CENTRAL VALUES OF \( L \)-FUNCTIONS AND HALF-INTEGRAL WEIGHT FORMS

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Abstract. We prove a relation between the Fourier coefficients of certain Hilbert modular forms of half-integral weight and central values of the corresponding Rankin \( L \)-functions. The result generalizes the classical theorem by Waldspurger. The approach is geometric and generalizes that of Gross and Hatcher.

1. Introduction

Let \( F \) be a totally real field of degree \( d \). For the purpose of this paper the class number \( h(F) \) of \( F \) is always assumed to be odd. Let \( f \) be a holomorphic Hilbert newform over \( F \) of weight \( 2k = (2k_1, 2k_2, \ldots, 2k_d) \) with \( k_i \geq 1 \) and level \( N = \mathfrak{p}^e \) for a prime ideal \( \mathfrak{p} \) of \( F \) and \( e \geq 1 \). Assume further that if \( d \) is odd, then so is \( e \). Let \( B \) be the quaternion algebra over \( F \) such that:

(1) If \( d \) is even, then \( B \) is only ramified at all infinite places.

(2) If \( d \) and \( e \) are both odd, then \( B \) is only ramified at infinite places and \( \mathfrak{p} \).

For \( D \gg 0 \) in \( F \), let \( \chi_D \) be the quadratic character of the idèle class group \( \mathbb{A}_F^\times/F^\times \) associated to the quadratic extension \( K_D = F(\sqrt{-D}) \). Suppose that the conductor of \( \chi_D \) is relatively prime to \( \mathfrak{p} \) and that \( K_D \) can be embedded in \( B \), that is, \( \chi_D(\mathfrak{p}) = -1 \). Write \( L(s, f_D) = L(s, f)\hat{L}(s, f \otimes \chi_D) \) for the (complete) \( L \)-function of \( f \) over \( K_D \). It is well known that the sign of the functional equation of \( L(s, f_D) \) is given by \((-1)^d\chi_D(\mathfrak{p}^e)\), which is equal to 1 by the above assumptions.

Let us give a brief account of the paper. In Section 2, among other things, we introduce a Gross curve \( X \) and a vector bundle \( V \) associated to the quaternion algebra \( B \). In Section 3 we construct a Hilbert modular form \( g \) of half-integral weight with coefficients in \( \text{Pic}(V) \). Let \( g_f \) be the \( f \)-isotypical component of \( g \), and denote the \( D \)-th Fourier coefficient of \( g_f \) by \( m_D \). Then Section 4, Theorem 2 gives the main result of this paper, which is that under certain conditions on the odd fundamental discriminants \(-D\) the central value of \( L(s, f_D) \) is given by

\[
L(1/2, f_D) = C_D \frac{\langle f, f \rangle}{\langle \nu_f, \nu_f \rangle} |m_D|^2,
\]

where \( C_D \) is an explicit positive constant and \( \langle \nu_f, \nu_f \rangle \) is some height pairing; see Section 4 for the precise statement.
Formula (1.1) can be viewed as a geometric generalization of the well-known result due to Waldspurger [12] (for \( F = \mathbb{Q} \)). A more general but probably less explicit generalization can be found in Shimura [10]. When \( F = \mathbb{Q} \), similar results have been obtained by Gross [2] (weight 2) and Hatcher [6] (higher weight). Our approach and results are extensions of theirs.

Now we make a few comments on the assumptions stated above. The oddness of the class number \( h(F) \) of \( F \) is used to guarantee the passage from the original central value formula, which is stated on \( B^x/F^x \) in [13], to the current setting on \( B^x \). This assumption may be dropped if one can prove a central value formula directly on \( B^x \). The assumption on the level of \( f \) is used to eliminate the dependence of the central value formula on the choice of the \( \text{Pic}(\mathcal{O}_{K_D}) \) orbit of special points. This assumption can be lifted if either a central value formula involving all special points is obtained, or if the Fourier coefficients of automorphic forms of more general type, such as Jacobi forms or vector-valued forms, are considered (see [4] for a similar situation). The assumption on the parity of \( d \) and \( e \) is to ensure that \( f \) comes from a form on \( B^x \) under the Jacquet-Langlands correspondence. It might be more natural to start with a form on \( B^x \) directly.

