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AN INDEX FORMULA FOR THE TWO VARIABLE JORDAN BLOCK

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ABSTRACT. On Hardy space $H^2(\mathbb{D}^2)$ over the bidisk, let (S_z, S_w) be the compression of the pair (T_z, T_w) to the quotient module $H^2(\mathbb{D}^2) \ominus M$. In this paper, we obtain an index formula for (S_z, S_w) when it is Fredholm. It is also shown that the evaluation operator $L(0)$ is compact on a Beurling type quotient module if and only if the corresponding inner function is a finite Blaschke product in w.

1. INTRODUCTION

In this paper, let $\mathbb{D}^2$ denote the open unit bidisk in $\mathbb{C}^2$ and let $\mathbb{T}^2$ denote the distinguished boundary of $\mathbb{D}^2$. The Hardy space $H^2(\mathbb{D}^2)$ is the closure of polynomials in $L^2(\mathbb{T}^2, \frac{1}{(2\pi)^2} d\theta_1 d\theta_2)$. On $H^2(\mathbb{D}^2)$, the Toeplitz operators T_z and T_w are unilateral shifts of infinity multiplicity. A closed subspace $M \subset H^2(\mathbb{D}^2)$ is called a submodule if it is invariant under the action of T_z and T_w. The quotient space $H^2(\mathbb{D}^2)/M$, naturally identified with $H^2(\mathbb{D}^2) \ominus M$, is endowed with a $\mathbb{C}[z, w]-$ module structure by

$$p \cdot f = p(S_z, S_w)f, \quad p \in \mathbb{C}[z, w], \quad f \in N,$$

where $S_z = P_N T_z|_N$, $S_w = P_N T_w|_N$, $\mathbb{C}[z, w]$ is the polynomial ring in two variables, and P_N stands for the orthogonal projection from $H^2(\mathbb{D}^2)$ onto N. Then (S_z, S_w) is a pair of commuting contractions, which is called the two variable Jordan block. We also denote the compression of T_z and T_w on M by R_z and R_w, respectively, that is,

$$R_z = T_z|_M, \quad R_w = T_w|_M.$$

It is easy to see that $R = (R_z, R_w)$ is a pair of commuting isometries.

In the one variable Hardy space $H^2(\mathbb{D})$, the classical Beurling theorem implies that every submodule is of the form $M = \theta H^2(\mathbb{D})$ for some inner function θ. In the one variable setting, the compression $S(\theta) = P_N T_z|_N$ is called the Jordan block (cf. [Be] [SF]). Here P_N is the orthogonal projection from $H^2(\mathbb{D})$ onto the associated quotient module N. It is well known that the index of $S(\theta)$ is always 0. However, the structure of submodules of $H^2(\mathbb{D}^2)$ is far more complex. For instance, W. Rudin displayed two submodules in [Ru], one which was of infinite rank, and the other

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which contained no nontrivial bounded functions. There has been a large amount of work on the operator theory and structure of Hardy submodules on the polydisk; see [CG], [CMY], [DP], [GM], [Ya1]-[Ya8] and references therein.

If we denote $H^2(\mathbb{D}) \ominus zH^2(\mathbb{D})$ by $H^2(w)$ and $H^2(\mathbb{D}) \ominus wH^2(\mathbb{D})$ by $H^2(z)$, then

$$H^2(w) = clos\{span\{w^j : j \geq 0\}\}, \quad H^2(z) = clos\{span\{z^j : j \geq 0\}\}.$$

We define a *left evaluation* operator $L(0)$ from $H^2(\mathbb{D}^2)$ to $H^2(w)$ and a *right evaluation* operator $R(0)$ from $H^2(\mathbb{D}^2)$ to $H^2(z)$ by

$$L(0)f(w) = f(0, w), \quad R(0) = f(z, 0), \quad f \in H^2(\mathbb{D}^2).$$

In [Ya3], the third author showed that if ϕ is inner, then $L(0)$ is Hilbert-Schmidt on $H^2(\mathbb{D}^2) \ominus \phi H^2(\mathbb{D}^2)$ if and only if ϕ is a finite Blaschke product in w. In this paper, we will show that the condition is also a necessary and sufficient condition for the compactness of $L(0)$ on $H^2(\mathbb{D}^2) \ominus \phi H^2(\mathbb{D}^2)$; here we will take a different approach.

