^{-n} M_\eta(x; 2^{n+1}\epsilon).
\]

**Proof.** Let \( x \in \mathbb{R} \) and \( \epsilon > 0 \). Then:

\[
|Q_\eta(x + i\epsilon)| \leq \sum_{n=1}^{\infty} \int_{\epsilon 2^n \leq |x - y| \leq \epsilon 2^{n+1}} \frac{|x - y|}{(x-y)^2 + \epsilon^2} \, d\eta(y)
\]
\[+ \int_{|x - y| \leq \epsilon} \frac{|x - y|}{(x-y)^2 + \epsilon^2} \, d\eta(y) \leq \frac{2}{\epsilon} \sum_{n=0}^{\infty} 2^{-n} M_\eta(x; 2^{n+1}\epsilon).\]

By a similar computation we obtain the same upper bound for \( P_\eta(x + i\epsilon) \). \( \square \)

We note that this result is an extended version of a lemma in \[10\] for UoH \( \eta \). Together with Proposition 3.2, Lemma 4.3 allows us to control the asymptotic behaviour of \( P_\kappa(x + i\epsilon) \) as \( \epsilon \to 0^+ \). We are thus in a position to prove Theorem 1.2.
5. PROOF OF THE MAIN THEOREM (THEOREM 1.2)

We shall divide the proof into two steps: Step 1 establishes the statement for \( \nu \) UnH. Theorem 1.3 then allows us to extend the result to the ac case (Step 2).

Step 1: Assume \( \nu \) to be UnH. If \( \alpha = 1 \), the statement follows directly from Theorem 3.5. Let \( \alpha < 1 \). We first examine the situation outside the support of the measure \( \mu \).

**Proposition 5.1.** Let \( 0 < \alpha < 1 \) and \( \nu \) UnH. Then \( \kappa \) is ac outside \( \operatorname{supp} \mu \).

**Proof.** Fixing \( x \notin \operatorname{supp} \mu \), there exist positive constants \( \Gamma_1 \) and \( \Gamma_2 \) such that

\[
|F_{\mu}(x + ie)|^2 \leq \Gamma_1, \quad P_{\mu}(x + ie) \geq \Gamma_2 \epsilon,
\]

for all \( \epsilon > 0 \) sufficiently small. Hence by Proposition 3.2 we obtain

\[
e^{1-\alpha} P_{\mu}(x + ie) \leq C_{\alpha} e^{1-\alpha} \left( \frac{|F_{\mu}(z)|^2}{P_{\mu}(z)} \right) \leq C_{\alpha} \left( \frac{\Gamma_1}{\Gamma_2} \right)^{1-\alpha},
\]

which implies the claim by Theorem 1.2. \( \square \)

**Remark 5.1.** By Theorem 4.3 (see the argument given in Step 2), the statement of Proposition 5.1 remains valid if \( \nu \) is (only) ac.

In order to analyze the situation within the support of \( \mu \), we first establish the following lemma:

**Lemma 5.2.** Let \( 0 < \alpha < 1 \) and \( \nu \) UnH. Fix \( 0 < \beta < 1 \). Then \( \kappa \) is \( \gamma c \) on the set \( T_{\mu;0+}^\beta \), where

\[
\gamma(\alpha, \beta) = \alpha - 2(1 - \beta)(1 - \alpha)
\]
as long as \( \beta > \max \left\{ 0, \frac{2 - 3\alpha}{2(1-\alpha)} \right\} \).

**Proof.** Let \( \beta < 1 \) be fixed. By Proposition 3.1 the statement is true outside \( \operatorname{supp} \mu \). Let \( x \in \operatorname{supp} \mu \) and assume \( \overline{D}_\mu(x) < \infty \) so that \( M_{\mu}(x; \delta) \leq A_x \delta^\beta, \forall \delta > 0 \). Thus,

\[
2 \epsilon \sum_{n=0}^{\infty} 2^{-n} M_{\mu}(x; 2^n + 1) \leq A_x 2^{1+\beta} \epsilon^{\beta-1} \sum_{n=0}^{\infty} 2^{-n(1-\beta)} < \infty.
\]

Note that finiteness of the upper bound in (5.3) requires \( \beta < 1 \).