2. GROSS CURVES AND VECTOR BUNDLES

Let \( B \) be the quaternion algebra over \( F \) defined as in Section 1(1) or (2). Let \( R \) be a fixed order of discriminant \( p^e \) over \( \mathcal{O}_F \) in \( B \), whose construction can be found for example in [3, Section 3] or [13, Section 3.2]. Notice that \( R \) is not unique even up to local conjugation. Let \( \widehat{F} = F \otimes \mathbb{Z} \), \( \widehat{B} = B \otimes \mathbb{Z} \) and \( \widehat{R} = R \otimes \mathbb{Z} \).

2.1. Gross curve \( X \). Let \( Y \) be the genus zero curve over \( F \) associated to \( B \). The points of \( Y \) over any \( F \)-algebra \( E \) are given by \( Y(E) = \{ \alpha \in B \otimes E \setminus \alpha \neq 0, \text{Tr}(\alpha) = \mathcal{N}(\alpha) = 0 \}/E^x \).

The Gross curve \( X \) is defined by the double coset
\[
X = B^x \setminus Y \times \widehat{B}^x / \widehat{R}^x
\]
Let \( \widehat{B}^x = \bigcup_{i=1}^n B^x g_i \widehat{R}^x \) be the double coset decomposition. Then there is an isomorphism
\[
X \cong \prod_{i=1}^n \Gamma_i \backslash Y = \prod_{i=1}^n X_i,
\]
where each \( \Gamma_i = (B^x \cap g_i \widehat{R}^x g_i^{-1})/\mathcal{O}_F^x \) is a finite group with order denoted by \( w_i \).

Suppose \( D \gg 0 \) is such that \( K_D = F(\sqrt{-D}) \) can be embedded into \( B \). Then, there is a canonical identification \( Y(K_D) \cong \text{Hom}_F(K_D, B) \) as follows: for any \( f \in \text{Hom}_F(K_D, B) \), let \( y \) be the image of the unique \( K_D \)-line on the quadric \( \{ \alpha \in B \otimes F \mid \alpha \neq 0, \text{Tr}(\alpha) = \mathcal{N}(\alpha) = 0 \} \) on which conjugation by \( f(K_D) \) acts by multiplication by the character \( k \mapsto k/k' \). Notice that \( f(K_D) \) has two fixed points on \( Y(K_D) \), one of which is \( y \) and the other one is \( \overline{y} \), the image of \( y \) under the complex conjugate of \( \text{Gal}(K_D/F) \). Special points on \( X \) over \( K_D \) are the images of \( Y(K_D) \times \widehat{B}^x / \widehat{R}^x \) in \( X(K_D) \).

Now suppose further that \( -D \in F \) is a fundamental discriminant (Definition 3.2) and \( p \nmid D \). A special point \( x = (y, g) \in X(K_D) \) is said to have discriminant \( -D \) if \( f(K_D) \cap g \widehat{R} g^{-1} = f(\mathcal{O}_{K_D}) \), where \( f : K \to B \) is the embedding corresponding to \( y \). If the component \( g \) of \( x \) is congruent to \( g_i \) in \( B^x \setminus \widehat{R}^x / \widehat{R}^x \), then the special
point \( x \) lies on the component \( X_i \). Let \( h_i(D) \) be the number of embeddings of \( \mathcal{O}_{K_D} \) into \( R_i \), modulo conjugation by \( R_i^\times \). Then there are exactly \( h_i(D) \) special points of discriminant \(-D\) on the component \( X_i \). There is a free action of the group \( \text{Pic}(\mathcal{O}_{K_D}) \cong K_D^x\backslash\hat{K}_D^x/\hat{O}_{K_D}^x \) on the set of special points of discriminant \(-D\) as follows. Let \( x = (y, g) \) be a special point of discriminant \(-D\) and let \( a \in \hat{K}_D^x \).

Let \( \hat{f} : \hat{K}_D^x \to \hat{B}^x \) be the homomorphism induced by the embedding \( f : K_D \to B \) that corresponds to \( y \). The action of \( a \) on \( x \) is then given by

\[
(2.3) \quad x_a = (y, \hat{f}(a)g).
\]

This action is well-defined and defines a free action of \( \text{Pic}(\mathcal{O}_{K_D}) \) on the set of special points of discriminant \(-D\); see \[2\] p. 133. The argument of \[2\] p. 133 shows that the orbit set classifies embeddings of \( \mathcal{O}_{K_D} \) into \( R_i \) modulo conjugation by \( R_i^\times \).

The latter set is known \[7\] Theorem 5.12 or \[1\] to have cardinality 2. On the other hand, the complex conjugate of \( \text{Gal}(K_D/F) \) on \( Y(K_D) \) induces an action on special points of discriminant \(-D\). Thus the product group \( \text{Gal}(K_D/F) \times \text{Pic}(\mathcal{O}_{K_D}) \) acts simply transitively on the set of all special points of discriminant \(-D\) on \( X \) by counting the number of points.