The core operator C^M for a submodule $M \subset H^2(\mathbb{D}^2)$ is defined by

$$C^M = I - R_z R_z^* - R_w R_w^* + R_z R_w R_z^* R_w^*.$$

For convenience, we will suppress the "M" in our writing. The core operator (also called the defect operator) has been been well studied on the bidisk or unit ball; see [Ar], [GY], [Guo] and [Ya7]-[Ya8]. Let $E_{-1}(C)$ and $E_1(C)$ be the corresponding eigenspace for C. The index of (S_z, S_w) can be determined with a mild condition. For example, it is indicated in [GRS] that if M contains a bounded function that does not vanish at $(0, 0)$, then $ind(S_z, S_w) = 0$. In [Ya6] and [Ya8], the author showed that if C is a Hilbert-Schmidt operator, then (R_z, R_w) and (S_z, S_w) are both Fredholm, and $ind(S_z, S_w) = -1 - ind(R_z, R_w) = dim E_1(C) - E_{-1}(C) - 1 = 0$. In this paper, we prove that if (S_z, S_w) is Fredholm, then

$$dim((M \ominus zM) \cap (M \ominus wM)) < \infty,$$

and the index of (S_z, S_w) also equals $dim E_1(C) - E_{-1}(C) - 1$. $[R_z^*, R_z][R_w^*, R_w]$ and $[R_w^*, R_z]$ are also important subjects in the study of Hardy submodule over the bidisk, and we will show that if (S_z, S_w) is Fredholm, then $[R_z^*, R_z][R_w^*, R_w]$ is compact if and only if $[R_w^*, R_z]$ is compact.

2. Evaluation operators

The evaluation operators, which turn functions of two variables into functions of one variable, serve as a bridge connecting the operator in $H^2(\mathbb{D}^2)$ to single operator theory. The evaluation operators also play an important role in the proof of Section 3.

We define a *difference quotient operator* D_z from $H^2(\mathbb{D}^2)$ to itself by $D_z f(z, w) = \frac{f(z,w) - L(0)f(w)}{z}$, $f \in H^2(\mathbb{D}^2)$. For any submodule M, D_z maps $M \ominus zM$ into N (cf. [Ya3]). Without confusion, the restriction of D_z on $M \ominus zM$ is also denoted by D_z. There are systemic studies on the evaluation operators and difference quotient operators; see [Ya3] and [Ya4].

In [Ya2], it is showed that $L(0)$ is always Hilbert-Schmidt on $M \ominus wM$; however, $L(0)$ may not be Hilbert-Schmidt on M. The following property gives a sufficient condition and may be of interest by itself.

Proposition 2.1. *Let M be a nonzero submodule in $H^2(\mathbb{D}^2)$. If $I - S_z^* S_z$ is in trace class, then the restriction of $L(0)$ on $M \ominus zM$ is not Hilbert-Schmidt.*

Proof. It is indicated in [Ya3] that $I-S_z^*S_z=D_zD_z^*$. Hence, if $I-S_z^*S_z$ is in trace class, then $D_zD_z^*$ is in trace class.

Let $\{e_i : i=1,2,\cdots,l\}$ be the orthogonal basis for N, where $l=dimN$ may be infinity, and let $\{f_i : i=1,2,\cdots,\infty\}$ be the orthogonal basis for $M\ominus zM$. Then

$$D_z^*e_i=\sum_{j=1}^{\infty}\langle T_ze_i,f_j\rangle f_j.$$

Hence

$$\begin{aligned}
Trace(D_zD_z^*) &= \sum_{i=1}^{l}\langle D_zD_z^*e_i,e_i\rangle=\sum_{i=1}^{l}\|D_z^*e_i\|^2\\
&= \sum_{i=1}^{l}\sum_{j=1}^{\infty}|\langle T_ze_i,f_j\rangle|^2\\
&= \sum_{j=1}^{\infty}\|P_NT_z^*f_j\|^2=\sum_{j=1}^{\infty}\|\frac{f_j-L(0)f_j}{z}\|^2\\
&= \sum_{j=1}^{\infty}(\|f_j\|^2-\|L(0)f_j\|^2)<\infty.
\end{aligned}$$

Since $\sum_{j=1}^{\infty}\|f_j\|^2=dim(M\ominus zM)=\infty$ for any submodule M, it follows that $L(0)|_{M\ominus zM}$ is not Hilbert-Schmidt. This completes the proof. $\square$

Example 1. If $\psi(w)$ is an analytic function with $\|\psi\|_\infty<1$ and $M=[z-\psi(w)]$, then on $H^2(\mathbb{D}^2)\ominus\psi H^2(\mathbb{D}^2)$, $S_z=\psi(S_w)$, and therefore $\|S_z\|\le\|\psi\|_\infty<1$. By Theorem 3.2 in [Ya5], $I-S_z^*S_z$ is in trace class, and hence $L(0)$ is not Hilbert-Schmidt on M.

