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# OPTIMAL $L^1$-BOUNDS FOR SUBMARTINGALES

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Abstract. The optimal function $f$ satisfying

$$\mathbb{E}|\sum_1^n X_i| \;\geq\; f(\mathbb{E}|X_1|,\ldots,\mathbb{E}|X_n|)$$

for every martingale $(X_1, X_1+X_2,\ldots,\sum_{i=1}^n X_i)$ is shown to be given by

$$f(a) \;=\; \max\left\{a_k - \sum_{i=1}^{k-1} a_i\right\}_{k=1}^n \cup \left\{\frac{a_k}{2}\right\}_{k=3}^n$$

for $a \in [0,\infty[^n$. A similar result is obtained for submartingales $(0, X_1, X_1+X_2,\ldots,\sum_{i=1}^n X_i)$.

The optimality proofs use a convex-analytic comparison lemma of independent interest.

## 1. Introduction and main results

This paper provides in two interesting special cases the optimal lower bounds for absolute moments of sums $S_n = \sum_{i=1}^n X_i$ given absolute moments of their increments $X_i$ and given some structural assumption on the process $(X_1,\ldots,X_n)$; see Theorems 1.1 and 1.2 below. These belong to the first few nontrivial results of their kind, despite a rather large literature on moment bounds in general. Let us introduce some notation for stating our results and for comparing them with previous ones.

For $n \in \mathbb{N} := \{1,2,3,\ldots\}$ we consider processes $(X_1,\ldots,X_n)$ of real-valued random variables and always put $S_k := \sum_{i=1}^k X_i$ for $k \in \{0,\ldots,n\}$, so that in particular $S_0 = 0$. We denote various structural assumptions on $(X_1,\ldots,X_n)$ by acronyms as follows:

IC: $X_1,\ldots,X_n$ are independent and centred,
MG: $(S_1,\ldots,S_n)$ is a martingale,
SMG: $(S_0,\ldots,S_n)$ is a submartingale or a supermartingale,

where "centred" means $\mathbb{E}X_i = 0$ for each $i$.

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For $r \in [1,\infty[$, $n \in \mathbb{N}$, $a \in [0,\infty[^n$, and with A denoting any of the assumptions IC, MG or SMG, we put

$$f_{r,\mathrm{A}}(a) := \inf\{\mathbb{E}|S_n|^r : \mathrm{A},\ \mathbb{E}|X_1|^r = a_1, \dots, \mathbb{E}|X_n|^r = a_n\}, \tag{1}$$

$$F_{r,\mathrm{A}}(a) := \sup\{\mathbb{E}|S_n|^r : \mathrm{A},\ \mathbb{E}|X_1|^r = a_1, \dots, \mathbb{E}|X_n|^r = a_n\}. \tag{2}$$

We sometimes abbreviate $f_{1,\mathrm{A}} =: f_{\mathrm{A}}$.

Thus, for example, $f_{\mathrm{MG}}(a)$ with $a \in [0,\infty[^n$ is by definition the best lower bound for $\mathbb{E}|S_n|$ given that $(S_1, \dots, S_n)$ is a martingale with $X_i = S_i - S_{i-1}$ satisfying $\mathbb{E}|X_i| = a_i$ for each $i \in \{1, \dots, n\}$.

Set theoretical correctness requires us to note that the above definitions (1) and (2), referring as they do to the nonexistent set of all processes satisfying assumption A, can be legalized by noting that the expectations and assumptions considered depend only on the laws of the processes $(X_1, \dots, X_n)$. These laws form a well-defined set on which a formally correct version of definitions (1) and (2) should be based, as in formula (14) below, which correctly defines the restriction of $f_{\mathrm{MG}}$ to $[0,\infty[^n$.

The explicit determination of the functions $f_{\mathrm{MG}}$ and $f_{\mathrm{SMG}}$ in the following two theorems is the main result of this paper.

