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# HYPERBOLIC LATTICE POINT PROBLEMS

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ABSTRACT. We prove some analogues of planar lattice point problems replacing $\mathbb{R}^2$ by the Poincaré model of the hyperbolic plane and using the orbit of a point under the modular group instead of the lattice generated by integral translations.

## 1. INTRODUCTION

I.M. Vinogradov and other authors considered in several works the problem of counting the number of lattice points (i.e., points in $\mathbb{Z}^2$) in the region limited by the graph of a positive function and an interval of the $X$-axis [4]. This is in fact a fundamental problem because the classic planar lattice point problem (approximate the number of lattice points in enlarging convex regions [6]) is reduced to it after dividing the boundary in several arcs and changing the role of the axes if necessary.

On the other hand, the spectacular development and applications of the spectral theory of automorphic forms, pioneered by A. Selberg [12], have motivated hyperbolic counting problems in which the base space is the Poincaré half-plane $(\mathbb{H}, ds)$,

$$\mathbb{H} = \big\{z \in \mathbb{C} \ : \ z = x + iy, \ x \in \mathbb{R}, \ y > 0\big\}, \qquad ds^2 = y^{-2}\big(dx^2 + dy^2\big),$$

and the role of the integral translations, generating the lattice in the Euclidean case, is assumed by an arithmetic discrete group of isometries of $\mathbb{H}$, in particular the modular group.

For instance, if $\mathcal{N}_R$ denotes the number of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(i)$ belongs to the hyperbolic circle $B(i,R) = \big\{z \in \mathbb{H} \ : \ \rho(z,i) < R\big\}$, where $\rho$ is the distance corresponding to $ds$, then

$$(1.1) \quad \mathcal{N}_R \sim \frac{6}{\pi}|B(i,R)| \qquad \text{as} \qquad R \to \infty, \qquad \text{where} \ \ |B(i,R)| = \iint_{B(i,R)} \frac{dxdy}{y^2}.$$

Note that $y^{-2}\, dxdy$ is the element of area of $ds$; indeed $|B(i,R)| = 4\pi \sinh^2(R/2)$. This seems to have been proved with an error term $O\big(e^{2R/3}\big)$ first by Selberg (unpublished) as a consequence of the pre-trace formula and revisited by other authors, but the upper bound for the error term remains unbeaten. A basic difference with

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the Euclidean setting is that most of the area and most of the elements of the orbit are concentrated along a thin band in the lower boundary. The formula

$$2\cosh\rho(z,w) - 2 = \frac{|z-w|^2}{\Im z \Im w} \tag{1.2}$$

reveals that $B(i,R)$ admits the Cartesian equation

$$B(i,R) = \{x+iy \in \mathbb{H} \,:\, x^2 + (y-1)^2 < 2(\cosh R - 1)y\}, \tag{1.3}$$

which represents an off-centered large Euclidean circle. To keep a closer analogy with the Euclidean situation it is convenient to introduce $T = 2\cosh R - 2$. The value of $T$ approximates $y_0^{-1}$, where $y_0$ is the minimal imaginary part of the points in $B(i,R)$. With this notation Selberg's result reads $\mathcal{N}_R = 6T + O(T^{2/3})$.

For a region $\Omega \subset \mathbb{H}$ we consider in general

$$\mathcal{N}(\Omega) = \#\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \,:\, \gamma(i) \in \Omega\}.$$

By the geometric properties of $(\mathbb{H}, ds)$ we expect a relation as in (1.1) between $\mathcal{N}(\Omega)$ and the hyperbolic area of $\Omega$ as $y_0 \to 0^+$. In this paper we state several results of this type, studying the error term as a function of a parameter $T$ related to the inverse of the minimal imaginary part.

Our approach employs Kloosterman sums to count elements of the orbit in regions limited by a graph. Paralleling the Euclidean planar situation we deduce results for other regions, some of them considered by other authors. Our emphasis here is on the simplicity and versatility, avoiding ad hoc manipulations for special equations.

Probably the closer analogue of the Euclidean situation is counting "hyperbolic lattice points" over the graph of a function (compare to §8 in [4]).