The compactness of $L(0)|_N$ and $R(0)|_N$ has a closed relationship with the essential normality of N; for example, see [GW2]. We will characterize the compactness of $L(0)$ on Beurling type quotient module completely. Let $M=\phi H^2(\mathbb{D}^2)$ be a Beurling type submodule, where ϕ is an inner function in $H^2(\mathbb{D}^2)$. Then $P_M=M_\phi M_\phi^*$ and $P_N=I-M_\phi M_\phi^*$ are the projections on M and N, respectively. Let P be the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$. The Toeplitz operator with symbol $f\in L^2(\mathbb{T}^2)$ is defined by $T_f(h)=P(fh)$, and the Hankel operator H_f is defined by $H_f(h)=(I-P)(fh)$, where $h\in H^2(\mathbb{D}^2)$. It is well known that

$$T_{fg}-T_fT_g=H_{\overline{f}}^*H_g, \tag{2.1}$$

for any $f,g\in L^2(\mathbb{T}^2)$.

Proposition 2.2. *Let $M=\phi H^2(\mathbb{D}^2)$ be a Beurling type submodule and $N=H^2(\mathbb{D}^2)\ominus\phi H^2(\mathbb{D}^2)$. Then $L(0)$ is compact on N if and only if ϕ is a finite Blaschke product in w.*

Proof. First we recall that for any $\lambda=(\lambda_1,\lambda_2)\in\mathbb{D}^2$, the normalized reproducing kernel of $H^2(\mathbb{D}^2)$ is

$$k_\lambda(z,w)=\frac{K_\lambda(z,w)}{\|K_\lambda\|}=\frac{\sqrt{(1-|\lambda_1|^2)(1-|\lambda_2|^2)}}{(1-\overline{\lambda_1}z)(1-\overline{\lambda_2}w)},$$

and $k_{\lambda_2}(w) = \frac{K_{\lambda_2}(w)}{\|K_{\lambda_2}\|} = \frac{\sqrt{(1-|\lambda_2|^2)}}{1-\overline{\lambda_2}w}$ is the normalized reproducing kernel of $H^2(w)$.

Now assume that $L(0)P_N$ is compact. By Lemma 1.2.1 in [Ya3], $L(\lambda_1)P_N$ is compact for any $\lambda_1 \in \mathbb{D}$; hence

$$\lim_{\lambda_2 \to \mathbb{T}} \|L(\lambda_1)P_N k_\lambda\| = 0. \tag{2.2}$$

Since $P_N k_\lambda = k_\lambda - M_\phi M_\phi^* k_\lambda = (1 - \overline{\phi(\lambda)}\phi)k_\lambda$, then $L(\lambda_1)P_N k_\lambda = \frac{1}{\sqrt{1-|\lambda_1|^2}}(1 - \overline{\phi(\lambda)}\phi(\lambda_1, w))k_{\lambda_2}$. Therefore,

$$\begin{aligned}
\|L(\lambda_1)P_N k_\lambda\|^2 &= \frac{1}{1-|\lambda_1|^2}\langle(1 - \overline{\phi(\lambda)}\phi(\lambda_1, w))k_{\lambda_2}, (1 - \overline{\phi(\lambda)}\phi(\lambda_1, w))k_{\lambda_2}\rangle \\
&= \frac{1}{1-|\lambda_1|^2}(1 - 2|\phi(\lambda)|^2 + |\phi(\lambda)|^2\langle M^*_{\phi(\lambda_1,w)}M_{\phi(\lambda_1,w)}k_{\lambda_2}, k_{\lambda_2}\rangle).
\end{aligned}$$

From formula (2.1), we have

$$M^*_{\phi(\lambda_1,w)}M_{\phi(\lambda_1,w)} = M_{\phi(\lambda_1,w)}M^*_{\phi(\lambda_1,w)} + H^*_{\overline{\phi(\lambda_1,w)}}H_{\overline{\phi(\lambda_1,w)}}.$$

