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RIEMANNIAN L^p CENTER OF MASS: EXISTENCE, UNIQUENESS, AND CONVEXITY

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Abstract. Let M be a complete Riemannian manifold and ν a probability measure on M. Assume $1 \leq p \leq \infty$. We derive a new bound (in terms of p, the injectivity radius of M and an upper bound on the sectional curvatures of M) on the radius of a ball containing the support of ν which ensures existence and uniqueness of the global Riemannian L^p center of mass with respect to ν. A significant consequence of our result is that under the best available existence and uniqueness conditions for the so-called "local" L^p center of mass, the global and local centers coincide. In our derivation we also give an alternative proof for a uniqueness result by W. S. Kendall. As another contribution, we show that for a discrete probability measure on M, under the existence and uniqueness conditions, the (global) L^p center of mass belongs to the closure of the convex hull of the masses. We also give a refined result when M is of constant curvature.

1. Introduction and history

Let us first give the following definition:

Definition 1.1. Let (M, d) be an n-dimensional complete Riemannian manifold with distance function $d(.,.)$.[1] Assume that ν is a probability measure (or mass distribution) defined on M. Denote the support of ν by $\mathrm{supp}(\nu)$. Define

$$f_p(x) = \begin{cases} \frac{1}{p}\int d^p(x,s)\mathrm{d}\nu(s), & 1 \leq p < \infty, \\ \max_{s\in\overline{\mathrm{supp}}(\nu)} d(x,s), & p = \infty, \end{cases} \tag{1.1}$$

where $\overline{\mathrm{supp}}(\nu)$ is the closure of the support of ν. When considering $p = \infty$ we assume that $\mathrm{supp}(\nu)$ is a bounded subset of M. Any minimizer of f_p in M is called a (global) Riemannian L^p center of mass or mean with respect to ν.

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[1]Throughout the paper, we assume that $d(.,.)$ is induced by the Levi-Civita connection associated with the Riemannian structure of M.

The author's study of the Riemannian center of mass is motivated by applications in the processing of manifold-valued data, i.e., processing data which lie in Riemannian manifolds. Interestingly, such forms of data appear in many applications; examples include robotics, computer graphics and animation, medical imaging, statistical analysis of shapes and many other areas. For such data, standard linear or nonlinear processing schemes are not valid, and one needs to redefine or generalize some standard concepts such as the mean or center of mass. Usually, in those applications, we deal with a finite number of data points, and the measure ν is a discrete probability measure with finite support. Although the L^2 center of mass is the most common one used in most applications, other values of p also can be useful. One can see that as p increases, the effect of outliers (those data points that are far from the cloud of points) in determining the mean becomes more pronounced. Therefore, using other values of p will give the user flexibility in different applications. The two extreme cases are $p = 1$ and $p = \infty$. Recall that $p = 1$ corresponds to the multivariate median in Euclidean space. For $p = 1$ the outliers have the minimum effect and in that sense the center is the most robust one. On the other hand for $p = \infty$ only the outliers contribute in determining the center. If the data points are samples drawn from a uniform distribution whose support is a ball of unknown radius and center, then the L^∞ center of mass of the data points gives the maximum likelihood estimator of the center of the support.

The focus of this paper is on very basic properties of the Riemannian L^p center of mass. We study the existence and uniqueness properties of the Riemannian L^p center of mass as well as the relative position of the center with respect to the convex hull of the support of the underlying probability measure. Before proceeding further, it is useful to review the long and interesting history of the notion of Riemannian center of mass.

1.1. **On the history of Riemannian center of mass.** Here, we give a brief chronological history of the notion of Riemannian center of mass and related existence and uniqueness results. Our account is based on the most widely used and available texts, and it might not be complete. According to many sources (e.g., [2, p. 235]) É. Cartan was the first to define and use the center of mass in the context of Riemannian geometry. In the 1920s, he defined the L^2 center of mass in complete simply connected manifolds of nonpositive curvature (a.k.a. Hadamard manifolds) and proved the existence and uniqueness of the center. He used the L^2 center of mass to prove that any compact subgroup of the isometry group of an Hadamard manifold has a fixed point. A proof of Cartan's theorem using the L^∞ center of mass instead can be found in [17, p. 225] or [16, p. 164]. Aside from an unpublished work by Calabi pointed out in [2, p. 233], the next usage is by Grove and Karcher in [10], where motivated by solving pinching problems, they define the L^2 center of mass in general Riemannian manifolds but for probability measures with support in small enough balls. As a result, Grove and Karcher give an extension of Cartan's result to arbitrary manifolds. As one expects and it will become clear later here, positive curvature and finite injectivity radius of M are what might bring about nonuniqueness of the center. In a series of papers [10, 11, 12] Grove, Karcher and Ruh improve, i.e., enlarge, the domain of existence and uniqueness and consider new applications of the L^2 center of mass. In [12], for compact Lie groups, the domain of uniqueness is enlarged and is shown to be a strongly convex

nonball set. Next, Karcher in [13] gives new proofs, applications and improvements of the previous results.

We need to elaborate on two points at this time: First, the definition of the L^2 center of mass used by Grove and Karcher is what we call the "local" definition. Denote an open ball of radius ρ and center $o \in M$ by $B(o, \rho)$. Given a probability measure ν with support in a small ball $B(o, \rho)$ the local L^2 center of mass is defined as a point in $\overline{B}(o, \rho)$ at which f_2 is minimized. In other words, the function under minimization is $f_2|_{\overline{B}}$, the restriction of f_2 to $\overline{B}(o, \rho)$. This definition serves the purposes of finding a unique zero of the gradient vector field of f_2 in B and deriving useful related estimates. However, as pointed out by Groisser in [9], the local definition does not immediately rule out the possibility that a fixed mass distribution might have different centers depending on which candidate ball we choose. We refer the reader to [9, p. 112] for a further discussion. On the other hand, the global definition does not suffer from this problem if we can establish uniqueness of the minimizer of f_p. Another benefit of resorting to global minimization of f_p is that we expect the global minimizer to be a "better" mean than a local minimizer. As has become standard, the global mean is also referred to as the Fréchet mean (e.g., [9, 15]).

Second, the detailed techniques used and the Jacobi field estimates derived to show the uniqueness of the local L^2 center of mass in [10, 11, 12] and in [13] are rather different, but the grand strategies used are very similar. Assume $\mathrm{supp}(\nu) \subset B(o, \rho)$. The strategy in [10, 11, 12] is to make sure that ρ is small enough such that the Hessian of $x \mapsto d^2(s, x)$ is positive definite at any critical point x of f_2 in $B(o, \rho)$ and for all $s \in B(o, \rho)$. The strategy in [13] is to choose ρ small enough such that the Hessian of $x \mapsto d^2(x, s)$ is positive definite for all $x, s \in B(o, \rho)$. In other words, both strategies use the fact that in order for the Hessian of f_2 to be positive definite at $x \in B(o, \rho)$, it is sufficient (but not necessary) for the Hessian of $x \mapsto d^2(x, s)$ to be positive definite for all $s \in B(o, \rho)$. One expects that the upper bounds on ρ derived based on the mentioned sufficient condition might not be very sharp and can be improved. Before delving into this issue, we briefly recall the result in [13]. We adopt the terminology of [6, p. 403] (or [7]), and we call a set $A \subset M$ strongly convex if any two points in A can be connected by a unique minimizing geodesic segment and the geodesic segment lies entirely in A. Let $\mathrm{inj}M$ denote the injectivity radius of M and let Δ be an upper bound on the sectional curvatures of M. If

$$\rho < r_{\mathrm{cx}} \triangleq \frac{1}{2} \min\{\mathrm{inj}M, \frac{\pi}{\sqrt{\Delta}}\}, \tag{1.2}$$

then $B(o, \rho)$ is a strongly convex ball (see [16, p. 177] or [6, p. 404]). We should emphasize that in this and all similar bounds, we shall interpret $\frac{1}{\sqrt{\Delta}}$ as ∞ if $\Delta \leq 0$. Denote the gradient vector field of f_2 by ∇f_2. Since $B(o, \rho)$ is strongly convex, $-\nabla f_2$ will be inward-pointing on the boundary of the ball (see Subsection 2.1, (2.5) and our proof in Subsection 2.2 for more details). Therefore, the minimum of $f_2|_{\overline{B}}$ cannot happen on the boundary of the ball and the minimizer must lie inside the ball. In order to guarantee the uniqueness of the minimizer, as explained before, Karcher requires $x \mapsto d^2(x, s)$ to be strictly convex for all $x, s \in B(o, \rho)$. Based on the Jacobi field estimates (see Subsection 2.1), this happens when $d(x, s) < \frac{\pi}{2\sqrt{\Delta}}$, which in turn is guaranteed if $\rho < \frac{\pi}{4\sqrt{\Delta}}$. Therefore, the local L^2 center of mass in