**Theorem 1.1.** *Let $f : [r,s] \longrightarrow \mathbb{R}$ with $0 < r < s \le 1$ strictly monotonic and differentiable and let $\Omega$ be the part of the strip $x \in \mathrm{Im}(f)$ limited from below by $\{(f(y), y) \,:\, y \in [r,s]\}$. Then there exists an absolute constant $C$ such that*

$$\left|\mathcal{N}(\Omega) - \frac{6}{\pi}|\Omega|\right| \le C\left(r^{-7/8} + \int_r^s y^{-7/8}|f'(y)|\,dy\right)\log\frac{2}{r},$$

*where $|\Omega| = \iint_\Omega y^{-2}\,dxdy$ is the hyperbolic area of $\Omega$.*

We do not specify if the boundary points are included or not in $\Omega$. It will be apparent in the proof that this is irrelevant. Taking this into account, note that the left hand side is unaffected by infinitesimal modifications of $f$. This allows us to relax the regularity of $f$, allowing for instance piecewise differentiable functions or even multiply defined functions at some point. The monotonicity can be relaxed with a convenient subdivision, but in this case the constant is not absolute and depends on the subdivision.

Our basic result focuses on curvilinear triangular regions of width at most 1. The rest of the results are based on it. The previous remarks also apply and come from its proof.

**Theorem 1.2.** *Let $F : [\alpha,\beta] \subset \mathbb{R}^+ \longrightarrow [-1/2,1/2]$ be monotonic and differentiable. Consider the region $\Omega_T = \{x+iy \ : \ \delta \le x \le F(Ty),\ Ty \in [\alpha,\beta]\}$ where $\delta = \min F$. Then for $T > 2$,*

$$\mathcal{N}(\Omega_T) = \frac{6}{\pi}|\Omega_T| + O\big(T^{7/8}\log T\big),$$

*where $|\Omega_T| = c_F T$ and the $O$-constant does not depend on $F$ if $\alpha$ remains bounded from below by a positive constant.*

In the proof we shall employ the optimal bound for individual Kloosterman sums but one expects more cancellation due to the extra summation. Conjecturally the natural error term in Theorem 1.2 is $O_\epsilon\big(T^{1/2+\epsilon}\big)$ for every $\epsilon > 0$, supported by the average results of [10] and [2] in the application to (1.1).

The case of circular sectors has been treated by several authors (we point out an interesting purely spectral approach in [11]).

**Corollary 1.3.** *Fix $\theta_0 \in [0,\pi)$ and let $\Xi_R(\theta_0) = \big\{z \in \mathbb{H} \ : \ \rho(z,i) < R,\ \theta_0 \le \theta(z) \le \pi\big\}$, where $\theta(z) \in (-\pi,\pi]$ is the angle $\widehat{0iz}$ determined by the geodesic segments joining $i$ with $0^+$ and with $z$. Then*

$$\mathcal{N}\big(\Xi_R(\theta_0)\big) = \frac{6}{\pi}|\Xi_R(\theta_0)| + O_{\theta_0}\big(Re^{7R/8}\big).$$

This result was proved in [1] with a slightly weaker error term. In our particular centered case, $z_0 = z_1 = i$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, our approach follows similar general lines but largely simplifies the arguments (see section 3 below for the general case).

Unlike in the Euclidean situation, there are no dilations in $\mathbb{H}$ (conformal geodesic-preserving homeomorphisms) other than isometries. Then one has to be cautious in the geometric interpretation of scaling functions and their relation with the convexity and the element of area.

Given $z \ne i$ let $\mathfrak{r}$ be the geodesic ray starting at $i$ with $z \in \mathfrak{r}$. For each $X > 0$ we define $D_X(z)$ as the unique element of $\mathfrak{r}$ such that $\cosh\rho\big(D_X(z),i\big) - 1 = X\big(\cosh\rho(z,i) - 1\big)$.

