IDEAL-TRIANGULARIZABILITY OF NIL-ALGEBRAS GENERATED BY POSITIVE OPERATORS

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Abstract. R. Drnovšek, D. Kokol-Bukovšek, L. Livshits, G. MacDonald, M. Omladič, and H. Radjavi constructed an irreducible set of positive nilpotent operators on $L^p[0,1]$ which is closed under multiplication, addition and multiplication by positive real scalars with the property that any finite subset is ideal-triangularizable. In this paper we prove the following:

1. Introduction and preliminaries

Let $E$ be a Banach lattice. By an operator on $E$ we mean a continuous linear transformation from $E$ into itself. An operator $T$ is called positive whenever $Tx \in E^+$ for all $x \in E^+$. For the terminology not explained in the text we refer the reader to the books [10] and [14].

A family $\mathcal{F}$ of operators on $E$ is said to be reducible if there exists a nontrivial closed subspace of $E$ that is invariant under every member of $\mathcal{F}$. Otherwise, we say that $\mathcal{F}$ is irreducible. If there exists a maximal subspace chain (i.e., a maximal totally ordered set of closed subspaces) whose elements are invariant under every member of $\mathcal{F}$, then $\mathcal{F}$ is said to be triangularizable. A family $\mathcal{F}$ of operators on $E$ is said to be ideal-reducible if there exists a nontrivial closed ideal of $E$ which is invariant under every operator in $\mathcal{F}$. Otherwise, we say that $\mathcal{F}$ is ideal-irreducible.

A subset $I$ of a semigroup $\mathcal{S}$ is said to be a semigroup ideal if $ST$ and $TS$ belong to $I$ for all $S \in \mathcal{S}$ and $T \in I$. For a subset $\mathcal{F}$ of a semigroup $\mathcal{S}$ we define $\mathcal{F}^n$ as the set of all products of at least $n$ elements of $\mathcal{F}$, and by $\text{sg}(\mathcal{F})$ we denote the semigroup generated by $\mathcal{F}$. It is obvious that for each $n \in \mathbb{N}$ the set $\mathcal{F}^n$ is a semigroup ideal in $\text{sg}(\mathcal{F})$.

The following proposition, which was proved in [3], is very useful in determining if a given semigroup of positive operators is ideal-reducible.

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A subset $I$ of a semigroup $\mathcal{S}$ is said to be a semigroup ideal if $ST$ and $TS$ belong to $I$ for all $S \in \mathcal{S}$ and $T \in I$. For a subset $\mathcal{F}$ of a semigroup $\mathcal{S}$ we define $\mathcal{F}^n$ as the set of all products of at least $n$ elements of $\mathcal{F}$, and by $\text{sg}(\mathcal{F})$ we denote the semigroup generated by $\mathcal{F}$. It is obvious that for each $n \in \mathbb{N}$ the set $\mathcal{F}^n$ is a semigroup ideal in $\text{sg}(\mathcal{F})$.

The following proposition, which was proved in [3], is very useful in determining if a given semigroup of positive operators is ideal-reducible.
Proposition 1.1. Let $E$ be a normed Riesz space, and let $S$ be a nonzero semigroup of positive operators on $E$. The following statements are equivalent:

1. $S$ is ideal-reducible;
2. there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(Sf) = \{0\}$;
3. there exist nonzero positive operators $A$ and $B$ on $E$ such that $ASB = \{0\}$;
4. some nonzero semigroup ideal of $S$ is ideal-reducible.

A family $F$ is ideal-triangularizable if there is a chain $C$ that is maximal as a chain of closed ideals of $E$ and that has the property that every ideal in $C$ is invariant under all the operators in $F$. Any such chain of closed ideals is said to be an ideal-triangularizing chain for the family $F$. By [2 Proposition 1.2], every maximal chain of closed ideals of a Banach lattice is also maximal as a chain of closed subspaces of a Banach space. Thus, an ideal-triangularizable family of operators is also triangularizable. The notion of ideal-triangularizability was introduced by Jahandideh in [9].

Let $F$ be a family of operators on $E$, and let $I$ and $J$ be closed ideals of $E$ satisfying $J \subseteq I$ that are invariant under every member of $F$. Then $F$ induces a family $\hat{F}$ of operators on the quotient Banach lattice $I/J$ as follows. For each $T \in F$ the operator $\hat{T}$ is defined on $I/J$ by

$$\hat{T}(x + J) = Tx + J.$$ 

Because $I$ and $J$ are invariant under $T$, the operator $\hat{T}$ is a well-defined operator on $I/J$. Any such $\hat{F}$ is called a collection of ideal-quotients of the family $F$. A set $\mathcal{P}$ of properties is said to be inherited by ideal-quotients if every family of ideal-quotients of a family of operators satisfying properties in $\mathcal{P}$ also satisfies the same properties.