$B(o,\rho)$ is unique if

$$\rho < \frac{1}{2}\min\{\mathrm{inj}M, \frac{\pi}{2\sqrt{\Delta}}\}. \tag{1.3}$$

For the unit sphere in $\mathbb{R}^3$ the above gives $\frac{\pi}{4}$ as the upper bound on the radius of the largest ball containing $\mathrm{supp}(\nu)$ to ensure the uniqueness of the center of mass. Nevertheless, it is obvious that for at least two point masses with equal weights lying in a hemisphere (i.e., if $\rho < \frac{\pi}{2}$), the midpoint of the shortest geodesic between them is the unique L^2 center of mass. It is shown by W. S. Kendall [14] that $\rho < \frac{\pi}{4\sqrt{\Delta}}$ is not necessary to ensure the uniqueness of the local L^2 center of mass in $B(o,\rho)$. In [14], he initially calls any local minimizer of f_2 on M a "Karcher mean". However, later in the paper, Kendall extends his definition to the situation where in the definition of f_2 the distance function d is replaced by its restriction to $B(o,\rho)$, i.e., $d|_B$. Groisser calls this latter mean the "solipsistic Karcher mean" [9]. Kendall shows that given $\mathrm{supp}(\nu) \subset B(o,\rho)$, if $\rho < \min\{\mathrm{inj}M, \frac{\pi}{2\sqrt{\Delta}}\}$, then there exists a unique solipsistic Karcher mean in $B(o,\rho)$ (see Theorem (7.3) and the remark above it in [14]).[2] A ball whose radius ρ satisfies Kendall's bound is weakly convex but not necessarily strongly convex [6, p. 405]; i.e., any two points in it can be connected by a unique geodesic which is the unique shortest curve between the two points among all curves in the ball and not the entire M, and that is why ρ can be larger than $\frac{1}{2}\mathrm{inj}M$ in Kendall's bound. It is interesting to note that the mentioned extended definition had appeared in [4], although probably it was not noticed by Kendall. The definition of the center of mass used in [6, p. 407] is as such. For our applications, this definition seems to add no clear benefit. Strong convexity of the ball is needed to ensure the existence of the Riemannian center of mass. Therefore, as also alluded to in [9], when one uses the global distance in defining f_2 (as we do in (1.1)), Kendall's result yields the bound in (1.2) to ensure both the existence and uniqueness of the local L^2 center of mass in $B(o,\rho)$ for a probability measure whose support is in $B(o,\rho)$. Observe that on the unit sphere in $\mathbb{R}^3$, by (1.2) the local L^2 center of mass is unique if $\rho < \frac{\pi}{2}$. As mentioned a long time ago in [10], interest in the Riemannian center of mass also exists among probabilists who investigate Brownian motions in manifolds and their relations to harmonic maps. Interestingly, Kendall's work belongs to this area of research. His proof of his uniqueness result requires some preparations and is rooted in the mentioned literature.

The next step in this journey is the result of Buss and Fillmore [5]. Their work is motivated by the use of the Riemannian L^2 center of mass in spherical interpolation for many applications including computer graphics and animation. Buss and Fillmore show that for a discrete probability measure with finite support on the unit sphere in $\mathbb{R}^n$, if the support is within a closed hemisphere and not contained entirely in the boundary, the global L^2 center of mass is unique and belongs to the hemisphere. They use an interesting reflection argument to show that the center must lie inside the ball. They also use an elegant technique of pairing

[2] To be accurate, we should mention that both Karcher's and Kendall's results, as originally stated by the authors, are rather different from and more general than what we presented here. They give the bounds in terms of more localized information about injectivity radii of points of M or sectional curvatures at those points instead of $\mathrm{inj}M$ and Δ, which contain global information. For example, Karcher requires $B(o,\rho)$ to be strongly convex and $\rho < \frac{\pi}{4\sqrt{\Delta_{B(o,\rho)}}}$, where $\Delta_{B(o,\rho)}$ is the supremum of the sectional curvatures of M in $B(o,\rho)$ [13]. See also Remark 2.5.

points in order to show that any stationary point of f_2 inside the hemisphere is a local minimizer; hence, they prove the uniqueness of the L^2 center. Their uniqueness proof can be considered as an alternative proof for Kendall's result on a constant curvature manifold and for finite support discrete probability measures. Although, they are not specific about it, Buss and Fillmore show that for such a measure on the unit sphere the "local" and "global" L^2 centers of mass coincide.

We mention that obtaining a compromised global existence and uniqueness result from a local existence and uniqueness result is not difficult. For example, assume we know that $\rho < r_*$ for some $r_* > 0$ guarantees the existence and uniqueness of the local L^p center of mass for a probability measure with support in $B(o,\rho)$ and that the center is the only stationary point of f_p $(1 < p < \infty)$ in $B(o,\rho)$. Then just by the triangle inequality we obtain the existence and uniqueness of the global L^p $(1 < p < \infty)$ center of mass for a measure with support in $B(o,\rho)$ where $\rho < \frac{1}{2}r_*$. Le in [15] uses a rather similar argument to show that in a general (i.e., nonconstant curvature) manifold, $\rho < \frac{1}{2}r_{\mathrm{cx}}$ guarantees existence and uniqueness of the global L^2 mean for a probability measure with support in $B(o,\rho)$. Le's work is motivated by applications in the statistical analysis of shapes.

In Theorem 2.1 of this paper, using comparison theorems from Riemannian geometry we extend (with some modifications) the techniques of Buss and Fillmore to a general Riemannian manifold M, and we derive a new bound for the existence and uniqueness of the global L^p center of mass with respect to an arbitrary probability measure on M. In particular, we improve the bound given by Le for the L^2 center of mass. The significance of our result is that under the best available conditions guaranteeing the uniqueness of the "local" L^p center of mass, the "local" and "global" L^p centers coincide; therefore, in the sense described before, ours is an uncompromised global existence and uniqueness result. Our proof of Theorem 2.1 also is an alternative (and maybe more geometrical) proof for Kendall's result, i.e., the bound (1.2).

1.2. **Organization of the paper.** This paper is organized as follows: In Section 2 we state and prove our existence and uniqueness theorem. In Subsection 2.1 major results needed in the subsequent proofs are gathered. We might use these results without specific reference. The actual proof is in Subsection 2.2, and at the end of this subsection we briefly discuss the uniqueness of the center of mass in the case of $1 \le p < 2$ and also a generalization of our existence and uniqueness theorem which can be used in incomplete manifolds. In Theorem 3.4 of Section 3 we study the convexity properties of the L^p mean in a general manifold, and in Theorem 3.1 we give a refined result for manifolds of constant curvature. We conclude the paper in Section 4 by some further discussions and open problems.

2. Existence and uniqueness

Our existence and uniqueness theorem is the following:

Theorem 2.1. *Let (M,d) be a complete Riemannian manifold with sectional curvatures upper bound of Δ and injectivity radius of* $\mathrm{inj}M$. *Define*

$$\rho_{\Delta,p} \triangleq \begin{cases} \frac{1}{2}\min\{\mathrm{inj}M, \frac{\pi}{2\sqrt{\Delta}}\} & \text{if } 1 \le p < 2, \\ \frac{1}{2}\min\{\mathrm{inj}M, \frac{\pi}{\sqrt{\Delta}}\}(= r_{\mathrm{cx}}) & \text{if } 2 \le p \le \infty. \end{cases} \tag{2.1}$$

Let ν be a probability measure on M such that $\mathrm{supp}(\nu) \subset B(o,\rho) \subset M$. *Assume $\rho < \rho_{\Delta,p}$. Then for $1 < p \leq \infty$, the L^p center of mass with respect to ν is a unique point and lies inside B. For $p = 1$, again a center exists and lies in B, and the center is unique except (possibly) in the degenerate case in which the ν-measure of some geodesic segment is* 1. *Moreover, for $1 < p < \infty$ the L^p center is in fact the only stationary point of f_p (see* (1.1)*) in $B(o,\rho)$. In addition, for $1 < p < 2$, f_p is strictly convex in $\overline{B}(o,\rho)$, and for $p = 1$ it is again strictly convex unless in the degenerate case mentioned before in which case f_1 will be only convex. Also, for $1 \leq p < 2$ when $\rho_{\Delta,p} \leq \rho < r_{\mathrm{cx}}$, an L^p center of mass with respect to ν exists and all such centers lie in $B(o,\rho)$.*