**Corollary 1.4.** *Let $\Omega \subset \mathbb{H}$ be a smooth compact region containing $i$ such that $D_X(\Omega)$ is convex Euclidean. Then*

$$\mathcal{N}\big(D_X(\Omega)\big) = \frac{6X}{\pi}|\Omega| + O\big(X^{7/8}\log X\big).$$

For $\Omega = B(i,1)$ and $X = 2\cosh R - 2$, this implies (1.1). This result can be seen as the hyperbolic version of the classical Euclidean planar lattice point problems [6].

Finally we extract a consequence with a more arithmetical flavor related to a divisor problem in the Gaussian domain.

**Corollary 1.5.** *Let $d_X(k)$ denote the number of Gaussian integers $z \in \mathbb{Z}[i]$ dividing $1+ki$ with $X < |z| \le 2X$. Then*

$$\sum_{k=0}^{K} d_X(k) = 12\frac{\log 2}{\pi}K + O\big(X^{7/4}\log X\big)$$

*uniformly in $K \le X^2/2$.*

## 2. The proofs

We separate for later reference an elementary result.

**Lemma 2.1.** *Let $f$ differentiable and monotonic in $[A,B]$. Then there exists an absolute constant $C$ such that for $B>2$,*

$$\Big|\sum_{A\le c^2+d^2\le B}^{*} f(c^2+d^2)-\frac{6}{\pi}\int_A^B f\Big|\le C\max(|f|)B^{1/2}\log B,$$

*where $\sum^*$ indicates that the integers $c$ and $d$ are coprime.*

*Proof.* Let $r_*(n)=\#\{(c,d)\in\mathbb{Z}^2\ :\ c^2+d^2=n,\ \gcd(c,d)=1\}$. Then by Möbius inversion, $r_*(n)=\sum_{d^2|n}\mu(d)r(n/d^2)$, where $r(n)$ is the number of representations of $n$ as a sum of two squares. The sum is $\sum r_*(n)f(n)$, and the result follows by partial summation from the trivial estimate for the circle problem $\sum_{n\le x} r(n)=\pi x+O(x^{1/2})$. □

A calculation proves that

$$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})\quad\Rightarrow\quad \gamma(i)=\frac{a}{c}-\frac{d}{c(c^2+d^2)}+\frac{i}{c^2+d^2}. \tag{2.1}$$

Given $c$ and $d$ coprime, the determinant equation $ad-bc=1$ determines $(a,b)$ up to adding an integral multiple of $(c,d)$, and this corresponds to composing $\gamma$ with an integral translation $z\mapsto z+n$, and therefore such a $(c,d)$ determines $\gamma(i)$ by imposing $|\Re\gamma(i)|\le 1/2$ (there are no points in the boundary).

After these considerations we can read Lemma 2.1 as the hyperbolic lattice point problem for the strip $\Sigma_T=\{x+iy\in[-\frac12,\frac12]\times[T^{-1},\infty)\}$, concluding

$$\mathcal{N}(\Sigma_T)=\frac{6}{\pi}|\Sigma_T|+O\big(T^{1/2}\log T\big)\qquad\text{as}\qquad T\to\infty. \tag{2.2}$$

At first sight one would think that the error term would be improved to $O\big(T^{\alpha}\big)$ for some $\alpha<1/2$ using nontrivial estimates for the circle problem in the proof of Lemma 2.1, but such improvements are related to the Riemann Hypothesis [9], [13] and seem to be out of reach with current methods.

*Proof of Theorem* 1.2. Define $T_+=T/\alpha$ and $T_-=T/\beta$. By (2.1)

$$\gamma(i)\in\Omega\quad\Leftrightarrow\quad 0\le\frac{a}{c}-\frac{d}{c(c^2+d^2)}-\delta\le F\big(\frac{T}{c^2+d^2}\big)-\delta,\qquad T_-\le c^2+d^2\le T_+.$$

The relation $ad-bc=1$ implies $a=\bar d$ for some $\bar d\cdot d\equiv 1 \pmod c$, and as we mentioned before, $\bar d$ is uniquely determined.