Then it follows that

$$\langle M^*_{\phi(\lambda_1,w)}M_{\phi(\lambda_1,w)}k_{\lambda_2}, k_{\lambda_2}\rangle = |\phi(\lambda)|^2 + \|H_{\overline{\phi(\lambda_1,w)}}k_{\lambda_2}\|^2.$$

Hence

$$\|L(\lambda_1)P_N k_\lambda\|^2 = \frac{1}{1-|\lambda_1|^2}((1 - |\phi(\lambda)|^2)^2 + \|H_{\overline{\phi(\lambda_1,w)}}k_{\lambda_2}\|^2),$$

which together with (2.2) gives us that

$$\lim_{\lambda_2 \to \mathbb{T}} |\phi(\lambda)| = 1, \quad \lambda_1 \in \mathbb{D}.$$

It follows that there is a subset E of $\mathbb{T}$ with measure 1 such that for any $\lambda_1 \in \mathbb{D}$,

$$|\phi(\lambda_1, w)| = 1, \quad w \in E,$$

and hence $\phi(\cdot, w)$ is a constant function with module 1 for every $w \in E$.

Let

$$\phi(z, w) = \sum_{i=0}^{\infty} \phi_i(w)z^i, \quad \phi_i \in H^2(w).$$

Then for every $w \in E$ and $k = 1, 2, \ldots$, we have

$$\frac{\partial^k \phi(z, w)}{\partial z^k}\Big|_{z=0} = \phi_k(w) = 0,$$

and hence $\phi_k = 0$, i.e., $\phi = \phi(w)$. It is easy to check that $H^2(w) \ominus \phi H^2(w) \subset H^2(\mathbb{D}^2) \ominus \phi H^2(\mathbb{D}^2)$ and $L(0)$ is an isometry acting on it. The compactness of $L(0)$ implies that $H^2(w) \ominus \phi H^2(w)$ is finite dimensional, which is possible only if ϕ is a finite Blaschke product in w.

On the other hand, if ϕ is a finite Blaschke product in w, by Corollary 4.2.2 in [Ya3] $L(0)$ is Hilbert-Schmidt on N. This completes the proof. $\square$

3. An index formula

In this section, we will prove that if (S_z, S_w) is Fredholm, then

$$dim((M \ominus zM) \cap (M \ominus wM)) < \infty,$$

and we will give an index formula for (S_z, S_w), i.e., $ind(S_z, S_w) = dimE_1(C) - E_{-1}(C) - 1$. First, let us recall some notation and useful tools.

Koszul complex. For every pair of commutating operators $A = (A_1, A_2)$ on a Hilbert space H, there is a short sequence

$$0 \to H \xrightarrow{d_1} H \oplus H \xrightarrow{d_2} H \to 0,$$

where

$$d_1 x = (-A_2 x, A_1 x), \quad d_2(x, y) = A_1 x + A_2 y, \quad x, y \in H.$$

It is easy to check that

$$d_2 d_1 = 0.$$

The tuple $A = (A_1, A_2)$ is said to be *Fredholm* if d_1 and d_2 both have closed range and

$$dim(Ker(d_1)) + dim(Ker(d_2) \ominus d_1(H)) + dim(H \ominus d_2(H \oplus H)) < \infty.$$

It is well known (cf. [Cu]) that the tuple $A = (A_1, A_2)$ is Fredholm if and only if

$$\widehat{A} = \begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix}$$

is Fredholm on $H \oplus H$, and in this case $ind(A_1, A_2) = ind\widehat{A}$.

We refer the reader to [Cu] for more detailed discussions of Fredholm complexes for n-tuples.

Two defect operators for $S = (S_z, S_w)$ are defined by

$$\Delta_S = I - S_z^* S_z - S_w^* S_w + S_z^* S_w^* S_z S_w$$

and

$$\Delta_{S^*} = I - S_z S_z^* - S_w S_w^* + S_z S_w S_z^* S_w^*.$$

C and Δ_S are useful tools in the study of Hardy submodules, and we refer the reader to [GY], [Ya7] and [Ya8] for more details.

Let η denote the function $P_N 1$. If M is nontrivial, then $1 \notin M$, i.e., $\eta \neq 0$. It is indicated in [Ya7] that Δ_{S^*} is a rank one operator, and in fact $\Delta_{S^*} = \eta \otimes \eta$. We write $Z(M) = \{(\lambda, \mu) \in \mathbb{D}^2 | f(\lambda, \mu) = 0, \text{ for all } f \in M\}$.