2.1. **A brief preparation.** First, let us fix some notation. By T_xM we mean the tangent space of the manifold M at $x \in M$. The exponential map of M at $x \in M$ is denoted by $\exp_x : T_xM \to M$. Within the injectivity domain of $\exp_x$, we denote its inverse by $\exp_x^{-1}$. If $q_1, q_2, q_3 \in M$ are close enough such that the minimizing geodesics connecting q_1 to q_2 and q_1 to q_3 are unique, then we denote the angle between the two geodesics by $\angle q_2q_1q_3$. With some abuse of notation we denote the value of the angle in radians by $\angle q_2q_1q_3$, as well. If q_1, q_2, and q_3 are close enough such that the three minimizing geodesics connecting them are unique, then we denote the unique triangle determined by q_1, q_2, and q_3 as its vertices and the mentioned geodesics as its sides by $\triangle q_1q_2q_3$. Note that if $\sum_{i=1}^{3} d(q_i, q_{i+1}) < 4r_{\mathrm{cx}}$ (with $q_4 \equiv q_1$) or if the points lie in a strongly convex set, then the mentioned geodesics are unique. Throughout the paper, we repeatedly use the structure of convex sets as derived by Cheeger and Gromoll in [7] without specific reference. The interior of any strongly convex set $C \subset M$, denoted by $\mathrm{int}C$ has the structure of a k-dimensional ($0 \leq k \leq n$) totally geodesic (embedded) submanifold of M with a (possibly nonsmooth or empty) boundary ∂C such that $\overline{C} = \mathrm{int}C \cup \partial C$. At a point $x \in \overline{C}$, we denote the tangent cone of C by C_x and define it as

$$C_x = \{\xi \in T_xM \,|\, \exp_x t\frac{\xi}{\|\xi\|} \in \mathrm{int}C \text{ for some positive } t < r_{\mathrm{cx}}\} \cup \{0\}, \tag{2.2}$$

where $\|.\|$ is the norm induced in T_xM by the Riemannian structure. C_x is a k-dimensional convex cone in T_xM, and $C_x \backslash \{0\}$ is (relatively) open in T_xM. If $x \in \partial C$, we say that a nonzero vector $\xi \in T_xM$ is inward-pointing (relative to C) if $\xi \in C_x$. If $x \in \mathrm{int}C$, then C_x is a k-dimensional subspace of T_xM; moreover, if the boundary of C is smooth at x, then C_x is an open half-space in T_xM.

We will perform comparisons between the sides of corresponding triangles in (M, d) and in the model space $(S^n_\Delta, \tilde{d})$, where S^n_Δ is the simply connected constant curvature Riemannian manifold of dimension n and $\tilde{d}$ is its associated distance function. In particular, we need the following hinge version of Alexandrov-Toponogov's comparison theorem [6, p. 420]:

Theorem 2.2. *Consider three points $q_1, q_2, q_3 \in M$ (q_1 distinct from q_2 and q_3) and assume $\sum_{i=1}^{3} d(q_i, q_{i+1}) < 4r_{\mathrm{cx}}$ (see* (1.2)*). Also let the minimizing geodesic from q_1 to q_2 and the minimizing geodesic from q_1 to q_3 make an angle α at q_1. Consider points $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \in S^n_\Delta$ such that $\tilde{d}(\tilde{q}_1, \tilde{q}_2) = d(q_1, q_2)$, $\tilde{d}(\tilde{q}_1, \tilde{q}_3) = d(q_1, q_3)$. Assume that the minimizing geodesic from $\tilde{q}_1$ to $\tilde{q}_2$ and the minimizing geodesic from $\tilde{q}_1$ to $\tilde{q}_3$ also make an angle α at $\tilde{q}_1$. Then we have $\tilde{d}(\tilde{q}_3, \tilde{q}_1) \leq d(q_3, q_1)$.*

We also will use the Hessian comparison theorem for distance functions in M and S^n_Δ. We review this theorem briefly (see [17, pp. 153-4] or [13] for further details). Let $s, x \in M$ and $\tilde{s}, \tilde{x} \in S^n_\Delta$ be such that $0 < d(x,s) = \tilde{d}(\tilde{x},\tilde{s}) < \min\{\mathrm{inj}M, \frac{\pi}{\sqrt{\Delta}}\}$. For convenience denote $x \mapsto d(s,x)$ by $x \mapsto d_s(x)$ and $\tilde{x} \mapsto \tilde{d}(\tilde{s},\tilde{x})$ by $\tilde{x} \mapsto \tilde{d}_{\tilde{s}}(\tilde{x})$. Assume that $t \mapsto \gamma(t)$ and $t \mapsto \tilde{\gamma}(t)$ are unit speed curves leaving x and $\tilde{x}$ such that they make equal angles of α with the unique minimizing geodesics from s and $\tilde{s}$ to x and $\tilde{x}$, respectively. We have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} d_s(\gamma(t))|_{t=0} \geq \frac{\mathrm{d}^2}{\mathrm{d}t^2} \tilde{d}_{\tilde{s}}(\tilde{\gamma}(t))|_{t=0} = \frac{\mathrm{cs}_\Delta\big(\tilde{d}_{\tilde{s}}(\tilde{x})\big)}{\mathrm{sn}_\Delta\big(\tilde{d}_{\tilde{s}}(\tilde{x})\big)} \sin^2\alpha, \tag{2.3}$$

where

$$\mathrm{sn}_\kappa(l) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}l) & \kappa > 0, \\ l & \kappa = 0, \\ \frac{1}{\sqrt{|\kappa|}}\sinh(\sqrt{|\kappa|}l) & \kappa < 0, \end{cases} \qquad \mathrm{cs}_\kappa(l) = \begin{cases} \cos(\sqrt{\kappa}l) & \kappa > 0, \\ 1 & \kappa = 0, \\ \cosh(\sqrt{|\kappa|}l) & \kappa < 0. \end{cases} \tag{2.4}$$

Notice the effect of the sign of Δ on the behavior of the Hessian. Also note that if $0 < d(x,s) < \min\{\mathrm{inj}M, \frac{\pi}{2\sqrt{\Delta}}\}$, then the Hessian of $x \mapsto d_s(x)$ is positive-semidefinite at x, and positive-definite except along the direction of the minimal geodesic from s to x. From this it follows that $x \mapsto d_s(x)$ in $B(s,\rho)$ where $\rho < \min\{\mathrm{inj}M, \frac{\pi}{2\sqrt{\Delta}}\}$ is strictly convex along nonradial geodesics and convex along radial geodesics. One can easily check, by using the properties of convex functions, that $x \mapsto d_s^p(x)$ is strictly convex in $B(s,\rho)$ where $1 < p < \infty$ and ρ is as before. For later use we recall that $x \mapsto d_s^p(x)$ is twice continuously differentiable at $x = s$ only for $2 \leq p < \infty$. In particular, similar to the Euclidean case when $1 < p \leq 2$, the Hessian of $x \mapsto d_s^p(x)$ is not well-defined at $x = s$, while its matrix representation is equal to the identity when $p = 2$ and equal to zero when $2 < p < \infty$.

2.2. **Proof of Theorem 2.1.** In the following subsections we provide the proof of Theorem 2.1.

2.2.1. *Insideness.* We show that the minimum of f_p on M, if it exists, should happen inside $B(o,\rho)$ provided $\rho < \rho_{\Delta,p}$. Observe that for $1 \leq p < \infty$ the gradient of f_p at any point $x \in \overline{B}(o,\rho)$ is

$$\nabla f_p(x) = -\int d^{p-2}(x,s)\exp_x^{-1} s \; \mathrm{d}\nu(s). \tag{2.5}$$

Note that at a point x on the boundary of $B(o,\rho)$, $-\nabla f_p(x)$ is an integral of inward-pointing vectors; hence, it itself must be inward-pointing. Therefore, a point on the boundary cannot be a minimizer of f_p for $1 \leq p < \infty$. Observe that $\overline{\mathrm{supp}}(\nu) \subset \overline{B}(o,\rho)$ and that $\overline{B}(o,\rho)$ is strongly convex. Moreover, because $x \mapsto d(o,x)$ is strictly convex along nonradial directions in $\overline{B}(o,\rho)$ the interior of a minimizing geodesic connecting any two points in $\overline{B}(o,\rho)$ belongs to the interior of the ball. Therefore, at any point x on the boundary of $B(o,\rho)$ any minimal geodesic from x to a point in $\overline{\mathrm{supp}}(\nu)$ will make an acute angle with the radial geodesic from x to o. Hence, by a first-order variation of the arc length, we can still reduce $f_\infty(x)$ by moving along this radial geodesic towards the center. Therefore, for $p = \infty$ a point on the boundary of $B(o,\rho)$ cannot minimize f_∞, as well.