The group \( \text{Pic}(X) \) of line bundles (or linearly equivalent divisor classes) on \( X \) is isomorphic to \( \mathbb{Z}^n \), and is generated by the classes \( e_i \) of degree 1 on each component \( X_i \). We define two divisor classes associated to a fundamental discriminant \(-D\) (and \( p \nmid D \)). Let \( u(D) = |\mathcal{O}_{K_D}^x/\mathcal{O}_R^x| \). The first one is given by

\[
(2.4) \quad c_D = \frac{1}{2u(D)} \sum_{\text{disc}(x)=-D} (x),
\]

where the sum is over all special points of discriminant \(-D\). The second one is given by the \( \text{Pic}(\mathcal{O}_{K_D}) \) orbit

\[
(2.5) \quad y = \frac{1}{u(D)} \sum_{a \in \text{Pic}(\mathcal{O}_{K_D})} (x_a) \in \text{Pic}(X),
\]

where \( x \in X \) is any fixed special point of discriminant \(-D\). Since \( x \) and \( y \) lie on the same component of \( X \), we obtain the following equality as divisor classes:

\[
(2.6) \quad c_D = y = \frac{1}{2u(D)} \sum_{i=1}^n h_i(D) e_i \in \text{Pic}(X).
\]

If \( a = \sum_{i=1}^n a_i e_i \) and \( b = \sum_{i=1}^n b_i e_i \) are two divisor classes in \( \text{Pic}(X) \), the height pairing between them is given by

\[
(2.7) \quad (a, b) = \sum_{i=1}^n w_i a_i b_i.
\]

2.2. Vector bundle \( V \) over \( X \). In the following, let \( v = 1, \ldots, d \) be the subscripts corresponding to infinite places of \( F \). Each local component \( B_v = B \otimes_F \mathcal{O}_v \) with the standard Hamiltonian quaternion algebra over \( \mathbb{R} \).

Let \( W = \mathbb{C}x \oplus \mathbb{C}y \) be the standard two-dimensional representation of \( \text{SU}(2) \) with an inner product given by \( [x,x] = [y,y] = 1 \) and \( [x,y] = 0 \). For each infinite place \( v = 1, \ldots, d \), the space \( \text{Sym}^{2k_v-2}(W) \) is an inner product space with basis \( \{x^{2k_v-2}, y^{2k_v-2} \} \). The induced inner product on \( \text{Sym}^{2k_v-2}(W) \) is such that \( [x^iy^j, x^iy^j] = i!j! \); see \[5\] Section 3 for more details. Notice that \( \text{Sym}^{2k_v-2}(W) \)
is also a representation of SO(3), thus can be regarded as a representation of $B_v^\times$. Let $U_v$ be the vector space of trace-free elements of $B_v$. Then $B_v^\times$ has a natural representation on $U_v$ by conjugation. An inner product on $U_v$ is given by $\langle u_1, u_2 \rangle = \frac{1}{2} \text{Tr}(u_1 \overline{u}_2)$. The space $\text{Sym}^{k_v-1}(U_v)$, as an inner product space, has an orthogonal decomposition \[ \text{Sym}^{k_v-1}(U_v) = \text{Sym}^{2k_v-2}(W) \oplus M_v. \] (2.8)

Let $W_{2k-1} = \bigoplus_v \text{Sym}^{2k_v-2}(W)$. The space $W_{2k-1}$ has an action of $B_v^\times$ through the diagonal embedding $B_v^\times \to (B \otimes \mathbb{R})^\times = \prod_v B_v^\times$. The vector bundle $V$ is then defined by

\[ V = B_v^\times \backslash Y \times W_{2k-1} \times \widehat{B_v^\times}/\widehat{R_v^\times}. \] (2.9)
If we write out the double coset quotient, then

\[ V \cong \prod_{i=1}^n \Gamma_i \backslash (Y \times W_{2k-1}) = \prod_{i=1}^n V_i. \] (2.10)

For each $i$, let $W_{2k-1}^{\Gamma_i}$ be the elements invariant under the action of $\Gamma_i$. Then $\text{Pic}(V)$ is defined by

\[ \text{Pic}(V) = \bigoplus_{i=1}^n \mathbb{C} e_i \otimes W_{2k-1}^{\Gamma_i}. \] (2.11)
Notice that $\text{Pic}(V)$ is already a vector space over $\mathbb{C}$. Letting $\nu = (y, g_i, w) \in V$, define the class of $\nu$ to be

\[ (\nu) = \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} e_i \otimes \gamma(w) \in \text{Pic}(V). \] (2.12)