Lemma 1.2 (The Ideal-triangularization Lemma). Let $\mathcal{P}$ be a set of properties inherited by ideal-quotients. If every family of operators on a Banach lattice of dimension greater than one which satisfies $\mathcal{P}$ is ideal-reducible, then every such family is ideal-triangularizable.

Definition 1.3. An algebra $A$ over a field $F$ is said to be nilpotent (resp. a nil-algebra) if there exists $n \in \mathbb{N}$ such that the product of an arbitrary $n$ elements in $A$ is equal to zero (resp. if every element in $A$ is nilpotent).

It is obvious that every nilpotent algebra is also a nil-algebra, but the converse statement is false in general. The following theorem of Nagata (11) and Higman (8) shows the converse statement when the nilpotency index of elements is bounded.

Theorem 1.4 (Nagata-Higman). Let $A$ be an algebra over a field $F$ of characteristic zero. Suppose that there is a natural number $n$ such that $a^n = 0$ for all $a \in A$. Then $a_1 \cdots a_{2^n-1} = 0$ for all $a_1, \ldots, a_{2^n-1} \in A$.

2. Results

The following proposition will be used in the proof of Theorem 2.2.

Proposition 2.1. Every nilpotent algebra generated by the set of positive operators acting on a Banach lattice is ideal-triangularizable.
Proof. Suppose that \( \mathcal{A} \) is a nilpotent algebra generated by the family \( \mathcal{F} \) of positive operators on a Banach lattice \( E \). By the Ideal-triangularization Lemma it suffices to show that \( \mathcal{A} \) is ideal-reducible. Since the semigroup \( \text{sg}(\mathcal{F}) \) linearly spans the algebra \( \mathcal{A} \), \( \mathcal{A} \) is ideal-reducible whenever \( \text{sg}(\mathcal{F}) \) is ideal-reducible. By the assumption, there exists \( n \in \mathbb{N} \) such that \( \mathcal{A}^n = \{0\} \) and so \( \mathcal{F}^n = \{0\} \). Without loss of generality we may assume that \( n \) is the smallest positive integer with the property \( \mathcal{F}^n = \{0\} \). If \( n = 1 \), then \( \mathcal{F} = \{0\} \), which is obviously an ideal-triangularizable family consisting of a single operator. Thus, we may assume that \( n > 1 \). This implies \( \mathcal{F}^{n-1} \neq \{0\} \) and so there exist positive operators \( A_1, \ldots, A_{n-1} \in \mathcal{F} \) such that \( A := A_1 \cdots A_{n-1} \neq 0 \). If \( I \) is the identity operator on \( E \), then \( A \cdot \text{sg}(\mathcal{F}) \cdot I = \{0\} \) and \( \text{sg}(\mathcal{F}) \) is ideal-reducible by Proposition 1.1.

The following theorem is the main theorem of this paper. Not only does it give two general results on ideal-triangularizability of nil-algebras generated by a set of positive operators, it also shows that every finitely generated nil-algebra generated by a set of positive operators has to be nilpotent. In general, there exist finitely generated nil-algebras that are not nilpotent (see [1]).

**Theorem 2.2.** Let \( \mathcal{A} \) be an algebra generated by the set \( \mathcal{F} \) of positive operators acting on a Banach lattice. Suppose that either of the following conditions holds:

1. There exists \( n \in \mathbb{N} \) such that \( \mathcal{A}^n = \{0\} \) for each \( A \) in the convex hull of the semigroup \( \text{sg}(\mathcal{F}) \).
2. \( \mathcal{A} \) is a finitely generated nil-algebra.

Then \( \mathcal{A} \) is nilpotent and ideal-triangularizable.

Proof. To see (1) assume that \( \mathcal{A}^n = \{0\} \) for each \( A \) in the convex hull of the semigroup \( \text{sg}(\mathcal{F}) \). Every operator in \( \mathcal{A} \) is a linear combination of operators from \( \text{sg}(\mathcal{F}) \). Suppose that \( \sum_{i=1}^k \lambda_i A_i \), where \( \lambda_i \in \mathbb{R} \) and \( A_i \in \text{sg}(\mathcal{F}) \), is an arbitrary operator from the algebra \( \mathcal{A} \). From the assumption it follows that

\[
(A_1 + \cdots + A_k)^n = 0.
\]

Expanding equality (2.1) and observing that every operator \( A_i \) is positive for all \( 1 \leq i \leq k \), we obtain that every summand in equality (2.1) is zero. Since the summands in the expansion of \((\sum_{i=1}^k \lambda_i A_i)^n\) differ from the summands in equality (2.1) by multiplicative constants, we have

\[
(\lambda_1 A_1 + \cdots + \lambda_k A_k)^n = 0.
\]

Therefore, for every \( A \in \mathcal{A} \) we have \( A^n = 0 \). By Theorem 1.4, the algebra \( \mathcal{A} \) is nilpotent and so it is ideal-triangularizable by Proposition 2.1.