Lemma 3.1. *Let M be a nontrivial submodule in $H^2(\mathbb{D}^2)$. If $S = (S_z, S_w)$ is Fredholm, then $dim((M \ominus zM) \cap (M \ominus wM)) \leq dimker\widehat{S} + 1 < \infty$. More precisely, if $(0,0) \in Z(M)$, then $dim((M \ominus zM) \cap (M \ominus wM)) = dimker\widehat{S}$, and if $(0,0) \notin Z(M)$, then $dim((M \ominus zM) \cap (M \ominus wM)) = dimker\widehat{S} + 1$.*

Proof. If $S = (S_z, S_w)$ is Fredholm, then

$$\widehat{S} = \begin{pmatrix} S_z & S_w \\ -S_w^* & S_z^* \end{pmatrix}$$

is Fredholm on $N \oplus N$. It is easy to check that

$$ker\widehat{S} = \{(f, g) \in N \oplus N | zf + wg \in M, -\overline{w}f + \overline{z}g \perp H^2(\mathbb{D}^2)\}.$$

It follows from [Ya8] that there is a bounded below map A from $ker\widehat{S}$ into $(M \ominus zM) \cap (M \ominus wM)$ defined by

$$A(f,g) = P_M(I + L(0) + R(0))(zf + wg),$$

where $(f,g) \in ker\widehat{S}$. Therefore, for any $h \in ((M \ominus zM) \cap (M \ominus wM)) \ominus RanA$, $(f,g) \in ker\widehat{S}$, we have

$$\begin{aligned}
\langle h, A(f,g)\rangle &= \langle h, (I + L(0) + R(0))(zf + wg)\rangle \\
&= \langle h, zf + zR(0)f\rangle + \langle h, wg + wL(0)g\rangle \\
&= \langle T_z^* h + R(0)T_z^* h, f\rangle + \langle T_w^* h + L(0)T_w^* h, g\rangle \\
&= 0.
\end{aligned}$$

Note that $T_z^* h \in N$ and $T_w^* h \in N$. It follows that

$$(T_z^* h + P_N R(0)T_z^* h, T_w^* h + P_N L(0)T_w^* h) \in (ker\widehat{S})^\perp.$$

In view of the Fredholmness of $\widehat{S}$ on $N \oplus N$, we conclude that $Ran\widehat{S}$ is closed and so is $Ran\widehat{S}^*$. Hence there exists $(f,g) \in N \oplus N$ such that

$$\widehat{S}^*(f,g) = (T_z^* h + P_N R(0)T_z^* h, T_w^* h + P_N L(0)T_w^* h),$$

i.e.,

$$S_z^* f - S_w g = T_z^* h + P_N R(0)T_z^* h, \tag{3.1}$$

$$S_w^* f + S_z g = T_w^* h + P_N L(0)T_w^* h. \tag{3.2}$$

If $h = 0$, then $S_z^* f - S_w g = 0$ and $S_w^* f + S_z g = 0$. Since S_z commutes with S_w, we have $S_z S_z^* f + S_w S_w^* f = 0$; that is, $f \in kerS_z^* \cap kerS_w^*$. Indeed, we have a linear map T from $((M \ominus zM) \cap (M \ominus wM)) \ominus RanA$ to the quotient space $N/(kerS_z^* \cap kerS_w^*)$ defined by

$$Th = f + kerS_z^* \cap kerS_w^*.$$

We now check that T is injective. If $f + kerS_z^* \cap kerS_w^* = 0$, i.e., $f \in kerS_z^* \cap kerS_w^*$, we have

$$\begin{aligned}
-S_w g &= T_z^* h + P_N R(0)T_z^* h, \\
S_z g &= T_w^* h + P_N L(0)T_w^* h.
\end{aligned}$$