The height pairing on $V \times V$ is defined as follows. Let $\nu_1 = (y_1, g_1, w_1)$ and $\nu_2 = (y_2, g_2, w_2)$. Then

\[ \langle \nu_1, \nu_2 \rangle = \begin{cases} 0, & i \neq j, \\ \sum_{\gamma \in \Gamma_i} [w_1, \gamma(w_2)]w_{2k-1}, & i = j. \end{cases} \] (2.13)
Formula (2.13) induces a height pairing on $\text{Pic}(V) \times \text{Pic}(V)$, which coincides with (2.7) if $k_v = 2$ for all $v$.

Let $D \gg 0$ be such that there is an embedding $f : K_D \to B$. At each infinite place $v$, let $u_v = \sqrt{-D_v} \in U_v$, and let $w_v$ be the component of $(u_v)^{k_v-1}$ in $\text{Sym}^{2k_v-2}(W)$ through (2.8). Write $w_v = \prod_v w_v \in W_{2k-1}$. A straight computation \[ \text{[}w_0^0, w_0^0\text{]} = \prod_v \frac{2^{2k_v-2}D_v^{k_v-1}(k_v-1)!^3}{(2k_v-2)!}. \] (2.14)
Suppose further that $-D$ is a fundamental discriminant with $p \nmid D$. A special point on $V$ of discriminant $-D$ is a point of the form $\nu = (y, g, w_0^0)$, where $x = (y, g)$ is a special point on $X$ of discriminant $-D$. The action of $\text{Pic}(\mathcal{O}_D)$ on special points of $X$ extends naturally to special points of $V$ through $\nu_a = (x_a, w_0^0)$; see (2.3). Similar to (2.6) define the special cycle of $V$ by

\[ \nu_D = \frac{1}{u(D)} \sum_{a \in \text{Pic}(\mathcal{O}_D)} (\nu_a) = \frac{1}{2u(D)} \sum_{i=1}^n h_i(D)(e_i \otimes w_0^0) \in \text{Pic}(V). \] (2.15)
2.3. Curve $X'$ and vector bundle $V'$. To utilize the central value formula of \[13\] we need to define a new curve $X'$ and a vector bundle $V'$ on it. The construction is parallel to that in Section 2.2. After the construction the relationship between them is studied.

The curve $X'$ is defined by

\[
X' = B^\times \backslash Y \times \hat{B}^\times / \hat{\mathbb{R}}^\times.
\]

If we write $\hat{B}^\times = \bigcup_{j=1}^m B^\times g_j^j \hat{\mathbb{R}}^\times$ for the double coset decomposition, then

\[
X' \cong \prod_{j=1}^m \Gamma'_j \backslash Y,
\]

where $\Gamma'_j \cong (B^\times \cap g_j^j \hat{\mathbb{R}}^\times g_j^{-1}) / F^\times$.

Thus the group of line bundles $\text{Pic}(X')$ of $X'$ is isomorphic to $\mathbb{Z}^m$. If $a' = \sum_{j=1}^m a_j e'_j$ and $b' = \sum_{j=1}^m b_j e'_j$, the height pairing between them is given by

\[
\langle a', b' \rangle' = \sum_{j=1}^m w'_j a_j b_j.
\]

Assumption. From now on we will assume that the class number $h(F)$ of $F$ is odd.

Lemma 2.1. The (multiplication) action of $\text{Pic}(\mathcal{O}_F) = F^\times \backslash \hat{\mathbb{R}}^\times / \hat{\mathcal{O}}_F^\times$ on $S = B^\times \backslash \hat{B}^\times / \hat{\mathbb{R}}^\times$ is free. The quotient space is given by

\[
B^\times \backslash \hat{B}^\times / \hat{\mathbb{R}}^\times.
\]

Proof. The second claim is obvious, and we only need to prove the first one. Let $t \in \hat{\mathbb{R}}^\times$ be a stabilizer of $g \in \hat{B}^\times$ such that $gt = agr$ with $a \in B^\times$ and $r \in \hat{\mathbb{R}}^\times$. Then $N(g)N(t) = N(a)N(g)N(r)$, which implies that $t^2 = N(t) = N(a)N(r) \in F^\times \hat{\mathcal{O}}_F$. Since $h(F)$ is assumed to be odd, we conclude that $t$ itself has to define a trivial class in $\text{Pic}(\mathcal{O}_F)$. \hfill $\square$

By Lemma 2.1 we can and will take $g'_j = g_j$ for $j = 1, \cdots, m$. Also, $n = mh(F)$.