Now, we show that for any point q outside $\overline{B}(o,\rho)$ there is a point q' inside the ball such that $f_p(q') < f_p(q)$. To this end, we show that there exists $q' \in B(o,\rho)$ for which $d(q',s) < d(q,s)$ $\forall s \in B(o,\rho)$. If $d(o,q) \geq 2\rho$, then we can choose $q' = o$. That is because from the triangle inequality, we have $d(q,s) \geq d(o,q) - d(o,s) > \rho$, and hence, $f_p(o) < f_p(q)$. So we only consider a q in the annular region $A = \{x \in M | \rho < d(o,x) < 2\rho\}$. Since $d(o,q) < 2\rho < \mathrm{inj}M$, there exists a unique minimal geodesic from q to o. This geodesic meets the boundary of $B(o,\rho)$ perpendicularly at a unique point q_c which lies between q and o. Denote the corresponding normal inward-pointing vector by $\mathbf{n}_{q_c}$. We choose q' as the reflection of q across the boundary of $B(o,\rho)$. Obviously, $d(q',s) < d(q,s)$ holds for $s = q'$, so in the remainder of this part we assume $s \neq q'$. We have $d(q_c,q) = d(q_c,q')$ and q' is between q_c and o on the minimal geodesic from q to o. A minimizing geodesic emanating from q_c to any point $s \in B(o,\rho)$ makes an angle less than $\frac{\pi}{2}$ with $\mathbf{n}_{q_c}$ at q_c. Note that since $d(q_c,s) < 2\rho < \mathrm{inj}M$ the mentioned geodesic is unique.

Next, we apply our comparison arguments. Consider the same configuration as the one in M, in the model space S^n_Δ: a ball of radius ρ centered at $\tilde{o} \in S^n_\Delta$ and a point $\tilde{q}$ outside the ball at distance $\tilde{d}(\tilde{o},\tilde{q}) = d(o,q)$ from the center. From $\tilde{q}'$ choose a geodesic which makes an angle equal to $\angle oq's$ with the geodesic from $\tilde{q}'$ to $\tilde{o}$ and choose the point $\tilde{s}$ on this geodesic such that $\tilde{d}(\tilde{q}',\tilde{s}) = d(q',s)$. By Theorem 2.2, we have $d(\tilde{o},\tilde{s}) < d(o,s)$, which means that $\tilde{s} \in B(\tilde{o},\rho)$. Therefore, $\tilde{\alpha}_s = \angle \tilde{q}'\tilde{q}_c\tilde{s} < \frac{\pi}{2}$. Depending on whether $\Delta > 0$ or $\Delta \leq 0$, we apply the standard or Spherical Law of Cosines formula, respectively, to the two triangles $\triangle \tilde{q}_c\tilde{q}'\tilde{s}$ and $\triangle \tilde{q}\tilde{q}_c\tilde{s}$ and conclude that

$$\tilde{d}(\tilde{q},\tilde{s}) > \tilde{d}(\tilde{q}',\tilde{s}). \tag{2.6}$$

Next, we apply Theorem 2.2 to triangles $\triangle qq's$ and $\triangle \tilde{q}\tilde{q}'\tilde{s}$. To verify that we can apply the theorem, note that

$$d(q,q') = 2d(q,q_c) \text{ and } d(q',s) \leq d(o,s) + d(o,q') < 2\rho - d(q,q_c). \tag{2.7}$$

If $d(q,s) \geq 2\rho - d(q,q_c)$, then we already have $d(q',s) < d(q,s)$; so we assume $d(q,s) < 2\rho - d(q,q_c)$. With this assumption, the perimeter of $\triangle qq's$ is smaller than $4r_{\mathrm{cx}}$. Note that by construction $\angle sq'q = \pi - \angle oq's$ and $\angle \tilde{s}\tilde{q}'\tilde{q} = \pi - \angle \tilde{o}\tilde{q}'\tilde{s}$; therefore, $\angle sq'q = \angle \tilde{s}\tilde{q}'\tilde{q}$. Applying Theorem 2.2 to the two triangles together with (2.6) yields $d(q,s) \geq \tilde{d}(\tilde{q},\tilde{s}) > d(q',s)$. Note that we only needed $\rho < r_{\mathrm{cx}}$ to prove this result; therefore, the preceding argument extends to the case where $1 \leq p < 2$ and $\rho < r_{\mathrm{cx}}$.

2.2.2. *Existence.* By the previous part we have $\min_{\overline{B}(o,\rho)} f_p < f_p(x)$ for all $x \in M \backslash \overline{B}(o,\rho)$ as long as $\rho < r_{\mathrm{cx}}$. But we argued that the minimum of f_p on $\overline{B}(o,\rho)$, cannot happen on the boundary of the ball. Therefore, the minimum of f_p on M happens inside $B(o,\rho)$ as long as $\rho < r_{\mathrm{cx}}$. This proves the existence of an L^p center of mass. Therefore, from (2.5), we observe that for $1 < p < \infty$ the gradient of f_p is continuous and the L^p center of mass is a stationary point of ∇f_p in $B(o,\rho)$, while for $p = 1$ the center is a critical point of f_1 in $B(o,\rho)$.

2.2.3. *Uniqueness.* We assume $1 \leq p < \infty$ and postpone the case of $p = \infty$ to later. For $1 < p < 2$, the claims about strict convexity of f_p and uniqueness of the center follow from the fact that $x \mapsto d^p(x,s)$ is strictly convex for $\forall x, s \in B(o,\rho)$ with $\rho < \rho_{\Delta,p}$. The same argument works for $2 \leq p < \infty$ if $\Delta \leq 0$. Similarly, for $p = 1$ the claims follow by noting that f_1 is strictly convex in $B(o,\rho)$ unless the

ν-measure of a geodesic segment is one, in which case f_1 is only convex. Now, we assume $p \geq 2$ and $\Delta > 0$.

We need to analyze the Hessian of f_p inside $B(o,\rho)$. Let $t \mapsto \gamma(t)$ be a unit speed geodesic leaving $q \in B(o,\rho)$ at $t = 0$. We denote the angle between the geodesic from $s \in B(o,\rho)\backslash\{q\}$ to q and γ at q by α_s. Then using the chain rule for differentiation and the fact that $\frac{\mathrm{d}}{\mathrm{d}t} d_s(\gamma(t))|_{t=0} = \cos\alpha_s$, we have

$$\frac{\mathrm{d}^2 f_p}{\mathrm{d}t^2}(\gamma(t))|_{t=0} = \int \left((p-1) d_s^{p-2}(q) \cos^2\alpha_s + d_s^{p-1}(q) \frac{\mathrm{d}^2}{\mathrm{d}t^2} d_s(\gamma(t))|_{t=0} \right) \mathrm{d}\nu(s). \tag{2.8}$$

Note that at $s = q$ the angle α_s is not well-defined, and this can be problematic if the ν-measure of a point is nonzero. However, one can check that, immaterial of α_s, for $p = 2$ the integrand is equal to 1 at $s = q$ and for $p > 2$ it is zero at $s = q$. Therefore, the mentioned ambiguity is unimportant. Now, the strategy is to show that any stationary point of f_p is a local minimizer. Note that by (2.3) and (2.8) we have

$$\frac{\mathrm{d}^2 f_p}{\mathrm{d}t^2}(\gamma(t))|_{t=0} \geq \int \left((p-1) \cdot d_s^{p-2}(q) \cdot \cos^2\alpha_s + d_s^{p-1}(q) \cdot \sqrt{\Delta} \cdot \cot\sqrt{\Delta} d_s(q) \cdot \sin^2\alpha_s \right) \mathrm{d}\nu(s). \tag{2.9}$$

Here again the ambiguity in α_s at $s = q$ is immaterial. If $q = o$, then $d(q,s) < r_{\mathrm{cx}}$, and $x \mapsto d^p(x,s)$ is strictly convex at $x = q$ for all $s \in B(o,\rho)$. Henceforth, assume $q \neq o$ is a stationary point of f_p. Since $\sqrt{\Delta} d_s(q) \cot\sqrt{\Delta} d_s(q) \leq 1$, from (2.9) one can conclude that

$$\frac{\mathrm{d}^2 f_p}{\mathrm{d}t^2}(\gamma(t))|_{t=0} \geq \int d_s^{p-1}(q) \sqrt{\Delta} \cot\sqrt{\Delta} d_s(q) \ \mathrm{d}\nu(s). \tag{2.10}$$

We need to show that the above integral is positive if $\nabla f_p(q) = 0$. Let γ_{qo} denote the unit speed minimal geodesic from q to o. Let β_s denote the angle between the geodesic from q to $s \in B(o,\rho)$ and γ_{qo}. The projection of $\nabla f_p(q)$ along the direction of γ_{qo} at q is zero. Therefore, we have

$$\int d^{p-1}(q,s) \cos\beta_s \ \mathrm{d}\nu(s) = 0. \tag{2.11}$$

Note that since $d^{p-1}(q,s) = 0$ at $s = q$, the ambiguity in defining β_s at $s = q$ is immaterial in the above relation.