To see (2) suppose that \( \mathcal{A} \) is generated by the family \( \mathcal{F} = \{T_1, \ldots, T_n\} \) of positive operators on \( E \). Since every operator in \( \mathcal{A} \) is nilpotent, there exists \( k \in \mathbb{N} \) such that \((T_1 + \cdots + T_n)^k = 0\). We claim that an arbitrary product of \( k \) operators from \( \mathcal{F} \) is zero. Let \( f : \{1, \ldots, k\} \rightarrow \{1, \ldots, n\} \) be any function. Then the element \( T_{f(1)} \cdots T_{f(k)} \) appears as a summand in the expansion of \((T_1 + \cdots + T_n)^k\). Since \((T_1 + \cdots + T_n)^k = 0\) and every summand in its expansion is a positive operator, it follows that \( T_{f(1)} \cdots T_{f(k)} = 0 \). Since every operator in the algebra \( \mathcal{A} \) is a linear combination of arbitrary products of operators from \( \mathcal{F} \), we obtain that an arbitrary product of \( k \) operators from the algebra \( \mathcal{A} \) is zero and therefore \( \mathcal{A} \) is a nilpotent algebra. Ideal-triangularizability of \( \mathcal{A} \) again follows by Proposition 2.1. \( \square \)
Corollary 2.3. Let $E$ be a Banach lattice and let $\mathcal{A}$ be a uniformly closed subalgebra of the algebra of all bounded operators on $E$. If $\mathcal{A}$ is generated by a set of positive operators and if $\mathcal{A}$ is a nil-algebra, then $\mathcal{A}$ is ideal-triangularizable.

Proof. Since $\mathcal{A}$ is a nil-Banach algebra, by [5] there exists $n \in \mathbb{N}$ such that $A^n = 0$ for each $A \in \mathcal{A}$. Ideal-triangularizability of $\mathcal{A}$ now follows from Theorem 2.2. □

Corollary 2.4. Let $\mathcal{A}$ be a nil-algebra generated by the set $\mathcal{F}$ of positive operators acting on a Banach lattice. Then any finite subset (of not necessarily positive operators) of $\mathcal{A}$ generates a nilpotent ideal-triangularizable algebra, and the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ is a nil-algebra.

Proof. Let $\mathcal{S}$ be a finite family of operators in $\mathcal{A}$. Then there exists a finite subfamily $\mathcal{F}_1 \subseteq \mathcal{F}$ such that $\mathcal{S}$ is contained in the algebra $\mathcal{B}$ generated by $\mathcal{F}_1$. By Theorem 2.3 the algebra $\mathcal{B}$ is ideal-triangularizable and nilpotent. Since the algebra generated by the family $\mathcal{S}$ is contained in the algebra $\mathcal{B}$, it has to be nilpotent and ideal-triangularizable. Applying [12, Corollary 1] we obtain that the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ is a nil-algebra. □

Recall that the algebra $\mathcal{A}$ over a field $F$ is Artinian if every decreasing sequence of its two-sided ideals eventually becomes constant.

Theorem 2.5. Let $\mathcal{A}$ be an Artinian algebra generated by the family $\mathcal{F}$ of positive operators. If there exists a nonempty subfamily $\mathcal{F}_1 \subseteq \mathcal{F}$ such that the ideal in $\mathcal{A}$ generated by $\mathcal{F}_1$ consists of nilpotent operators, then $\mathcal{A}$ is ideal-reducible. If $\mathcal{A}$ is a nil-algebra, then $\mathcal{A}$ is ideal-triangularizable.

Proof. To see the first part of the theorem, assume that the ideal $\mathcal{J}$ in the algebra $\mathcal{A}$ generated by a nonempty subfamily $\mathcal{F}_1 \subseteq \mathcal{F}$ of $\mathcal{F}$ consists of nilpotent operators. By Hopkins’s theorem (see [7]), the ideal $\mathcal{J}$ is nilpotent. This implies that the semigroup ideal generated by the family $\mathcal{F}_1$ in the semigroup $\text{sg}(\mathcal{F})$ is ideal-reducible, and by Proposition 1.1 it follows that $\text{sg}(\mathcal{F})$ is ideal-reducible. Finally, $\mathcal{A}$ is ideal-reducible.