Remark 2.1. The same argument shows that the natural homomorphism $\text{Pic}(\mathcal{O}_F) \rightarrow \text{Pic}(\mathcal{O}_K)$ is injective for any quadratic extension $K/F$.

Lemma 2.2. For each $j = 1, \cdots, m$, there is a natural isomorphism from $\Gamma'_j$ to $\Gamma'_j$. More precisely, the natural homomorphism

\[
\left( B^\times \cap g_j \hat{\mathbb{R}}^\times g_j^{-1} \right) / \mathcal{O}_F^\times \cong \left( B^\times \cap g_j \hat{\mathbb{R}}^\times \hat{\mathbb{R}}^\times g_j^{-1} \right) / F^\times
\]

is an isomorphism. Thus $w'_j = w_j$ for $j = 1, \cdots, m$.

Proof. The homomorphism in (2.19) is induced by the identity map. To see that it is surjective, suppose $b \in B^\times \cap g_j \hat{\mathbb{R}}^\times g_j^{-1}$. Then we need to find $b_1 \in B^\times \cap g_j \hat{\mathbb{R}}^\times g_j^{-1}$ such that $b_1 = bf$ for some $f \in F^\times$. Since $h(F)$ is odd, and taking the norms of $b = g_j trg_j^{-1} \in B^\times$ we get $N(b) = t^2N(r) \in F^\times$, which implies that $t \in F^\times \hat{\mathcal{O}}_F'$, so $b_1 = bf$ for some $f \in F$. For injectivity, suppose $b_1, b_2 \in B^\times \cap g_j \hat{\mathbb{R}}^\times g_j^{-1}$ such that $b_1 = f b_2$. Since $b_1 b_2^{-1} \in g_j \hat{\mathbb{R}}^\times g_j^{-1}$, we get that $f \in F^\times \cap g_j \hat{\mathbb{R}}^\times g_j^{-1} = \mathcal{O}_F^\times$. \hfill $\square$
Corollary 2.3. The natural projection
\[ \pi : X = B^\times \backslash Y \times \hat{B}^\times / \hat{R}^\times \to X' = B^\times \backslash Y \times \hat{B}^\times / \hat{F}^\times \hat{R}^\times \]
is an étale covering of equal degree \( h(F) \).

Corollary 2.4. Let \( c'_D = \pi(c_D) \in \text{Pic}(X') \). Then
\[ \pi^*(c'_D) = c_D. \]
Thus \((c_D, c'_D) = \deg(\pi)(c'_D, c'_D)' = h(F)(c'_D, c'_D)' \) by the projection formula.

Notice that the centers of \((B \otimes \mathbb{R})^\times \) and \( B^\times \) act trivially on \( W_{2k-1} \) and \( U_v \) respectively. Thus, the vector bundle \( V \) descends to a vector bundle on \( X' \) and is denoted by \( V' \). The group \( \text{Pic}(V') \) can also be defined similarly. \( \text{Pic}(V) \) is equipped with a height pairing, denoted again by \( (\cdot, \cdot)' \). The projection of a special point \( \nu \) of fundamental discriminant \(-D\) on \( V \) defines a special point \( \nu' \) on \( V' \). Hence the special cycle \( \nu_D \)\(^{(2.14)}\) descends to a special cycle on \( V' \):
\[
\nu' = \sum_{a \in \text{Pic}(\mathcal{O}_D)/\text{Pic}(\mathcal{O}_F)} (\nu'_a) = \frac{1}{2u(D)} \sum_{j=1}^{m} h_j(D)(c'_j \otimes w_0^D) \in \text{Pic}(V').
\]

Similar to Corollary 2.4 we have the following.