Let \( \gamma < 1 \). Using Proposition 3.2 and Lemma 4.5, estimate (5.3) yields

\[
e^{1-\gamma} P_{\kappa}(x + ie) \leq B_{x,\beta} \left( \frac{e^{2(\beta-1) + \frac{1-\gamma}{1-\alpha}}}{P_{\mu}(x + ie)} \right)^{1-\alpha}.
\]

By Theorem 4.2 and Proposition 2.1, \( \kappa \) will be \( \gamma c \) on the set \( \{ x : \limsup_{\epsilon \to 0^+} e^{1-\gamma} P_{\kappa}(x + ie) < \infty \} \). Choose \( \gamma \) such that \( 2(\beta - 1) + \frac{1-\gamma}{1-\alpha} = 1 \), i.e. \( \gamma = \alpha - 2(1 - \beta)(1 - \alpha) \). Since,

\[
e^{1-\gamma} P_{\kappa}(x + ie) \to \int \frac{1}{(x - y)^2} d\mu(y) \quad \text{as} \quad \epsilon \to 0^+
\]

and \( \int \frac{1}{(x - y)^2} d\mu(y) > 0 \) for \( x \in \operatorname{supp} \mu \), we obtain that \( \kappa \) is \( \gamma c \) on the set \( T_{\mu;0+}^\beta \) with \( \gamma \) determined by (5.2). Finally, \( \gamma > 0 \) is ensured by requiring \( \beta > \max \left\{ 0, \frac{2 - 3\alpha}{2(1-\alpha)} \right\} \). \( \square \)

In summary we now obtain the claim for \( \nu \) UnH: Let \( \delta = \alpha(1 - \epsilon) \), \( 0 < \epsilon < 1 \). It suffices to prove the statement for \( \epsilon \) sufficiently small. Let \( \beta \) be such that \( \gamma(\alpha, \beta) = \delta \), i.e. \( \beta = 1 - \frac{\alpha}{2(1-\alpha)} \epsilon \). Choosing \( \epsilon \) sufficiently small we can ensure that \( \beta > \frac{2 - 3\alpha}{2(1-\alpha)} \), which is required to apply Lemma 5.2.
For such a choice of $\epsilon$ and $\beta$, Lemma 5.2 implies that for any Borel set $B$ with $h^\delta(B) = 0$,

\begin{equation}
\kappa(B) = \int \mu_{\lambda, \text{sing}}(B \cap T_{\mu; \infty}^\beta) d\nu(\lambda) \leq \int \mu_{\lambda, \text{sing}}(T_{\mu; \infty}^1) d\nu(\lambda) = 0 .
\end{equation}

Applying Propositions 2.2 and 2.1, $\mu_{\lambda, \text{sing}}(T_{\mu; \infty}^1) = 0$ for $\lambda \neq 0$, which by continuity of $\nu$ implies the last equality in (5.5).

**Step 2:** Let $0 < \alpha < 1$ and $\delta < \alpha$. If $\nu$ is $\alpha c$, then by Theorem 4.3 given $\epsilon > 0$ there are measures $\nu_1 \perp \nu_2$, $\nu = \nu_1 + \nu_2$ such that $\nu_1$ is $U\alpha$ and $\nu_2(\mathbb{R}) < \epsilon$.

Let $B \subseteq \mathbb{R}$ be a Borel set with $h^\delta(B) = 0$. Then, $\int \mu_{\lambda}(B) d\nu_1(\lambda) = 0$ by Step 1, whence

$$\kappa(B) = \int \mu_{\lambda}(B) d\nu_2(\lambda) < \epsilon .$$

An analogous argument shows that $\kappa$ is absolutely continuous if $\alpha = 1$, which concludes the proof of Theorem 1.2.

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