It yields that

$$\begin{aligned}
\frac{h - h(0,w)}{z} + \frac{h(z,0) - h(0,0)}{z} + wg &\in M, \\
\frac{h - h(z,0)}{w} + \frac{h(0,w) - h(0,0)}{w} - zg &\in M,
\end{aligned}$$

i.e.,

$$h - h(0,w) + h(z,0) - h(0,0) + zwg \in zM, \tag{3.3}$$

$$h - h(z,0) + h(0,w) - h(0,0) - zwg \in wM. \tag{3.4}$$

Adding the left parts of (3.3) and (3.4), we obtain

$$2h - 2h(0,0) \in M.$$

Then $h(0,0) \in M$, which in turn implies that $h(0,0) = 0$, since M is a nontrivial submodule. Note that $h \in (M \ominus zM) \cap (M \ominus wM)$, and it follows from (3.3) and (3.4) that

$$\langle h - h(0,w) + h(z,0) + zwg, h\rangle = 0$$

and

$$\langle h - h(z,0) + h(0,w) - zwg, h\rangle = 0,$$

i.e.,

$$\|h\|^2 - \|L(0)h\|^2 + \|R(0)h\|^2 + \langle zwg, h\rangle = 0 \tag{3.5}$$

and

$$\|h\|^2 - \|R(0)h\|^2 + \|L(0)h\|^2 - \langle zwg, h\rangle = 0. \tag{3.6}$$

Adding (3.5) and (3.6), we obtain $2\|h\|^2 = 0$. Then $h = 0$ and T is injective.

Now we prove that the dimension of $RanT$ is less than 1. Adding (3.1) multiplied by S_z with (3.2) multiplied by S_w, we have

$$(S_zS_z^* + S_wS_w^*)f = S_zT_z^*h + S_zP_NR(0)T_z^*h + S_wT_w^*h + S_wP_NL(0)T_w^*h. \tag{3.7}$$

Calculating the four items on the right-hand side of (3.7) one by one, we have

$$\begin{aligned} S_zT_z^*h &= S_z\frac{h - h(0,w)}{z} = -P_Nh(0,w), \\ S_zP_NR(0)T_z^*h &= S_zR(0)T_z^*h \\ &= S_z\frac{h(z,0) - h(0,0)}{z} \\ &= P_Nh(z,0) - h(0,0)\eta. \end{aligned}$$

Similarly, we have $S_wT_w^*h = S_w\frac{h-h(z,0)}{w} = -P_Nh(z,0)$ and $S_wP_NL(0)T_w^*h = P_Nh(0,w) - h(0,0)\eta$. Hence

$$(S_zS_z^* + S_wS_w^*)f = -2h(0,0)\eta.$$

Note that $\Delta_{S^*} = I - S_zS_z^* - S_wS_w^* + S_zS_wS_z^*S_w^* = \eta\otimes\eta$, and it follows that $(I + S_zS_wS_z^*S_w^*)f = (\eta\otimes\eta)f - 2h(0,0)\eta$. Since $I + S_zS_wS_z^*S_w^*$ is an invertible operator, we obtain that $dimRanT \le 1$, and thus $dim(((M\ominus zM)\cap(M\ominus wM))\ominus RanA) \le 1$. That is,

$$dim((M\ominus zM)\cap(M\ominus wM)) \le dimker\widehat{S} + 1 < \infty.$$

If $(0,0)\in Z(M)$, then for every $h\in M$, $h(0,0) = 0$. By the previous arguments, for any $h\in((M\ominus zM)\cap(M\ominus wM))\ominus RanA$, $h = 0$, and hence $dim((M\ominus zM)\cap(M\ominus wM)) = dimRanA = dimker\widehat{S}$.

Note that $(0,0)\notin Z(M)$ if and only if $1\notin N$. We claim that $P_M1\notin RanA$; otherwise, there is $h = zf + wg\in M$ such that $P_M1 = P_M(I + L(0) + R(0))h$, where $(f,g)\in ker\widehat{S}$. Then $\langle h, P_M1\rangle = \langle h, P_M(I + L(0) + R(0))h\rangle$, i.e., $\|h\|^2 + \|L(0)h\|^2 + \|R(0)h\|^2 = 0$, which implies that $h = 0$ and hence $P_M1 = 0$, which contradicts $(0,0)\notin Z(M)$. It is obvious that $P_M1\in(M\ominus zM)\cap(M\ominus wM)$. Then the orthogonal projection of P_M1 from $(M\ominus zM)\cap(M\ominus wM)$ into $((M\ominus zM)\cap(M\ominus wM))\ominus RanA$ is not zero, that is, $dim(((M\ominus zM)\cap(M\ominus wM))\ominus RanA) \ge 1$. Combining the previous results, we have $dim((M\ominus zM)\cap(M\ominus wM)) = dimRanA + 1 = dimker\widehat{S} + 1$.