Our goal is to show that (2.11) is enough to guarantee positive definiteness of the Hessian of f_p at q. We do this using comparison theorems. For the moment assume that q has zero ν-measure, so that we can ignore the case $s = q$ in the following. Now, consider a ball $B(\tilde{o},\rho) \subset S^n_\Delta$. Choose a point $\tilde{q}$ inside the ball such that $\tilde{d}(\tilde{o},\tilde{q}) = d(o,q)$. Corresponding to a particular point $s \in B(o,\rho)$ choose a point $\tilde{s} \in S^n_\Delta$ such that $\tilde{d}(\tilde{q},\tilde{s}) = d(q,s)$ and $\angle\tilde{s}\tilde{q}\tilde{o} = \angle sqo = \beta_s$. Applying Theorem 2.2 to the triangles $\triangle\tilde{s}\tilde{q}\tilde{o}$ and $\triangle sqo$, we observe that $\tilde{d}(\tilde{o},\tilde{s}) \leq d(o,s) < \rho$, and therefore $\tilde{s} \in B(\tilde{o},\rho)$. Also note that since $d(o,s), d(o,q) < \rho$ and $d(q,s) < 2\rho$, the perimeter of $\triangle sqo$ is less than $4r_{\mathrm{cx}}$ and we are allowed to use Theorem 2.2.

Next, consider Figure 1 which shows the situation inside $B(\tilde{o},\rho)$. Note that $B(\tilde{o},\rho)$ is inside the larger hemisphere $H = B(\tilde{o}, \frac{\pi}{2\sqrt{\Delta}})$. Continue $\tilde{\gamma}_{\tilde{q}\tilde{o}}$, the radial geodesic segment from $\tilde{q}$ to $\tilde{o}$, till it meets the boundary of the hemisphere at $\tilde{z}$. Also continue the geodesic from $\tilde{q}$ to $\tilde{s}$ to meet the boundary of H at $\tilde{y}_{\tilde{s}}$. This latter geodesic meets the boundary of $B(\tilde{o},\rho)$ at $\tilde{y}'_{\tilde{s}}$. For the moment assume that s (equivalently $\tilde{s}$) does not lie on γ_{qo} (equivalently $\tilde{\gamma}_{\tilde{q}\tilde{o}}$) or its extensions from either

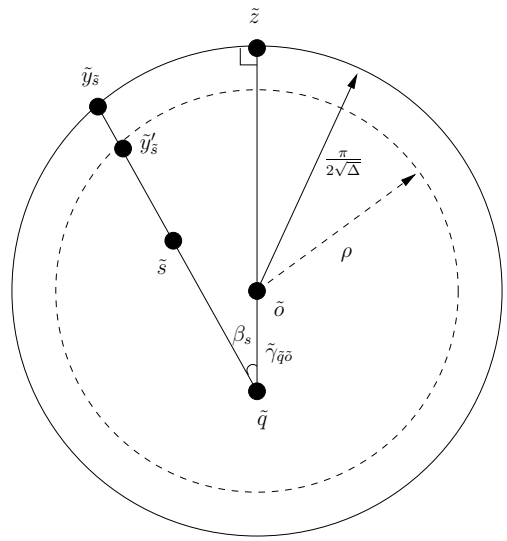

Figure 1: Virtual top view of the hemisphere $H = B(\tilde{o}, \frac{\pi}{2\sqrt{\Delta}}) \subset S^n_\Delta$ and the smaller ball $B(\tilde{o}, \rho)$. See Subsection 2.2.3 for more details.

side. Note that while the boundary of $B(\tilde{o}, \rho)$ is not totally geodesic, the boundary of the hemisphere H is totally geodesic. Consequently, we make the important observation that $\angle \tilde{q}\tilde{z}\tilde{y}_{\tilde{s}} = \frac{\pi}{2}$ (this angle is well-defined by the assumption we just made). Next, the application of the Spherical Law of Cosines twice in the triangle $\triangle \tilde{q}\tilde{y}_{\tilde{s}}\tilde{z}$, once for the right angle $\angle \tilde{q}\tilde{z}\tilde{y}_{\tilde{s}} = \frac{\pi}{2}$ and once for the angle $\angle \tilde{y}_{\tilde{s}}\tilde{q}\tilde{z} = \beta_s$ yields the crucial relation

$$\cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{y}_{\tilde{s}}) = \cos \beta_s \cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{z}). \tag{2.12}$$

We can relax the assumption just made on s, by noting that the above identity is valid as long as $s \neq q$. The above relation enables us to relate β_s to $\tilde{d}(\tilde{q}, \tilde{s})$ via a constant (i.e., $\tilde{d}(\tilde{q}, \tilde{z})$) independent of $\tilde{s}$. Note that

$$\cot t_1 - \cot t_2 \leq -(t_1 - t_2),\ 0 < t_2 \leq t_1 \leq \pi \tag{2.13}$$

and that

$$\tilde{d}(\tilde{q}, \tilde{y}_{\tilde{s}}) - \tilde{d}(\tilde{q}, \tilde{s}) = \tilde{d}(\tilde{s}, \tilde{y}_{\tilde{s}}). \tag{2.14}$$

Also note that $\tilde{d}(\tilde{s}, \tilde{y}_{\tilde{s}}) \geq \tilde{d}(\tilde{y}'_{\tilde{s}}, \tilde{y}_{\tilde{s}})$ and $\tilde{d}(\tilde{y}'_{\tilde{s}}, \tilde{y}_{\tilde{s}}) \geq \frac{\pi}{2\sqrt{\Delta}} - \rho$ (see Figure 1). Therefore, we have

$$\tilde{d}(\tilde{q}, \tilde{y}_{\tilde{s}}) - \tilde{d}(\tilde{q}, \tilde{s}) \geq \frac{\pi}{2\sqrt{\Delta}} - \rho. \tag{2.15}$$

Since $\tilde{d}(\tilde{q}, \tilde{s}) < \tilde{d}(\tilde{q}, \tilde{y}_{\tilde{s}}) < \frac{\pi}{\sqrt{\Delta}}$, from (2.13) and the above we conclude that

$$\cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{s}) \geq \cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{y}_{\tilde{s}}) + (\frac{\pi}{2} - \rho\sqrt{\Delta}). \tag{2.16}$$

Combining this with (2.12) and noting that $\rho < \frac{\pi}{2\sqrt{\Delta}}$, we have

$$\cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{s}) \geq \cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{z}) \cos \beta_s + (\frac{\pi}{2} - \rho\sqrt{\Delta}) > \cot \sqrt{\Delta}\tilde{d}(\tilde{q}, \tilde{z}) \cos \beta_s. \tag{2.17}$$

By our construction, the above implies that

$$\cot\sqrt{\Delta}d(q,s) > \cot\sqrt{\Delta}\tilde{d}(\tilde{q},\tilde{z})\cos\beta_s. \tag{2.18}$$

Plugging the above in (2.10) and using (2.11), we conclude that

$$\frac{\mathrm{d}^2 f_p}{\mathrm{d}t^2}(\gamma(t))|_{t=0} > \cot\sqrt{\Delta}\tilde{d}(\tilde{q},\tilde{z})\sqrt{\Delta}\int d^{p-1}(q,s)\cos\beta_s \mathrm{d}\nu(s) = 0; \tag{2.19}$$

that is, the Hessian of f_p is positive definite at q. One can check that the above remains valid even in the singular case that the ν-measure of q is nonzero. Since on the boundary of $B(o,\rho)$, $-\nabla f_p$ is inward-pointing, the Poincaré-Hopf Index Theorem implies that q is the only zero of ∇f_p in B. Hence, by our insideness argument in Subsection 2.2.1, q is the unique minimum of f_p in M.

It remains to consider $p=\infty$. Let $q, q' \in B(o,\rho)$ be two distinct minimizers of f_∞ and assume that $r = f_\infty(q) = f_\infty(q')$. Note that we must have $r < r_{\mathrm{cx}}$. The closed balls $\overline{B}(q,r)$ and $\overline{B}(q',r)$ must contain $\overline{\mathrm{supp}}(\nu)$. By the strict convexity of the distance function along nonradial directions, the distance between any point in $\overline{\mathrm{supp}}(\nu)$ and the midpoint of the unique minimizing geodesic from q and q' is smaller than r. This contradicts the minimality of both q and q'. Hence, the L^∞ center must be unique. This completes the proof of Theorem 2.1. □

Remark 2.3. Note that we have $\rho_{\Delta,p} = \infty$ only when $\Delta \le 0$ and $\mathrm{inj}M = \infty$. However, even in this case, Theorem 2.1 does not allow ν to be of unbounded support. The subtle point is that if $\mathrm{supp}(\nu)$ is unbounded, f_p might be infinite on the entire M, in which case our proof of the theorem fails. To avoid this situation one can require $f_p(x)$ $(1 \le p < \infty)$ to be finite at some $x \in M$ (see, e.g., Theorem 2.1 in [3]). This condition forces f_p to be finite on the entire M. Then in the case $\Delta \le 0$ and $\mathrm{inj}M = \infty$, the existence and uniqueness properties of the L^p $(1 \le p < \infty)$ center of mass with respect to a probability measure with unbounded support follow from the global convexity properties of the distance function on M.