Proposition 2.5. Let \( \nu_D \) and \( \nu'_D \) be defined in \((2.14)\) and \((2.21)\) respectively. Then
\[ (\nu_D, \nu'_D) = h(F)(\nu'_D, \nu'_D)' . \]

3. Hilbert Modular Forms of Half-integral Weight

Let \( B^0 \) be the subspace of trace-free elements of \( B \), and let \( U = \prod_v B^0 \otimes_v \mathbb{R} \). For each infinite place \( v \), let \( \mu^1_v, \mu^2_v, \mu^3_v \) be a fixed basis of \( B^0 \otimes_v \mathbb{R} \). A homogeneous polynomial \( P = \prod_v P_v \) of degree \( k - 1 \) on \( U \) is said to be spherical harmonic if for each \( v \),
\[ \sum_{i=1}^3 \frac{\partial^2}{\partial x_{i,v}^2} P_v = 0. \]

Lemma 3.1. For each \( \nu \in \text{Pic}(V) \), there is a spherical harmonic polynomial \( P_{i,v}^\nu = \prod_v P_{i,v}^\nu \) of degree \( k - 1 \) on \( U \) such that for every \( D \gg 0 \), one has
\[ (\mu, (e_i \otimes w^D_v)) = \prod_v P_{i,v}^\nu(\sqrt{-D_v}) = \prod_v P_{i,v}^\nu(x_{1,v}, x_{2,v}, x_{3,v}), \]
where \( \sqrt{-D_v} = x_{1,v} \mu^1_v + x_{2,v} \mu^2_v + x_{3,v} \mu^3_v \).


Let \( R_i, i = 1, \cdots, n \) be the order associated to \( g_i \in B^\times \backslash \hat{B}^\times / \hat{R}^\times \), that is, \( R_i = B \cap g_i \hat{R} g_i^{-1} \). Let
\[ S_i = \mathcal{O}_F + 2R_i \]
be a suborder of \( R_i \). Write \( S^0_i \) for the subset of trace-free elements of \( S_i \). For each \( D \gg 0 \) in \( \mathcal{O}_F \), define \( A_i(D) = \{ b \in S^0_i \mid N(b) = -b^2 = D \} \) and \( a_i(D) = |A_i(D)| \).
Definition 3.2. Let \( D \gg 0 \) be in \( \mathcal{O}_F \) and \( K_D = F(\sqrt{-D}) \). Then \(-D\) is called a fundamental discriminant if \( \mathcal{O}_{K_D} \) has relative discriminant \((D)\) over \( \mathcal{O}_F \), and

\[
O_K = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2}\mathcal{O}_F
\]

for some \( a \in \mathcal{O}_F \).

Proposition 3.3. Suppose the relative discriminant of \( K_D \) over \( F \) is \((D)\), and suppose \( K_D \) splits over all primes of \( F \) dividing 2. Then \(-D\) is a fundamental discriminant.

Proof. First, notice that \( x = \frac{a + \sqrt{-D}}{2} \) for \( a \in \mathcal{O}_F \) is in \( \mathcal{O}_{K_D} \) if and only if \( 4|(a^2 + D) \).

Let \((2) = p_1^{e_1} \cdots p_s^{e_s}\) be the prime decomposition of \((2)\) in \( \mathcal{O}_F \). Since each \( p_i \) splits in \( K_D \), the equation \( y_i^2 = -D \) has solutions in \( \mathcal{O}_{F, p_i} \), which modulo \( p_i^{2e_i} \) implies that \( y_i^2 \equiv -D \mod(p_i^{2e_i}) \) has solutions for all \( i \). Now the Chinese Remainder Theorem tells us that there is an \( a \in \mathcal{O}_F \) such that \( a \equiv y_i \) for all \( i \). Such an \( a \) satisfies the required property \( a^2 \equiv -D \mod(4) \). Hence \( x = \frac{a + \sqrt{-D}}{2} \in \mathcal{O}_{K_D} \).

Let \( \mathcal{O} = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2}\mathcal{O}_F \). As \( (\frac{a + \sqrt{-D}}{2})^2 = \frac{a^2 - D}{4} + \frac{a\sqrt{-D}}{2} = -\frac{a^2 - D}{4} + \frac{a + \sqrt{-D}}{2} \in \mathcal{O} \), so \( \mathcal{O} \) is an order in \( K \). Moreover, the discriminant of \( \mathcal{O} \) is \((D)\), which is the same as that of \( \mathcal{O}_K \). Therefore \( \mathcal{O}_K = \mathcal{O} = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2}\mathcal{O}_F \) and \(-D\) is a fundamental discriminant.

Proposition 3.3 implies that there are plenty of fundamental discriminants. For instance, let \( D \) be square-free such that \((D, d_F) = 1\) and \(-D \equiv 1\) \((\mod 8)\). Then \(-D\) is a fundamental discriminant.

Proposition 3.4. Suppose \(-D\) is a fundamental discriminant. Then, for each \( i = 1, \cdots, n \),

\[
\frac{a_i(D)}{u_i} = \frac{h_i(D)}{u(D)},
\]

where \( h_i(D) \) is the number of (optimal) embeddings of \( \mathcal{O}_K \) into \( R_i \), modulo conjugation by \( R_i^* \).