This completes the proof of Lemma 3.1. $\square$

Theorem 3.2. *Let M be a submodule in $H^2(\mathbb{D}^2)$. If $S = (S_z, S_w)$ is Fredholm, then $ind(S_z, S_w) = dimE_1(C) - dimE_{-1}(C) - 1$.*

Proof. Curto's lemma implies that $ind(S_z, S_w) = dimker\widehat{S} - dimker\widehat{S}^*$. If $(0,0) \in Z(M)$, then $1 \in N$, which implies that $1 \in kerS_z^* \cap kerS_w^*$. Since $kerS_z^* \cap kerS_w^* \subset \mathbb{C}$, we have $dim(kerS_z^* \cap kerS_w^*) = 1$ and $dimker\widehat{S}^* = 1 + dim(kerS_z \cap kerS_w)$. If $(0,0) \notin Z(M)$, then $kerS_z^* \cap kerS_w^* = \{0\}$ and hence $dimker\widehat{S}^* = dim(kerS_z \cap kerS_w)$. By Lemma 3.1 we have, in any case, that

$$ind(S_z, S_w) = dim((M \ominus zM) \cap (M \ominus wM)) - dim(kerS_z \cap kerS_w) - 1.$$

By Corollary 3.4 in [GY], $E_1(C) = (M \ominus zM) \cap (M \ominus wM)$ and $E_{-1}(C) = (zM \cap wM) \ominus zwM$. It is indicated in [GW1] and [Ya1] that $dim(kerS_z \cap kerS_w) = dim((zM \cap wM) \ominus zwM)$, and this completes the proof of Theorem 3.2. ☐

For a submodule M, we define two operators F_z on $M \ominus zM$ and F_w on $M \ominus wM$ by

$$F_z f = [R_z^*, R_z]wf, \quad F_w g = [R_w^*, R_w]zf,$$

where $f \in M \ominus zM$, $g \in M \ominus wM$, respectively.

Since F_z and F_w display parallel properties, we just list some properties of F_z; more details can be found in [Ya4].

Lemma 3.3 ([Ya4]). *Let M be a submodule. Then the following hold:*

(1) *$RanF_z = (zM + wM) \ominus zM$, $kerF_z = z(kerS_z \cap kerS_w)$.*
(2) *If $f \in M \ominus zM$, then $f - F_z^* F_z f = [R_w^*, R_z][R_z^*, R_w]f$, $f - F_z F_z^* f = [R_z^*, R_z][R_w^*, R_w]f$.*
(3) *For every $f \in M \ominus zM$, $[R_w^*, R_w][R_z^*, R_z]F_z f = -F_w[R_z^*, R_w]f$, and for every $f \in M \ominus wM$, $[R_z^*, R_z][R_w^*, R_w]F_w f = -F_z[R_w^*, R_z]f$.*

Corollary 3.4. *Let M be a submodule in $H^2(\mathbb{D}^2)$, $N = H^2(\mathbb{D}^2) \ominus M$. If (S_z, S_w) is Fredholm, then $[R_z^*, R_z][R_w^*, R_w]$ is compact if and only if $[R_w^*, R_z]$ is compact.*

Proof. If (S_z, S_w) is Fredholm, combining Lemma 3.1 and Lemma 3.3 (1), we have $dim((M \ominus zM) \ominus \overline{RanF_z}) = dim((M \ominus zM) \cap (M \ominus wM)) < \infty$, and $dimkerF_z = dim(ker(S_z) \cap kerS_w) < \infty$.

If $[R_w^*, R_z]$ is compact, then by Lemma 3.3 (2), $RanF_z$ is closed and then F_z is Fredholm. Lemma 3.3 (3) implies that $[R_z^*, R_z][R_w^*, R_w]$ is compact.

If $[R_z^*, R_z][R_w^*, R_w]$ is compact, the parallel properties of F_w and the above arguments imply that $[R_w^*, R_z]$ is compact.

This completes the proof of Corollary 3.4. ☐

The closedness of $zM + wM$ is worth studying for many reasons. We have been unable to prove whether the Fredholmness of (S_z, S_w) implies the closedness of $zM + wM$. If so, then the Fredholmness of (S_z, S_w) implies the Fredholmness of (R_z, R_w) and the Fredholmness of F_z; moreover, $ind(S_z, S_w) = -1 - ind(R_z, R_w) = -1 - indF_z$.

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