Remark 2.4. If $\Delta > 0$, $\rho_{\Delta,p}$ for $p \ge 2$ can be dramatically larger than $\rho_{\Delta,p}$ for $1 \le p < 2$. One might wonder why we could not have a similar bound for $1 \le p < 2$. Ignoring the manageable technicalities arising from f_p not being twice continuously differentiable for $1 \le p < 2$, the main obstacle in this case is that from (2.9) we could not get a bound similar to (2.10) in which the integrand is independent of α_s. Still it might be possible to derive a better $\rho_{\Delta,p}$ for $1 \le p < 2$. Nevertheless, it is easy to construct an example to show that $\rho < r_{\mathrm{cx}}$ will not work for at least values of p equal to or close to 1. Consider the unit sphere in $\mathbb{R}^3$ with its standard metric. Let o denote the north pole. Consider three points x_1, x_2, x_3 which are located at (arc) distance r from o such that the minimal geodesics from o to the x_i's make equal angles of $\frac{\pi}{3}$ with each other. Let ν be a discrete probability measure such that $\nu(x_i) = \frac{1}{3}, i = 1,2,3$. By symmetry the triangle $\triangle x_1x_2x_3$ is an equilateral triangle whose L^2 (and in fact $L^p, p \ge 2$) center of mass is o for $r < \frac{\pi}{2}$ by Theorem 2.1. At the same time, as long as $r < \frac{\pi}{4}$, again by symmetry and the same theorem o is the only L^p mean for $1 \le p < 2$. Denote the side length of the triangle by a. The Spherical Law of Cosines determines the dependence of a on r as

$$\cos a = \cos^2 r - \frac{1}{2}\sin^2 r. \tag{2.20}$$

Note that $f_1(o) = 3r$ and $f_1(x_i) = 2a$ for $i = 1,2,3$. It is easy to check that there is a number $r_1(\approx 0.4248\pi)$, such that if $r = r_1$, then $f_1(x_i) = f_1(o)$ for $i = 1,2,3$;

hence, o can no longer be the unique L^1 mean. A direct numerical evaluation of f_1 shows that the four points o, x_1, x_2 and x_3 minimize f_1. Note that at x_i $(i = 1, 2, 3)$ the gradient of f_1 is not defined, whereas at o we have $\nabla f_1(o) = 0$. For r a little bit larger than r_1, we have $f_1(x_i) < f_1(o)$ $(i = 1, 2, 3)$. This also shows that, in this example, for $p > 1$ but close to 1 the L^p center cannot be unique.

Remark 2.5. The numbers $\mathrm{inj}M$ and Δ give some global or coarse scale information about the geometry of M, but they might mask more local or fine scale geometrical information. Consequently, Theorem 2.1 in its current form, which is typical of global results in global Riemannian geometry, can give too pessimistic estimates. However, our proof of this theorem can yield a more general version of the theorem. Let $\mathrm{inj}B(o, 2\rho)$ and $\Delta_{B(o,2\rho)}$ denote the infimum of the injectivity radii of points in $B(o, 2\rho)$ and the supremum of the sectional curvatures of M in $B(o, 2\rho)$, respectively. A careful look at the proof of Theorem 2.1 and the proofs of the comparison theorems used shows that if $\mathrm{inj}M$ and Δ are replaced, respectively, by $\mathrm{inj}B(o, 2\rho)$ and $\Delta_{B(o,2\rho)}$, then the claims of Theorem 2.1 still hold, while M is only required to be a connected Riemannian manifold and not complete, necessarily. This more general version of the theorem might be useful in applications where M is incomplete or singular at certain points but it has complete or smooth regions; see Remark 2.7 in [9]. Also note that if $\mathrm{inj}M$ and Δ are replaced by $\mathrm{inj}B(o, \rho)$ and $\Delta_{B(o,\rho)}$, respectively, then the claims of the theorem hold true for the local L^p center of mass instead of the global one (cf. Theorem 1.2 in [13]). This leaves it open whether with $\mathrm{inj}B(o, \rho)$ and $\Delta_{B(o,\rho)}$ instead of $\mathrm{inj}M$ and Δ, respectively, the claims of Theorem 2.1 hold true. We conjecture that the answer is negative.

3. Convexity

We adopt the following definition:

Definition 3.1. Let A be a subset of a Riemannian manifold. The convex hull of A, if it exists, is the intersection of all strongly convex sets containing A.

In [9], Groisser asks the question whether (global) minimization of f_2 automatically finds a zero of the vector field ∇f_2 which is inside the convex hull of the support of the underlying probability measure ν. We shall give an affirmative answer to this question with two technical precautions. First, to avoid the unnecessary technicality that an arbitrary probability measure can cause, we assume that ν is a discrete probability measure. For most applications this is a natural assumption. Second, we show that the L^p center belongs to the closure of the convex hull rather than the convex hull itself. The reason for this and the technicalities involved will become clear in the rest of this section.

Let $\nu(x_i) = \nu_i > 0$ for $1 \le i \le N$ and $\sum_{i=1}^N \nu_i = 1$, where $\{x_i\}_{i=1}^N \subset B(o, \rho)$ and $\rho < \rho_{\Delta,p}$. We allow for $N = \infty$, and in such a case by writing "for $k \le i \le N$" we mean "for all integers $i \ge k$." We also require the set $\{x_i\}_{i=1}^N$ to be bounded. The L^p center of mass of $\{x_i\}_{i=1}^N$ (with respect to ν) exists uniquely by Theorem 2.1. We denote the center by $\bar{x}_p$. Note that $\bar{x}_p$ belongs to the intersection of all strongly convex balls containing $\{x_i\}_{i=1}^N$. One would like the center of mass to belong to the convex hull of $\{x_i\}_{i=1}^N$. The proof of this fact is not immediate. The boundary of the convex hull of $\{x_i\}_{i=1}^N$ is not smooth, necessarily. An alternative object with smooth boundary is the minimal ball of $\{x_i\}_{i=1}^N$ defined as:

Definition 3.2. A closed ball of smallest radius containing the bounded set $\{x_i\}_{i=1}^N \subset M$ is called a minimal ball of $\{x_i\}_{i=1}^N$.

Remark 3.3. Very often, a minimal ball is referred to as a "circumscribed ball." However, note that, e.g., for $N = 3$ points in the Euclidean plane the minimal ball of $\{x_i\}_{i=1}^3$ might not coincide with the usual "circumscribed circle" of $\{x_i\}_{i=1}^3$. To avoid any confusion we chose the name "minimal ball".

From Theorem 2.1, one can see that if $\{x_i\}_{i=1}^N \subset B(o,\rho)$ with $\rho < r_{\mathrm{cx}}$, then the minimal ball of $\{x_i\}_{i=1}^N$ is a unique strongly convex ball with center $\bar{x}_\infty$ and radius $f_\infty(\bar{x}_\infty)$. The following theorem describes the relative position of $\bar{x}_p$ with respect to the convex hull and minimal ball of $\{x_i\}_{i=1}^N$. We mention that in [5], convexity properties of the L^2 center of mass on the sphere are addressed; however, the techniques used cannot be used in the case of an arbitrary manifold.

Theorem 3.4. *Let ν be a discrete probability measure with (finite or countable but bounded) support $\{x_i\}_{i=1}^N \subset B(o,\rho)$, where $\rho < \rho_{\Delta,p}$ (see (2.1)). Denote the L^p $(1 \le p \le \infty)$ center of mass with respect to ν by $\bar{x}_p$. In general, $\bar{x}_p$ belongs to the closure of the convex hull of $\{x_i\}_{i=1}^N$. More precisely,*

1. *For $1 < p < \infty$, if at least one of the x_i's belongs to the interior of the convex hull of $\{x_i\}_{i=1}^N$, then $\bar{x}_p$ also belongs to the interior of the convex hull; otherwise, it belongs to the closure of the convex hull.*
2. *For $p = 1$ and $p = \infty$, $\bar{x}_p$ belongs to the closure of the convex hull of $\{x_i\}_{i=1}^N$.*

Moreover, for $1 < p \le \infty$, $\bar{x}_p$ belongs to the interior of the minimal ball of $\{x_i\}_{i=1}^N$; and for $p = 1$, $\bar{x}_1$ might belong to the boundary of the ball as well as to its interior. In addition, the L^p $(1 < p \le \infty)$ center of mass with respect to any discrete probability measure whose support is a nonsingleton subset of $\{x_i\}_{i=1}^N$ belongs to the interior of the minimal ball of $\{x_i\}_{i=1}^N$.

