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IDEAL-TRIANGULARIZABILITY OF NIL-ALGEBRAS GENERATED BY POSITIVE OPERATORS

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Abstract. R. Drnovšek, D. Kokol-Bukovšek, L. Livshits, G. MacDonald, M. Omladič, and H. Radjavi constructed an irreducible set of positive nilpotent operators on $L^p[0,1)$ which is closed under multiplication, addition and multiplication by positive real scalars with the property that any finite subset is ideal-triangularizable. In this paper we prove the following:

(1) every algebra of nilpotent operators which is generated by a set of positive operators on a Banach lattice is ideal-triangularizable whenever the nilpotency index of its operators is bounded;
(2) every finite subset of an algebra of nilpotent operators which is generated by a set of positive operators on a Banach lattice is ideal-triangularizable.

1. Introduction and preliminaries

Let E be a Banach lattice. By an *operator* on E we mean a continuous linear transformation from E into itself. An operator T is called *positive* whenever $Tx \in E^+$ for all $x \in E^+$. For the terminology not explained in the text we refer the reader to the books [10] and [14].

A family $\mathcal{F}$ of operators on E is said to be *reducible* if there exists a nontrivial closed subspace of E that is invariant under every member of $\mathcal{F}$. Otherwise, we say that $\mathcal{F}$ is *irreducible*. If there exists a maximal subspace chain (i.e., a maximal totally ordered set of closed subspaces) whose elements are invariant under every member of $\mathcal{F}$, then $\mathcal{F}$ is said to be *triangularizable*. A family $\mathcal{F}$ of operators on E is said to be *ideal-reducible* if there exists a nontrivial closed ideal of E which is invariant under every operator in $\mathcal{F}$. Otherwise, we say that $\mathcal{F}$ is *ideal-irreducible*.

A subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be a *semigroup ideal* if ST and TS belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$. For a subset $\mathcal{F}$ of a semigroup $\mathcal{S}$ we define $\mathcal{F}^n$ as the set of all products of at least n elements of $\mathcal{F}$, and by $\mathrm{sg}(\mathcal{F})$ we denote the semigroup generated by $\mathcal{F}$. It is obvious that for each $n \in \mathbb{N}$ the set $\mathcal{F}^n$ is a semigroup ideal in $\mathrm{sg}(\mathcal{F})$.

The following proposition, which was proved in [3], is very useful in determining if a given semigroup of positive operators is ideal-reducible.

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Proposition 1.1. *Let E be a normed Riesz space, and let $\mathcal{S}$ be a nonzero semigroup of positive operators on E. The following statements are equivalent:*

(1) *$\mathcal{S}$ is ideal-reducible;*
(2) *there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(\mathcal{S}f) = \{0\}$;*
(3) *there exist nonzero positive operators A and B on E such that $A\mathcal{S}B = \{0\}$;*
(4) *some nonzero semigroup ideal of $\mathcal{S}$ is ideal-reducible.*

A family $\mathcal{F}$ is *ideal-triangularizable* if there is a chain $\mathcal{C}$ that is maximal as a chain of closed ideals of E and that has the property that every ideal in $\mathcal{C}$ is invariant under all the operators in $\mathcal{F}$. Any such chain of closed ideals is said to be an *ideal-triangularizing* chain for the family $\mathcal{F}$. By [2, Proposition 1.2], every maximal chain of closed ideals of a Banach lattice is also maximal as a chain of closed subspaces of a Banach space. Thus, an ideal-triangularizable family of operators is also triangularizable. The notion of ideal-triangularizability was introduced by Jahandideh in [9].

Let $\mathcal{F}$ be a family of operators on E, and let I and J be closed ideals of E satisfying $J \subseteq I$ that are invariant under every member of $\mathcal{F}$. Then $\mathcal{F}$ induces a family $\hat{\mathcal{F}}$ of operators on the quotient Banach lattice I/J as follows. For each $T \in \mathcal{F}$ the operator $\hat{T}$ is defined on I/J by

$$\hat{T}(x + J) = Tx + J \ .$$

Because I and J are invariant under T, the operator $\hat{T}$ is a well-defined operator on I/J. Any such $\hat{\mathcal{F}}$ is called a *collection of ideal-quotients* of the family $\mathcal{F}$. A set $\mathcal{P}$ of properties is said to be *inherited by ideal-quotients* if every family of ideal-quotients of a family of operators satisfying properties in $\mathcal{P}$ also satisfies the same properties.