Let $\psi(x)=x-[x]-1/2$. Then for each $0<t\le 1$, the function $\psi(x-t)-\psi(x)+t$ is the 1-periodic extension of the characteristic function of $[0,t]$. In our case we take $t=F\big(T/(c^2+d^2)\big)-\delta$ to write $\mathcal{N}(\Omega_T)$ as

$$\mathcal{N}(\Omega_T)=\sum_{T_-\le c^2+d^2\le T_+}^{*}\bigg(F\big(\frac{T}{c^2+d^2}\big)-\delta+\psi\Big(x(c,d)-F\big(\frac{T}{c^2+d^2}\big)+\delta\Big)-\psi\big(x(c,d)\big)\bigg),$$

where $x(c,d)=\bar d/c-d/c(c^2+d^2)-\delta$, and we have employed the notation of Lemma 2.1.

Taking $f(t) = F(T/t) - \delta$ in Lemma 2.1 the contribution of the first two terms is

$$\frac{6T}{\pi}\int_{\beta^{-1}}^{\alpha^{-1}} \big(F(1/t)-\delta\big)\,dt + O\big(T^{1/2}\log T\big) = \frac{6}{\pi}|\Omega_T| + O\big(T^{1/2}\log T\big).$$

Note that the $O$-constant degenerates when $\alpha \to 0^+$ because in this case $T^+/T \to \infty$. The same occurs in the rest of the $O$-estimates.

It remains to prove

$$\sideset{}{^*}\sum_{T_- \le c^2+d^2 \le T_+} \psi\big(\frac{\bar{d}}{c} + g(c,d)\big) = O\big(T^{7/8}\log T\big) \tag{2.3}$$

for $g(c,d) = 1/2 - d/c(c^2+d^2)$ and $g(c,d) = -d/c(c^2+d^2) - F\big(T/(c^2+d^2)\big)$.

It is well-known (see for instance [8], p. 5) that for each $M \in \mathbb{Z}^+$ there exist two trigonometric polynomials, $P_-$ and $P_+$ such that $P_- \le \psi \le P_+$ and $P_\pm(x) = \sum_{|m|\le M} a_m^\pm e(mx)$ with $a_0^\pm \ll M^{-1}$ and $a_m^\pm \ll m^{-1}$ for $m \ne 0$. Then

$$\sideset{}{^*}\sum_{T_- \le c^2+d^2 \le T_+} \psi\big(\frac{\bar{d}}{c} + g(c,d)\big) \ll \frac{T}{M} + \sum_{m=1}^{M}\frac{1}{m}|S_m| \tag{2.4}$$

with

$$S_m = \sideset{}{^*}\sum_{T_- \le c^2+d^2 \le T_+} e\big(m\frac{\bar{d}}{c} + mg(c,d)\big).$$

Note that due to the use of continuous upper and lower bounds for $\psi(u)$ it makes no difference whether or not the points of the boundary are included in $\Omega_T$ because they correspond to integral values of $u$.

For each $c$ fixed we apply Abel's lemma in $d$ to separate the term $e\big(mg(c,d)\big)$, getting

$$S_m \ll m\sum_{1\le c\le T_+^{1/2}} \big|\sum_{d\in I_c} e\big(m\frac{\bar{d}}{c}\big)\big|,$$

where $I_c \subset [T_-^{1/2}, T_+^{1/2}]$ is an interval depending on $c$. Completing the sum to $d \in (A,B] \supset I_c$ with $c|B-A \ll T^{1/2}$ (use for instance Lemma 12.1 in [7]) we deduce

$$S_m \ll m(\log T)\sum_{1\le c\le T_+^{1/2}} \frac{T^{1/2}}{c}\big|\sideset{}{^*}\sum_{d=1}^{c} e\big(m\frac{\bar{d}}{c} + n\frac{d}{c}\big)\big|$$

for some $n = n(c) \in \mathbb{Z}$. By Weil's bound (see for instance Lemma 2 in [5]) the absolute value is bounded by $(m,c)^{1/2}c^{1/2}\tau(c)$. Using the elementary estimates

$$\sum_{m\le M}(m,c)^{1/2} \le M\sigma_{-1/2}(c) \qquad \text{and} \qquad \sum_{n\le x}\tau(n)\sigma_{-1/2}(n) = O(x\log x)$$