Proof. Denote the convex hull of $\{x_i\}_{i=1}^N$ by C. Note that $\overline{C}$, the closure of C, is a subset of the strongly convex set $\overline{B}(o,\rho)$ (recall that $\rho < r_{\mathrm{cx}}$). This is enough to guarantee that $\overline{C}$ also is strongly convex. We need the following lemma:

Lemma 3.5. *Under the conditions in Theorem 3.4 on ν we have:*

1. *For $1 < p < \infty$, if at least one of x_i's, say x_1, belongs to the interior of the convex hull of $\{x_i\}_{i=1}^N$, then at every point on the boundary of the convex hull the vector field $-\nabla f_p$ is inward-pointing.*
2. *For $1 < p \le \infty$, at every point on the boundary of the minimal ball of $\{x_i\}_{i=1}^N$ (and any closed ball containing $\{x_i\}_{i=1}^N$ whose radius is smaller than r_{cx}) the vector field $-\nabla f_p$ is inward-pointing. For $p = \infty$, also $\bar{x}_\infty$ cannot lie on the boundary of any closed ball of radius smaller than r_{cx} containing $\{x_i\}_{i=1}^N$.*

Proof. Assume that x_1 belongs to the interior of C. Let x be a point on the boundary of C. Let $\gamma_i : [0, d(x,x_i)] \to M$ be the normal minimizing geodesic from x to x_i with initial (unit norm) velocity $\dot{\gamma}_i(0)$, for $1 \le i \le N$. Recall the definitions and facts about C_x in Subsection 2.1. Note that at least $\dot{\gamma}_1(0)$ belongs to the interior of C_x, and the rest of the $\dot{\gamma}_i(0)$'s belong to $\overline{C}_x$, the closure of C_x. Also note that $d(x,x_1) > 0$. By (2.5), $-\nabla f_p(x)$ is a linear combination of $\dot{\gamma}_i(0)$'s where the weight of $\dot{\gamma}_1(0)$ is positive and the weights of the rest of the $\dot{\gamma}_i(0)$'s are nonnegative. It

follows from the properties of convex cones that $-\nabla f_p(x)$ belongs to (the interior of) C_x. Hence, $-\nabla f_p(x)$ is inward-pointing along the boundary of the convex hull. This proves the first claim.

For the second part, note that the interior of any minimizing geodesic connecting any two points in the minimal ball (including points on the boundary) must belong to the interior of the minimal ball. Thus, for any point x on the boundary of the minimal ball, including those x_i's that lie on the boundary, each $\dot{\gamma}_i(0)$ (defined as before) is inward-pointing x on the boundary of the minimal ball; so is $-\nabla f_p(x)$. Our argument remains valid if the minimal ball of $\{x_i\}_{i=1}^N$ is replaced by any other closed ball containing $\{x_i\}_{i=1}^N$ whose radius is smaller than r_{cx}. We have already showed the claim of the lemma for $p = \infty$ in the beginning of Subsection 2.2.1. $\square$

Assuming that x_1 is in the interior of the convex hull, by this lemma, the vector field $-\nabla f_p$ will be inward-pointing on the boundary of the convex hull. Therefore, the minimum of $f_p|_{\overline{C}}$, the restriction of f_p to $\overline{C}$, must happen at a point q in the interior of the convex hull. This point must be a stationary point of $f_p|_{\overline{C}}$, i.e., where the projection of $\nabla f_p(q)$ onto $T_q\mathrm{int}C$ is zero. Since $\overline{C}$ is strongly convex, the unique minimizing geodesic from q to x_i (which might lie on the boundary) belongs to $\overline{C}$ for $1 \le i \le N$. Using this and since the interior of C is totally geodesic, we see that each velocity vector $\exp_q^{-1} x_i \in T_qM$ $(1 \le i \le N)$ has no component out of $T_q\mathrm{int}C$. Therefore, we must have $\nabla f_p(q) = 0$ (see (2.5)). From Theorem 2.1, ∇f_p has $\bar{x}_p$ as its only zero in $B(o, \rho)$. Therefore, $\bar{x}_p$ must belong to $\mathrm{int}C$. Since $-\nabla f_p$ is inward-pointing on the boundary of the minimal ball for $1 < p < \infty$, the argument used in Theorem 2.1 immediately shows that $\bar{x}_p$ belongs to the interior of the minimal ball of $\{x_i\}_{i=1}^N$. Note that this is independent of whether any of the x_i's belong to the interior of the minimal ball. The preceding argument can also be used to prove the final claim of the theorem.

The rest of the proof is comprised of limit arguments to address all the remaining cases. One can show that $\bar{x}_p$ depends continuously on x_1 and on p $(1 < p \le \infty)$; see Remark 3.6 below. For $p = 1$, provided we exclude the degenerate case mentioned in Theorem 2.1, again we have continuous dependence on p. We use the continuity as follows. First, for $1 < p < \infty$, let all the x_i's belong to the boundary of the convex hull and let $\langle x_1^k \rangle_k$ be a sequence of interior points of C converging to x_1. Consider a sequence of discrete probability measures $\langle \nu^k \rangle_k$ such that $\nu^k(x_1^k) = \frac{1}{2}\nu(x_1)$, $\nu^k(x_1) = \frac{1}{2}\nu(x_1)$, and $\nu^k(x_i) = \nu(x_i)$ for $2 \le i \le N$. The supports of all these measures have C as their convex hulls. Denote the L^p mean with respect to ν^k by $\bar{x}_p^k$. By continuity we have $\bar{x}_p^k \to \bar{x}_p$ as $x_1^k \to x_1$; hence, $\bar{x}_p$ belongs to $\overline{C}$.

Second, continuous dependence on p immediately implies that $\bar{x}_\infty$ must belong to $\overline{C}$. By the same argument, $\bar{x}_1$ belongs to $\overline{C}$, provided we exclude the degenerate situation. Also quite similarly, we conclude that $\bar{x}_1$ must belong to the closure of the minimal ball of $\{x_i\}_{i=1}^N$. For the degenerate situation even though the minimizers of f_1 might not be unique, they indeed belong to the interiors of the convex hull and the minimal ball. This completes the proof of the theorem. $\square$

Remark 3.6. The fact used about continuous dependence of $\bar{x}_p$ on each x_i and on p is not difficult to prove. For $2 \le p < \infty$, it follows from the inverse function theorem and the fact that $\bar{x}_p$ solves $\nabla f_p(\bar{x}_p) = 0$. For $1 \le p < 2$ and $p = \infty$, a rather more involved argument is needed to prove the continuous dependencies. We

just sketch a proof: Define $\hat{f}_p : \overline{B}(o,\rho) \to \mathbb{R}$ by $\hat{f}_p(x) = (pf_p(x))^{\frac{1}{p}}$ for $1 \le p < \infty$ and by $\hat{f}_\infty(x) = f_\infty(x)$ for $p = \infty$. Note that $\hat{f}_p$ and f_p both have $\bar{x}_p$ as their unique minimizers. One can check that for $1 \le p < \infty$, $\hat{f}_p$ is Lipschitz continuous with Lipschitz constant 1. For a sequence of reals $\langle p_k \rangle_k$ with limit p, since $\langle \hat{f}_{p_k} \rangle_{p_k}$ is a uniformly bounded and equicontinuous sequence of functions, one can extract a subsequence converging uniformly to f_p on $\overline{B}$. Using this fact, and since $\hat{f}_p$ and each of the $\hat{f}_{p_k}$'s have $\bar{x}_p$ and $\bar{x}_{p_k}$ as their unique minimizers, respectively, one can show that $\bar{x}_{p_k} \to \bar{x}_p$. Obviously, in order to guarantee uniqueness we could only consider $p_k \uparrow 2$ and also $p_k \downarrow 1$ with the extra nondegeneracy assumption. By a similar argument, continuous dependence of $\bar{x}_p$ on every x_i for $1 \le p < 2$ can be proved.

Remark 3.7. Since in the proof of the theorem we did not use the minimality of either the convex hull or the minimal ball, the "minimal ball" in the statement of the theorem can be replaced by "any *closed* ball of radius $\rho < \rho_{p,\Delta}$ containing the set $\{x_i\}_{i=1}^N$" (cf. Theorem 2.1).

Remark 3.8. Note that the L^p $(1 < p \le \infty)$ center of mass of a subset of $\{x_i\}_{i=1}^N$ belongs to the interior of the minimal ball of $\{x_i\}_{i=1}^N$ according to this theorem; however, the center might not belong to the interior of the convex hull of $\{x_i\}_{i=1}^N$ even in $\mathbb{R}^n$. This shows the stronger strict inclusion property of the minimal ball compared to the convex hull.