Lemma 1.2 (The Ideal-triangularization Lemma)**.** *Let $\mathcal{P}$ be a set of properties inherited by ideal-quotients. If every family of operators on a Banach lattice of dimension greater than one which satisfies $\mathcal{P}$ is ideal-reducible, then every such family is ideal-triangularizable.*

Definition 1.3. An algebra $\mathcal{A}$ over a field F is said to be nilpotent (resp. a nil-algebra) if there exists $n \in \mathbb{N}$ such that the product of an arbitrary n elements in $\mathcal{A}$ is equal to zero (resp. if every element in $\mathcal{A}$ is nilpotent).

It is obvious that every nilpotent algebra is also a nil-algebra, but the converse statement is false in general. The following theorem of Nagata ([11]) and Higman ([8]) shows the converse statement when the nilpotency index of elements is bounded.

Theorem 1.4 (Nagata-Higman)**.** *Let $\mathcal{A}$ be an algebra over a field F of characteristic zero. Suppose that there is a natural number n such that $a^n = 0$ for all $a \in \mathcal{A}$. Then $a_1 \cdots a_{2^n-1} = 0$ for all $a_1, \ldots, a_{2^n-1} \in \mathcal{A}$.*

2. Results

The following proposition will be used in the proof of Theorem 2.2.

Proposition 2.1. *Every nilpotent algebra generated by the set of positive operators acting on a Banach lattice is ideal-triangularizable.*

Proof. Suppose that $\mathcal{A}$ is a nilpotent algebra generated by the family $\mathcal{F}$ of positive operators on a Banach lattice E. By the Ideal-triangularization Lemma it suffices to show that $\mathcal{A}$ is ideal-reducible. Since the semigroup $\mathrm{sg}(\mathcal{F})$ linearly spans the algebra $\mathcal{A}$, $\mathcal{A}$ is ideal-reducible whenever $\mathrm{sg}(\mathcal{F})$ is ideal-reducible. By the assumption, there exists $n \in \mathbb{N}$ such that $\mathcal{A}^n = \{0\}$ and so $\mathcal{F}^n = \{0\}$. Without loss of generality we may assume that n is the smallest positive integer with the property $\mathcal{F}^n = \{0\}$. If $n = 1$, then $\mathcal{F} = \{0\}$, which is obviously an ideal-triangularizable family consisting of a single operator. Thus, we may assume that $n > 1$. This implies $\mathcal{F}^{n-1} \neq \{0\}$ and so there exist positive operators $A_1, \ldots, A_{n-1} \in \mathcal{F}$ such that $A := A_1 \cdots A_{n-1} \neq 0$. If I is the identity operator on E, then $A \cdot \mathrm{sg}(\mathcal{F}) \cdot I = \{0\}$ and $\mathrm{sg}(\mathcal{F})$ is ideal-reducible by Proposition 1.1. □

The following theorem is the main theorem of this paper. Not only does it give two general results on ideal-triangularizability of nil-algebras generated by a set of positive operators, it also shows that every finitely generated nil-algebra generated by a set of positive operators has to be nilpotent. In general, there exist finitely generated nil-algebras that are not nilpotent (see [1]).

Theorem 2.2. *Let $\mathcal{A}$ be an algebra generated by the set $\mathcal{F}$ of positive operators acting on a Banach lattice. Suppose that either of the following conditions holds:*

(1) *there exists $n \in \mathbb{N}$ such that $A^n = 0$ for each A in the convex hull of the semigroup $\mathrm{sg}(\mathcal{F})$;*
(2) *$\mathcal{A}$ is a finitely generated nil-algebra.*

Then $\mathcal{A}$ is nilpotent and ideal-triangularizable.

Proof. To see (1) assume that $A^n = 0$ for each A in the convex hull of the semigroup $\mathrm{sg}(\mathcal{F})$. Every operator in $\mathcal{A}$ is a linear combination of operators from $\mathrm{sg}(\mathcal{F})$. Suppose that $\sum_{i=1}^k \lambda_i A_i$, where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathrm{sg}(\mathcal{F})$, is an arbitrary operator from the algebra $\mathcal{A}$. From the assumption it follows that

$$(A_1 + \cdots + A_k)^n = 0. \tag{2.1}$$

Expanding equality (2.1) and observing that every operator A_i is positive for all $1 \leq i \leq k$, we obtain that every summand in equality (2.1) is zero. Since the summands in the expansion of $\left(\sum_{i=1}^k \lambda_i A_i\right)^n$ differ from the summands in equality (2.1) by multiplicative constants, we have

$$(\lambda_1 A_1 + \cdots + \lambda_k A_k)^n = 0.$$

Therefore, for every $A \in \mathcal{A}$ we have $A^n = 0$. By Theorem 1.4, the algebra $\mathcal{A}$ is nilpotent and so it is ideal-triangularizable by Proposition 2.1.