Proof. First let \( f : \mathcal{O}_K \to R_i \) be an embedding. Write \( \mathcal{O}_K = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2}\mathcal{O}_F \).

The element \( b = f(\sqrt{-D}) \in R_i \) then satisfies \( \text{Tr}(b) = 0 \) and \( N(b) = -D \). Since \( \frac{a + b}{2} = f(\frac{a + \sqrt{-D}}{2}) \in R_i \), we have \( b \in A_i(D) \).

Conversely let \( b \in A_i(D) \). Thus \( a' + b \in 2R_i \) for some \( a' \in \mathcal{O}_F \). In particular, \( \frac{a' + b}{2} \) is integral over \( \mathcal{O}_F \), which implies that \( \frac{a' + \sqrt{-D}}{2} \in K_D \) is integral over \( \mathcal{O}_K \), that is, \( \frac{a' + \sqrt{-D}}{2} \in \mathcal{O}_K \). By comparing discriminants we get

\[
\mathcal{O}_K = \mathcal{O}_F + \frac{a' + \sqrt{-D}}{2}\mathcal{O}_F.
\]

Thus we obtain an embedding \( f : \mathcal{O}_K \to R_i \) by letting \( f \left( \frac{a' + \sqrt{-D}}{2} \right) = \frac{a' + b}{2} \).

The group \( \Gamma_i = R_i^*/\mathcal{O}_F^* \) acts on \( A_i(D) \) and the set of embeddings by conjugation. Hence we have proved that

\[
\frac{|A_i(D)/\Gamma_i|}{h_i(D)} = h_i(D).
\]

Now (3.2) follows from (3.3) because the order of the stabilizer of an element \( b \in A_i(D) \) under the action of \( \Gamma_i \) is equal to \( u(D) \). \(\square\)
Proposition 3.5. The series

\[
(g) = \sum_{D > 0, \sigma D = 0} \left( \sum_{i=1}^{n} \frac{a_i(D)}{2w_i} (e_i \otimes w_0^D) \right) q^D
\]
defines a Hilbert modular form of weight \( k + \frac{1}{2} \) with coefficients in \( \text{Pic}(V) \). Here \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{H}^d \), and \( q^D = \exp(2\pi i \text{Tr} D z) \). Moreover, if \( -D \) with \( p \nmid D \) is a fundamental discriminant, then the \( D \)-th Fourier coefficient of \( g \) equals \( \nu_D \).

Proof. By Lemma 3.1, the \( i \)-th series in (3.4) is a theta series with spherical harmonic coefficients attached to the quadratic space \( S_0^i \). The modularity and weight of such a series is well known when \( F = \mathbb{Q} \); see for instance [8]. For a general field \( F \), the proof can be found in [11, Section 5].

If \( -D \) is a fundamental discriminant, the equality between the \( D \)-th Fourier coefficient of \( g \) and \( \nu_D \) follows by comparing (2.15) and (3.2). \( \square \)

The term for \( D = 0 \) in (3.4) is nonzero only if \( k_v = 2 \) for all \( v \).

4. Central values and Fourier coefficients of half-integral weight forms

In this section we first recall a formula for the central value \( L(1/2, f_D) \) which is expressed in terms of the height pairing between special cycles of \( \text{Pic}(V) \). Then we relate these central values to the Fourier coefficients of the Hilbert modular form of half-integral weight constructed in Proposition 3.5.

From now on let \( -D \) with \( p \nmid D \) be a fundamental discriminant, and assume there exists a special point of discriminant \( -D \) on \( X \).

Theorem 1. The central value of \( L(s, f_D) \) is given by

\[
L(1/2, f_D) = C_D \langle f_D, f \rangle \langle \nu_D, f \rangle,
\]
where \( C_D \) is given by

\[
C_D = C_1 h(F) d^{3/2} d^{-1/2} K_D \prod_v \frac{(k_v - 1)!}{2D_v^{1-k_v}}
\]

with \( C_1 \) defined in [13, (1.4)], \( \langle f, f \rangle \) is the Petersson inner product, and \( \nu_D, f \) is the \( f \)-isotypical component of \( \nu_D \).