Remark 3.9. Another way of defining the notion of mean for manifold-valued data is the so-called extrinsic mean [3], as opposed to the Riemannian mean, which might be called the intrinsic mean. The notion of extrinsic mean is based on the idea of embedding the underlying manifold in a large Euclidean space and finding the closest-point projection operator (defined almost everywhere; see Remark 3.10, below) from the ambient space onto the embedded manifold. A simple example shows that the extrinsic mean, in general, lacks the convexity property which the Riemannian L^p mean enjoys. In Figure 2, M is a smooth closed curve passing through the origin, symmetric with respect to the vertical axis and is considered as a submanifold of $\mathbb{R}^2$. For convenience assume the lower part of M in the lower half-plane is a semicircle. Here $\mathrm{inj}M$ is half of the length of the closed curve and the sectional curvature of M is zero; therefore, $\rho_{\Delta,p} = \frac{\mathrm{inj}M}{2}$. Now, consider two symmetrically located points x_1 and x_2 in M close to the horizontal axis as shown in Figure 2. Obviously, the equal weight extrinsic L^2 mean of $\{x_1, x_2\}$ coincides with the origin (denoted by $\bar{x}_{ex}$). At the same time, in M, the midpoint of the minimal geodesic segment between x_1 and x_2 is $\bar{x}$, which belongs to the convex hull of the two points, while $\bar{x}_{ex}$ does not belong to the convex hull. It would be interesting to study the conditions under which, in a symmetric submanifold of $\mathbb{R}^n$ [8], the extrinsic mean belongs to the convex hull of the data points.

Remark 3.10 (Related to Remark 3.9). For a submanifold of a Euclidean space (as a Riemannian manifold), it is well-known that the set of points, in the ambient space, for which the closest-point projection onto the submanifold is not unique, is a closed set of zero Lebesgue measure (see, e.g., Theorem 3.2 in [3]). In general, this fact should not be interpreted as an advantage of the extrinsic mean over the intrinsic mean: In Figure 2, note that there must be two points on the straight line segment between x_1 and x_2 such that each has two closest-point projections

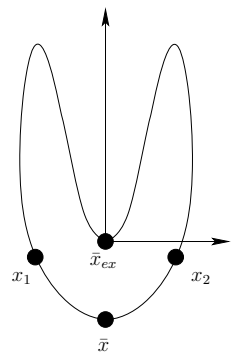

Figure 2: An example showing lack of convexity for the extrinsic mean.

on M (one on the upper part of M in the upper half-plane and one on the shortest geodesic segment between x_1 and x_2 on the lower part of M); therefore, there exist weights $0 < \nu_1, \nu_2 < 1$ with respect to which $\{x_1, x_2\}$ has two extrinsic L^2 means, while its intrinsic L^2 mean (with respect to all weights) is unique. Moreover, note that due to this nonuniqueness, the extrinsic mean of $\{x_1, x_2\}$ (with respect to the specific weights ν_1 and ν_2) lacks continuity both with respect to the data points and to the weights; consequently, if we have data points and weights which are close to these specific data points and weights, then a small change in the data points or the weights might cause the extrinsic mean (which is almost surely uniquely defined) to jump from the upper part of M to its lower arc connecting x_1 and x_2 (or vice versa), and this happens while the intrinsic mean of such data points is unique and well-behaved.

3.1. **A refined result for constant curvature manifolds.** The limit argument used in the proof of Theorem 3.4 led us to conclude that under the conditions in Theorem (3.4) the L^p center belongs to the closure of the convex hull of $\{x_i\}_{i=1}^N$. We might ask whether the center of mass can belong to the "convex hull" or "the interior of the convex hull" of $\{x_i\}_{i=1}^N$. For a finite number of points in $\mathbb{R}^n$ (and any manifold of constant curvature), the convex hull is a closed set; however, the exact structure of the convex hull of a finite set of points in a manifold of nonconstant curvature is not known. Specifically, it is not known whether the convex hull is closed or not [2, p. 231]. So even when ν is of finite support, we could not replace "closure of the convex hull" with "convex hull" immediately. Another related question is whether one can rule out the possibility of the L^p $(1 < p < \infty)$ center of mass belonging to the boundary of the convex hull. Note that for $p = 1$ and $p = \infty$ the answer will be negative even in the case of Euclidean space. However, in Theorem 3.11, we shall give an affirmative answer for $1 < p < \infty$ in the case of manifolds of constant curvature. The key property that helps us show this is the fact that constant curvature manifolds possess generic totally geodesic submanifolds of lower dimension, which is rare in the case of manifolds of nonconstant sectional curvature. We recall the following axiom of the plane from [17, p. 136]: Let M be a manifold of constant curvature of dimension n. Let W be any r-dimensional subspace of T_qM, $q \in M$. Then the set $S_\rho(W) = \{\exp_q \xi | \xi \in W, \|\xi\| < \rho\}$ is an r-dimensional

totally geodesic submanifold of M for every $\rho < \mathrm{inj}M$. Building upon the proof of Theorem 3.4 and Lemma 3.5 we have:

Theorem 3.11. *Let M be a complete Riemannian manifold of constant curvature Δ. Assume $1 < p < \infty$, define $\rho_{\Delta,p}$ as in* (2.1) *and let $\rho < \rho_{\Delta,p}$. Let ν be a discrete probability measure with support $\{x_i\}_{i=1}^N \subset B(o,\rho)$. Then the L^p center of mass with respect to ν belongs to the interior of the convex hull of $\{x_i\}_{i=1}^N$.*

Proof. As before, denote the convex hull by C. Recall our definitions and arguments in the proof of Theorem 3.4 and Lemma 3.5. We only need to show that $-\nabla f_p(x)$ is inward-pointing at each $x \in \partial C$. For each such x, as mentioned before, $\exp_x^{-1} x_i$ belongs to $\overline{C}_x$ for $1 \le i \le N$. For convenience, let $\eta_i = d^{p-2}(x, x_i) \exp_x^{-1} x_i$. Note that $-\nabla f_p(x) = \sum_i \nu_i \eta_i$ (see (2.5)). We assume that all the η_i's belong to the boundary of the cone C_x; otherwise, similar to Theorem 3.4, $-\nabla f_p(x)$ will be inward-pointing at $x \in \partial C$. First, assume that the dimension of C is n. Let all the η_i's belong to a face F of dimension $k < n$. Consequently, the η_i's and hence the $\exp_x^{-1} x_i$'s belong to a k-dimensional subspace $W \subset T_pM$, which in turn implies that $\{x_i\}_{i=1}^N$ belongs to a totally geodesic submanifold of dimension smaller than n. This is a contradiction since we assumed the dimension of the convex hull to be n. Therefore there is no face of dimension less than n which contains all the η_i's. If $\sum_i \nu_i \eta_i$ does not belong to the interior of the hull, it belongs to a face F' of dimension $k' < n$. It follows from the properties of closed convex cones and their faces that $\eta_i \in F'$ for $1 \le i \le N$, which is again a contradiction. Therefore, if the dimension of the convex hull is n, then $-\nabla f_p(x)$ must be inward-pointing at every $x \in \partial C$. Next, if the dimension of C (or C_x) is less than n, then $\{x_i\}_{i=1}^N$ lies in a totally geodesic submanifold of lower dimension and we can apply the same argument to this submanifold. Note that our claim is trivial if the data points lie on a geodesic, i.e, a totally geodesic submanifold of dimension one. Therefore, on the boundary of the convex hull of $\{x_i\}_{i=1}^N$, the vector field $-\nabla f_p$ is inward-pointing. □

Note that the assumption of $\nu_i > 0$ is crucial for this result to hold. The main obstacle in extending the preceding argument to a manifold of nonconstant sectional curvature is that in such a case, a face of the cone C_x does not necessarily map to a k-dimensional ($1 < k < n$) totally geodesic submanifold via $\exp_x$. The author has been unable either to surpass this obstacle or to give an example in which the L^p ($1 < p < \infty$) center of mass of a finite (or countable) set points belongs to the boundary of the convex hull.

4. Conclusions and open problems

The bound $\rho < \rho_{\Delta,p} = r_{\mathrm{cx}}$ in Theorem 2.1 on the radius of the ball for $p \ge 2$ is quite satisfactory, since based on the available literature it seems that this bound is also the best available estimate for the convexity radius of M. However, it is left open whether the derived bound for $1 \le p < 2$ can be improved.

Our proof of Theorem 2.1 is heavily based on the fact that $\mathrm{supp}(\nu)$ lies in a ball of radius $\rho < \rho_{\Delta,p}$. We do not know whether the uniqueness of the L^p center of mass for $p \ge 2$ holds if in the statement of Theorem 2.1 the ball $B(o,\rho)$ is replaced by any strongly convex set in M. As mentioned in Remark 2.4, for values of p close to and equal to one, this cannot be true.

Not directly related to the topics of this paper but very useful are estimates that relate the norm of the vector field ∇f_p at a point x to the distance of x from the center of mass (e.g., (4.8) in [11] or Theorem (1.5) in [13] for $p = 2$). These estimates were derived based on the sufficient but not necessary restrictions on ρ which we mentioned in Subsection 1.1. It would be very useful to derive similar estimates for the improved bound (1.2).

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