To see (2) suppose that $\mathcal{A}$ is generated by the family $\mathcal{F} = \{T_1, \ldots, T_n\}$ of positive operators on E. Since every operator in $\mathcal{A}$ is nilpotent, there exists $k \in \mathbb{N}$ such that $(T_1 + \cdots + T_n)^k = 0$. We claim that an arbitrary product of k operators from $\mathcal{F}$ is zero. Let $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$ be any function. Then the element $T_{f(1)} \cdots T_{f(k)}$ appears as a summand in the expansion of $(T_1 + \cdots + T_n)^k$. Since $(T_1 + \cdots + T_n)^k = 0$ and every summand in its expansion is a positive operator, it follows that $T_{f(1)} \cdots T_{f(k)} = 0$. Since every operator in the algebra $\mathcal{A}$ is a linear combination of arbitrary products of operators from $\mathcal{F}$, we obtain that an arbitrary product of k operators from the algebra $\mathcal{A}$ is zero and therefore $\mathcal{A}$ is a nilpotent algebra. Ideal-triangularizability of $\mathcal{A}$ again follows by Proposition 2.1. □

Corollary 2.3. *Let E be a Banach lattice and let $\mathcal{A}$ be a uniformly closed subalgebra of the algebra of all bounded operators on E. If $\mathcal{A}$ is generated by a set of positive operators and if $\mathcal{A}$ is a nil-algebra, then $\mathcal{A}$ is ideal-triangularizable.*

Proof. Since $\mathcal{A}$ is a nil-Banach algebra, by [5] there exists $n \in \mathbb{N}$ such that $A^n = 0$ for each $A \in \mathcal{A}$. Ideal-triangularizability of $\mathcal{A}$ now follows from Theorem 2.2. □

Corollary 2.4. *Let $\mathcal{A}$ be a nil-algebra generated by the set $\mathcal{F}$ of positive operators acting on a Banach lattice. Then any finite subset (of not necessarily positive operators) of $\mathcal{A}$ generates a nilpotent ideal-triangularizable algebra, and the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ is a nil-algebra.*

Proof. Let $\mathcal{S}$ be a finite family of operators in $\mathcal{A}$. Then there exists a finite subfamily $\mathcal{F}_1 \subseteq \mathcal{F}$ such that $\mathcal{S}$ is contained in the algebra $\mathcal{B}$ generated by $\mathcal{F}_1$. By Theorem 2.2, the algebra $\mathcal{B}$ is ideal-triangularizable and nilpotent. Since the algebra generated by the family $\mathcal{S}$ is contained in the algebra $\mathcal{B}$, it has to be nilpotent and ideal-triangularizable. Applying [12, Corollary 1] we obtain that the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ is a nil-algebra. □

Recall that the algebra $\mathcal{A}$ over a field F is *Artinian* if every decreasing sequence of its two-sided ideals eventually becomes constant.

Theorem 2.5. *Let $\mathcal{A}$ be an Artinian algebra generated by the family $\mathcal{F}$ of positive operators. If there exists a nonempty subfamily $\mathcal{F}_1 \subseteq \mathcal{F}$ such that the ideal in $\mathcal{A}$ generated by $\mathcal{F}_1$ consists of nilpotent operators, then $\mathcal{A}$ is ideal-reducible. If $\mathcal{A}$ is a nil-algebra, then $\mathcal{A}$ is ideal-triangularizable.*

Proof. To see the first part of the theorem, assume that the ideal $\mathcal{J}$ in the algebra $\mathcal{A}$ generated by a nonempty subfamily $\mathcal{F}_1$ of $\mathcal{F}$ consists of nilpotent operators. By Hopkins's theorem (see [7]), the ideal $\mathcal{J}$ is nilpotent. This implies that the semigroup ideal generated by the family $\mathcal{F}_1$ in the semigroup $\mathrm{sg}(\mathcal{F})$ is ideal-reducible, and by Proposition 1.1 it follows that $\mathrm{sg}(\mathcal{F})$ is ideal-reducible. Finally, $\mathcal{A}$ is ideal-reducible.