(for the second see §1.6 in [7]) we conclude that the summation in the right hand side of (2.4) is $O\big(MT^{3/4}\log^2 T\big)$. Choosing $M = T^{1/8}/\log T$ we obtain (2.3). $\square$

*Proof of Theorem* 1.1. Let $[u,v] = \mathrm{Im}(f)$ and assume that $f$ is increasing (the other case is symmetric). First we consider the case $u+1/2, v+1/2 \in \mathbb{Z}$. Define $g(x) = f^{-1}(x+u)$ and $F_k(y) = f(g(k)y)$, where $f^{-1}$ denotes the inverse function of $f$. Note that $r \le g \le 1$ and $F_k$ applies $[1, g(k+1)/g(k)]$ onto $[u+k, u+k+1]$.

Theorem 1.2 (after the translation $z \mapsto z-k-u-1/2 \in \mathrm{SL}_2(\mathbb{Z})$) with $F=F_k$, $T=1/g(k)$ and (2.2) with $T=1/g(k+1)$ prove that the contribution to $\mathcal{N}(\Omega)$ of the strip $x\in[u+k,u+k+1]$ admits an error term $O\big((g(k))^{-7/8}\log\frac{2}{r}\big)$. Adding all of these error terms and using the monotonicity we have

$$\big(\log\frac{2}{r}\big)\sum_{k=0}^{v-u-1}\big(g(k)\big)^{-7/8} \le \big(\log\frac{2}{r}\big)\bigg(r^{-7/8}+\int_0^{v-u}\big(g(x)\big)^{-7/8}\,dx\bigg).$$

The change of variables $x=f(y)-u$ gives the expected bound.

If $u+1/2\notin\mathbb{Z}$ or $v+1/2\notin\mathbb{Z}$ we still apply the previous argument in the maximal interval $I=[n-1/2,m-1/2]\subset[u,v]$ with $n,m\in\mathbb{Z}$ (take $I=\emptyset$ if it does not exist). The at most two strips of width less than one corresponding to $x\in[u,v]-I$ are treated directly with Theorem 1.2. Note that the right borders of these strips can be well approximated by functions as in Theorem 1.2 (with $\beta$ large). $\square$

*Proof of Corollary* 1.3. The equation (1.3) implies that $B(i,R)\cap\{x\ge 0,\ y\le 1\}$ is included in the strip $\Sigma=\{0\le x\le 2\sinh(R/2)\}$. A simple calculation with the element of area proves that the hyperbolic areas $\big|B(i,R)\cap\{y\le 1\}\big|$ and $\big|\Sigma\cap\{y\le 1\}\big|$ are $O\big(e^{R/2}\big)$, which is absorbed by the error term. On the other hand, $\gamma(i)$ never reaches $\{y>1\}$ for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$. Then $\mathcal{N}\big(\Xi_R(\theta_0)\big)$ is identical to $\mathcal{N}\big(\Omega\big)$, where $\Omega$ is the part of the strip $\Sigma$ limited from below by the lower boundary of $\Xi_R(\theta_0)\cap\{y\le 1\}$.

According to (1.3), the contribution of the part limited by the circle is estimated choosing $f(y)=\big(4y\sinh^2(R/2)-(y-1)^2\big)^{1/2}$ in Theorem 1.1, giving an error term $O\big(Re^{7R/8}\big)$ that dominates the part bounded by the geodesic $\theta(z)=\theta_0$, represented by a fixed semicircle, or by the horizontal line $y=1$. $\square$