To see the second part of the theorem, observe that the algebra $\mathcal{A}$ is nilpotent and is therefore ideal-triangularizable by Proposition 2.1. □

In [15] Zhong constructed an irreducible multiplicative semigroup of positive nilpotent operators with unbounded nilpotency index acting on $L^2[0, 1]$. In [4] the authors improved Zhong’s result. They constructed an irreducible multiplicative semigroup of square-zero positive operators on $L^p[0, 1]$ ($1 \leq p < \infty$) in which any finite number of elements generates an ideal-triangularizable semigroup. They even constructed an irreducible set of positive nilpotent operators on $L^p[0, 1]$ which is closed under multiplication, addition and multiplication by positive real scalars with the property that any finite subset is ideal-triangularizable. We recall the construction from [4], as this example shows the following:

(1) in Theorem 2.2 it is not enough to assume that the convex hull of the semigroup $\text{sg}(\mathcal{F})$ consists of nilpotent operators;
(2) there exists an algebra of nilpotent operators generated by the family of positive operators acting on $L^p[0, 1]$ such that any finitely generated subalgebra is ideal-triangularizable but the whole algebra is irreducible.
Example 2.6. Let $1 \leq p < \infty$ and let the interval $[0, 1)$ be equipped with the Lebesgue measure. For $\varphi \in L^\infty[0, 1)$, let $M_\varphi$ denote the multiplication operator on $L^p[0, 1]$ defined by $(M_\varphi f)(x) = \varphi(x)f(x)$ for all $f \in L^p[0, 1)$. For $\alpha \in (-1, 1)$ let $U_\alpha$ denote the translation operator on $L^p[0, 1)$ defined by $(U_\alpha f)(x) = f(x + \alpha)$ for all $f \in L^p[0, 1)$, where $\oplus$ denotes the translation modulo 1. Let $\chi_{[0, 1)}$ denote the characteristic function of the measurable subset $E \subseteq [0, 1)$. Now we will construct an irreducible nil-algebra generated by the set of positive operators. Define $S_0 = \{0\}$,

$$S_1 = \left\{ 0, U_1, M_{\chi_{[1/2, 1)}} \right\},$$

and similarly, for $n = 2, 3, \ldots$, define

$$S_n = \{0\} \cup \left\{ U_{\frac{k}{2^n}}, M_{\chi_{[1/2^n, 1)}} : E = \left( \frac{k}{2^n}, \frac{k+1}{2^n} \right) \right\} \text{ with } j, k \text{ odd integers}.$$

In [4] Theorem 2.1 it was proved that $S_n \cdot S_n = \{0\}$ and $S_n \cdot S_m \subseteq S_{\max(n,m)}$. It follows that the set $S = \bigcup_{n=0}^{\infty} S_n$ is a countable semigroup of positive operators on $L^p[0, 1)$. Let $\mathcal{A}$ be the algebra generated by the semigroup $S$. It can be shown as in [4] Theorem 3.1 that $\mathcal{A}$ is an irreducible nil-algebra.

We conclude this paper with an application of the idea of the proof of Proposition 2.1 to obtain a different proof of [6] Theorem 4.1.

Theorem 2.7. Let $X$ be a Banach space and $\mathcal{A}$ an algebra of bounded nilpotent operators on $X$. If there exists $n \in \mathbb{N}$ such that $A^n = 0$ for every operator $A \in \mathcal{A}$, then $\mathcal{A}$ is triangularizable.

Proof. By the Triangularization Lemma [13, 7.1.11] it suffices to show that $\mathcal{A}$ is reducible. Let $n_0$ be the smallest natural number such that $\mathcal{A}^{n_0} = \{0\}$. If $n_0 = 1$, then $\mathcal{A} = \{0\}$ and is obviously triangularizable. Otherwise we can proceed as in the proof of Theorem 2.7 to obtain a nonzero operator $T$ on $X$ such that $TA = 0$ for all $A \in \mathcal{A}$. Since $\mathcal{A}$ is nonzero, there exists a nonzero vector $x \in X$ such that $Y = Ax \neq \{0\}$. Since $TA = 0$ and $T$ is nonzero, $Y$ is not dense in $X$ and so the closure $\overline{Y}$ of $Y$ is a nontrivial closed subspace which is invariant under the algebra $\mathcal{A}$.

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References


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