Proof. By [13, Theorem 1.2],

\[
L(1/2, f_D) = C_1 d^{3/2} d^{-1/2} \langle f_D, f \rangle \langle \nu_D, f \rangle \langle \gamma \rangle,
\]
where \( C_1 \) is the rational number given by [13, (1.4)] and \( \langle \cdot, \cdot \rangle_G \) is the geometric pairing defined in [13, Section 3.1] through a multiplicity function \( M_{\infty} = \prod_v M_v \), which is defined as follows. Choose a decomposition \( B = K_D + K_D^\perp \), and let \( \xi(a + bj) = \frac{N(bj)}{N(a + bj)} \). Then

\[
M_v(\gamma_v) = \frac{2^{2k_v-1} (k_v - 1)!^2}{(2k_v - 2)!} P_{k_v - 1}(1 - 2\xi(\gamma_v)),
\]
where \( P_n(t) = \sum_{j=0}^{n} \binom{n}{j} \binom{n + j}{n} \left( t - \frac{1}{2} \right)^j \) is the standard \( n \)-th Legendre polynomial.
By [5, (7.6)] one has

\[ [w_D^0, \gamma(w_D^0)] = \prod_v P_{k_v-1}(1 - 2\xi(\gamma_v)) \cdot \prod_v 2^{2k_v-2}D_{k_v}^{-1}(k_v - 1)!^3 \cdot (2k_v - 2)! \]

\[ = \left( \prod_v \left( \frac{(k_v - 1)!}{2D_v^{1-k_v}} \right) \right) M_\infty(\gamma). \]

Comparing the height pairing (2.13) and the geometric pairing in [13, Section 3.1], we obtain the following relation between them:

\[ \langle \nu'_{D,f}, \nu'_{D,f} \rangle' = \left( \prod_v \left( \frac{(k_v - 1)!}{2D_v^{1-k_v}} \right) \right) \langle \nu'_{D,f}, \nu'_{D,f} \rangle_G. \]

By Proposition 2.5

\[ \langle \nu_{D,f}, \nu_{D,f} \rangle = h(F) \left( \prod_v \left( \frac{(k_v - 1)!}{2D_v^{1-k_v}} \right) \right) \langle \nu_{D,f}, \nu_{D,f} \rangle_G; \]

which combined with (4.3) completes the proof of (4.1).

\[ \square \]

Remark 4.1. (1) Strictly speaking, the special cycle in [13] is defined over a curve associated to the group 
\[ G = B \times F / \hat{B} \times \hat{F} \]

with \( \hat{B} \times \hat{F} / \hat{O} \times \hat{O} \). This isomorphism is induced by 
\[ \hat{B} \times \hat{F} / \hat{O} \times \hat{O} = G(\hat{A}_f) / \Delta, \]

(2) The existence of a special point of discriminant \(-D\) implies that the functional equation of \( L(s, f_D) \) has sign +1 (due to the assumption on the parity of \( d \) and \( e \)). Hence the result of [13] can be applied.

(3) There are naturally defined Hecke operators acting on all the curves and line bundles defined in Section 2. By the Jacquet-Langlands correspondence, \( f \) determines an eigenform on the group \( B^\times / F^\times \), which has the same Hecke eigenvalues as \( f \). Hence it is legitimate to speak of \( f \)-isotypical components on Pic(V).

Let \( \nu_f \in \text{Pic}(V) \otimes \mathbb{C} \) be a nonzero element in the \( f \)-isotypical component. By the strong multiplicity-one theorem, such a \( \nu_f \) is unique up to a scalar multiple. Let

\[ g(\nu_f) = \sum_D m_D q^D = \langle g, \nu_f \rangle. \]

Now we state the main result of this paper.

**Theorem 2.** Let \(-D\) with \( p \nmid D \) be an odd fundamental discriminant such that there is a special point on \( X \) of discriminant \(-D\). Then

\[ L(1/2, f_D) = C_D \left( \frac{\langle f, f \rangle_{\langle \nu_f, \nu_f \rangle}}{\langle \nu_f, \nu_f \rangle} \right) |m_D|^2, \]

where \( C_D \) is given by (4.2).
Proof. By Proposition 3.5 and (4.8),
\[ m_D = \langle \nu_D, \nu_f \rangle = \langle \nu_D, \nu_f \rangle. \]
Thus
\[ \nu_D, f = \frac{m_D}{\langle \nu_f, \nu_f \rangle} \nu_f \in \text{Pic}(V). \]
So
\[ \langle \nu_D, \nu_D, f \rangle = \frac{|m_D|^2}{\langle \nu_f, \nu_f \rangle}. \]
Now (4.9) follows immediately from (4.1).
\[ \square \]
The form \( g(\nu_f) \) is a Niwa-Shintani lifting of \( f \). Hence (4.9) can be regarded as an explicit geometric generalization of the result of [12].

References


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