*Proof of Corollary* 1.4. Using the convexity and considering the two pairs of vertical and horizontal support lines of $D_X(\Omega)$ we can divide its boundary in four graphs $x=f_i(y)$ with $f_i:[r_i,s_i]\longrightarrow\mathbb{R}$ monotonic. Let us say that $f_1$ and $f_2$ correspond to the lower boundary and $f_3$ and $f_4$ to the upper one, and let $\Omega_i$ be the regions in Theorem 1.1 associated to these functions. Then

$$\mathcal{N}\big(D_X(\Omega)\big)=\mathcal{N}(\Omega_1)+\mathcal{N}(\Omega_2)-\mathcal{N}(\Omega_3)-\mathcal{N}(\Omega_4), \tag{2.5}$$

where the boundaries are accordingly included or excluded to avoid repetitions. The element of area in the modified geodesic polar coordinates $(\widetilde{r},\theta)$ with $\widetilde{r}(z)=2\cosh\rho(z,i)-2$ and $\theta(z)$ as in Corollary 1.3 is $d\widetilde{r}\,d\theta$; then $\big|D_X(\Omega)\big|=X|\Omega|$.

By the compactness of $\Omega$, $\Omega\subset B(i,R_0)$ for some $R_0$; hence $D_X(\Omega)\subset B(i,R)$ with $2\cosh R-2=cX$ for some $c$ depending only on $\Omega$. Recalling (1.3) we deduce that $D_X(\Omega)$ is contained in the half-strip $\big\{|x|\le c_1X^{1/2},\ y\ge c_2X^{-1}\big\}$ for some constants $c_1,c_2>0$, in particular $r_i\gg X^{-1}$. As in the proof of Corollary 1.3 we can dismiss the part $\{y>1\}$ in the sets appearing in (2.5) because its contribution is absorbed by the error term; consequently, we assume $s_i\le 1$. By Theorem 1.1 it remains to prove

$$\int_{r_i}^{s_i} y^{-7/8}|f_i'(y)|\,dy=O\big(X^{7/8}\big).$$

The inclusion $D_X(\Omega)\subset B(i,R)$ and (1.3) assure $|f_i(y)|<\sqrt{cyX-(y-1)^2}$, and integrating by parts the bound for the integral follows. $\square$

*Proof of Corollary* 1.5. Letting $z = d + ci \in \mathbb{Z}[i]$, then $z|1 + ki$ means that there exists $a + bi \in \mathbb{Z}[i]$ such that $(a + bi)(d + ci) = 1 + ik$ or equivalently $ad - bc = 1$, $ac + bd = k$. By (2.1) we have

$$d_X(k) = \#\big\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ : \ (2X)^{-2} \le \Im\gamma(i) < X^{-2},\ \Re\gamma(i) = k\Im\gamma(i)\big\},$$

and defining $F : [1/4, 1] \longrightarrow [-1/2, 1/2]$ as $F(y) = Ky/X^2$,

$$\sum_{k=0}^{K} d_X(k) = \mathcal{N}\big(\Omega\big) \qquad \text{with } \Omega = \big\{x+iy \ : \ (2X)^{-2} \le y < X^{-2},\ 0 \le x \le F(X^2y)\big\}.$$

With the notation of Theorem 1.2 we have $\delta = F(1/2) = K/(4X^2)$. Redefining $F$ in a small neighborhood of $1/4$ we can apply this theorem with $\delta = 0$, keeping $|\Omega_T|$ arbitrarily close to $|\Omega|$. Finally a calculation proves that $|\Omega| = 2K \log 2$. □

## 3. Some extensions

We have restricted ourselves to $\mathrm{SL}_2(\mathbb{Z})$ for simplicity, but the same arguments apply for congruence subgroups $\Gamma$ dividing the constant $6/\pi$ by $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$.

We illustrate the situation for $\Gamma_0(N)$. The constant $6/\pi$ for $\mathrm{SL}_2(\mathbb{Z})$ comes from Lemma 2.1, and the new constant $\mathcal{C}_\Gamma$ for $\Gamma = \Gamma_0(N)$ is obtained by imposing consequently $N \mid c$ in the summation. This variant of Lemma 2.1 is again elementary. We work out the details for $f = 1$ in $[0, x]$; the general case follows by partial summation:

$$\sideset{}{^*}\sum_{\substack{c^2+d^2\le x\\ N|c}} 1 = \sum_k \mu(k) \sum_{\substack{c^2+d^2\le x\\ N|c,\ k|c,\ k|d}} 1 = \sum_k \mu(k) \sum_{\substack{c^2+d^2\le x\\ \mathrm{lcm}(N,k)|c,\ k|d}} 1 = \sum_k \frac{\pi x \mu(k)}{\mathrm{lcm}(N,k)k} + O\big(\sqrt{x}\log x\big).$$