To see the second part of the theorem, observe that the algebra $\mathcal{A}$ is nilpotent and is therefore ideal-triangularizable by Proposition 2.1. □

In [15] Zhong constructed an irreducible multiplicative semigroup of positive nilpotent operators with unbounded nilpotency index acting on $L^2[0,1]$. In [4] the authors improved Zhong's result. They constructed an irreducible multiplicative semigroup of square-zero positive operators on $L^p[0,1)$ $(1 \leq p < \infty)$ in which any finite number of elements generates an ideal-triangularizable semigroup. They even constructed an irreducible set of positive nilpotent operators on $L^p[0,1)$ which is closed under multiplication, addition and multiplication by positive real scalars with the property that any finite subset is ideal-triangularizable. We recall the construction from [4], as this example shows the following:

(1) in Theorem 2.2 it is not enough to assume that the convex hull of the semigroup $\mathrm{sg}(\mathcal{F})$ consists of nilpotent operators;
(2) there exists an algebra of nilpotent operators generated by the family of positive operators acting on $L^p[0,1)$ such that any finitely generated subalgebra is ideal-triangularizable but the whole algebra is irreducible.

Example 2.6. Let $1 \leq p < \infty$ and let the interval $[0,1)$ be equipped with the Lebesgue measure. For $\varphi \in L^\infty[0,1)$, let M_φ denote the multiplication operator on $L^p[0,1)$ defined by $(M_\varphi f)(x) = \varphi(x)f(x)$ for all $f \in L^p[0,1)$. For $\alpha \in (-1,1)$ let U_α denote the translation operator on $L^p[0,1)$ defined by $(U_\alpha f)(x) = f(x \oplus \alpha)$ for all $f \in L^p[0,1)$, where $\oplus$ denotes the translation modulo 1. Let χ_E denote the characteristic function of the measurable subset $E \subseteq [0,1)$. Now we will construct an irreducible nil-algebra generated by the set of positive operators. Define $\mathcal{S}_0 = \{0\}$,

$$\mathcal{S}_1 = \left\{0, U_{\frac{1}{2}} M_{\chi_{[\frac{1}{2},1)}}\right\},$$

and similarly, for $n = 2, 3, \ldots$, define

$$\mathcal{S}_n = \{0\} \cup \left\{U_{\frac{j}{2^n}} M_{\chi_E} : \ E = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \text{ with } j,k \text{ odd integers}\right\}.$$

In [4, Theorem 2.1] it was proved that $\mathcal{S}_n \cdot \mathcal{S}_n = \{0\}$ and $\mathcal{S}_n \cdot \mathcal{S}_m \subseteq \mathcal{S}_{\max\{n,m\}}$. It follows that the set $\mathcal{S} = \bigcup_{n=0}^\infty \mathcal{S}_n$ is a countable semigroup of positive operators on $L^p[0,1)$. Let $\mathcal{A}$ be the algebra generated by the semigroup $\mathcal{S}$. It can be shown as in [4, Theorem 3.1] that $\mathcal{A}$ is an irreducible nil-algebra.

We conclude this paper with an application of the idea of the proof of Proposition 2.1 to obtain a different proof of [6, Theorem 4.1].

Theorem 2.7. *Let X be a Banach space and $\mathcal{A}$ an algebra of bounded nilpotent operators on X. If there exists $n \in \mathbb{N}$ such that $A^n = 0$ for every operator $A \in \mathcal{A}$, then $\mathcal{A}$ is triangularizable.*

Proof. By the Triangularization Lemma [13, 7.1.11] it suffices to show that $\mathcal{A}$ is reducible. Let n_0 be the smallest natural number such that $\mathcal{A}^{n_0} = \{0\}$. If $n_0 = 1$, then $\mathcal{A} = \{0\}$ and is obviously triangularizable. Otherwise we can proceed as in the proof of Theorem 2.2 to obtain a nonzero operator T on X such that $TA = 0$ for all $A \in \mathcal{A}$. Since $\mathcal{A}$ is nonzero, there exists a nonzero vector $x \in X$ such that $Y = \mathcal{A}x \neq \{0\}$. Since $TA = 0$ and T is nonzero, Y is not dense in X and so the closure $\overline{Y}$ of Y is a nontrivial closed subspace which is invariant under the algebra $\mathcal{A}$. □

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