Then

$$\mathcal{C}_\Gamma = \pi \sum_k \frac{\mu(k)\gcd(N,k)}{Nk^2} = \frac{\pi}{N}\prod_{p|N}\big(1 - \frac{1}{p}\big)\prod_{p\nmid N}\big(1 - \frac{1}{p^2}\big).$$

Multiplying by $6\pi^{-2}\prod\big(1-p^{-2}\big)^{-1} = 1$ we obtain $\mathcal{C}_\Gamma = 6(\pi N)^{-1}\prod_{p|N}\big(1+p^{-1}\big)^{-1}$, which coincides with $6/\big(\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]\big)$.

Note that, on the other hand, the condition $N \mid c$ does not affect the treatment of the error term with Kloosterman sums. Hence we have proved

**Theorem 3.1.** *Letting* $\mathcal{N}(\Omega) = \#\big\{\gamma \in \Gamma \ : \ \gamma(i) \in \Omega\big\}$, *then under the hypothesis of Theorem* 1.2 *we have*

$$\mathcal{N}_\Gamma(\Omega_T) = \frac{6|\Omega_T|}{\pi[SL_2(\mathbb{Z}) : \Gamma]} + O_\Gamma\big(T^{7/8}\log T\big)$$

*for every Hecke congruence subgroup* $\Gamma = \Gamma_0(N)$.

For $\Gamma = \Gamma(N)$ one has to require $N \mid c$ and $N \mid d - 1$. A technical difference is that $(c, d)$ determines $\gamma$ by imposing $|\Re\gamma(i)| \le N/2$ instead of $|\Re\gamma(i)| \le 1/2$ due to the congruence condition on $b$ (see the comments after Lemma 2.1). Introducing the scaling factor $N^{-1}$, the same reasoning as before (in fact it is slightly easier) gives

$$\mathcal{C}_\Gamma = N^{-1}\sum_{(k,N)=1}\frac{\pi\mu(k)}{(Nk)^2} = \frac{\pi}{N^3}\prod_{p\nmid N}\big(1 - \frac{1}{p^2}\big) = \frac{6}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}$$

and Theorem 3.1 also applies to $\Gamma = \Gamma(N)$.

It is also possible to consider other orbits with identical error terms because (2.1) generalizes to

$$\gamma(x+iy) = \frac{a}{c} - \frac{cx+d}{\big((cx+d)^2+c^2y^2\big)c} + \frac{yi}{(cx+d)^2+c^2y^2}.$$

The new expression is incorporated into $g(c,d)$ and extracted in the same way by partial summation in (2.4). Every congruence subgroup $\Gamma$ admits a coset decomposition $\Gamma = \bigcup \Gamma(N)\gamma_j$. Hence $\mathcal{N}_\Gamma(\Omega_T)$ is the sum of the values of $\mathcal{N}_{\Gamma(N)}(\Omega_T)$ for the orbits containing $\gamma_j(i)$. From $[\mathrm{SL}_2(\mathbb{Z}):\Gamma]^{-1} = [\Gamma:\Gamma(N)][\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]^{-1}$ we conclude that Theorem 3.1 is valid for every congruence subgroup.

Note that unlike in the Euclidean case the relation between elements of the group and elements of the orbit is not one to one. In $\mathrm{SL}_2(\mathbb{Z})$ we have $\gamma(z) = (-\gamma)(z)$, and even considering $\mathrm{PSL}_2(\mathbb{Z})$ the points in the orbits of $i$ and $(1+i\sqrt{3})/2$ have nontrivial stability groups.

Part of the literature in planar lattice point theory is devoted to counting primitive points (also called visible points). A hyperbolic analogue was introduced in [3], where the case of the circle is treated in connection with the orchard problem. It is not clear if it is possible to extend the results of the present paper in this direction for $\Omega$'s lacking special symmetries.

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