**Theorem 1.1** (The martingale case)**.** *For $n \in \mathbb{N}$ and $a \in [0,\infty[^n$, we have*

$$f_{\mathrm{MG}}(a) = \max\Big\{a_k - \sum_{i=1}^{k-1} a_i\Big\}_{k=1}^{n} \cup \Big\{\frac{a_k}{2}\Big\}_{k=3}^{n}. \tag{3}$$

**Theorem 1.2** (The submartingale case)**.** *For $n \in \mathbb{N}$ and $a \in [0,\infty[^n$, we have*

$$f_{\mathrm{SMG}}(a) = \max\Big\{a_k - \sum_{i\neq k} a_i\Big\}_{k=1}^{n} \cup \Big\{\frac{a_k - \sum_{i<k} a_i}{2}\Big\}_{k=1}^{n-1} \cup \Big\{\frac{a_k - \sum_{i>k} a_i}{3}\Big\}_{k=3}^{n} \cup \Big\{\frac{a_k}{4}\Big\}_{k=3}^{n-1}. \tag{4}$$

Here and below $\{b_k\}_{k=n_1}^{n_2}$ denotes the possibly empty set $\{b_k : k \in \mathbb{N}, n_1 \le k \le n_2\}$ and summations as in (4) are with respect to $i \in \{1, \dots, n\}$ subject to the indicated conditions. Put less formally, Theorem 1.1, for example, says that inequality (12) below is optimal if just MG is assumed and if the right hand side is only allowed to depend on $\mathbb{E}|X_1|, \dots, \mathbb{E}|X_n|$.

Theorems 1.1 and 1.2 are proved in Section 3 using in particular a comparison lemma for certain convex functions, namely Lemma 2.2 supplied in the preparatory Section 2, which might be of independent interest. The remainder of this section contains a few remarks and some comparisons with known results.

In Theorem 1.1 we could have written $\{\frac{a_k}{2}\}_{k=1}^{n}$ instead of $\{\frac{a_k}{2}\}_{k=3}^{n}$ either by changing the proof a bit or by noting that $\frac{a_1}{2} \le a_1 - \sum_{i=1}^{1-1} a_i$ and $\frac{a_2}{2} = \frac{1}{2}(a_1 + a_2 - a_1) \le \max\{a_1, a_2 - a_1\}$. However, in the formulation chosen, the restriction of $f_{\mathrm{MG}}$ to $[0,\infty[^n$ is presented as a pointwise supremum of $n + (n-2)_+$ linear functions of which no proper subset would suffice. (To check this, one may consider $a = (a_i)_{i=1}^n$ with $a_i = \delta_{k,i}$ to prove that $a_k - \sum_{i=1}^{k-1} a_i$ cannot be omitted for $k \in \{1, \dots, n\}$ and $a_i = \delta_{1,i} + \delta_{2,i} + 3\delta_{k,i}$ to show that $\frac{a_k}{2}$ cannot be omitted for $k \in \{3, \dots, n\}$.) A similar remark applies to Theorem 1.2. (To prove here that none of the linear functions can be omitted, one may consider $a_i = \delta_{k,i}$ for $k \in \{1, \dots, n\}$, $a_i = 3\delta_{k,i} + 2\delta_{n,i}$ for $k \in \{1, \dots, n-1\}$, $a_i = \delta_{1,i} + \delta_{2,i} + 2\delta_{k,i}$ for $k \in \{3, \dots, n\}$,

and $a_i = \delta_{1,i} + \delta_{2,i} + 3\delta_{k,i} + \delta_{n,i}$ for $k \in \{3, \ldots, n-1\}$.) In particular, Theorems 1.1 and 1.2 imply the inequalities

$$\frac{1}{2} \max_{k=1}^{n} a_k \quad \leq \quad f_{\mathrm{MG}}(a) \ \leq \ \max_{k=1}^{n} a_k, \tag{5}$$

$$\frac{1}{4} \max_{k=1}^{n} a_k \quad \leq \quad f_{\mathrm{SMG}}(a) \ \leq \ \max_{k=1}^{n} a_k \tag{6}$$

for $n \in \mathbb{N}$ and $a \in [0, \infty[^n$, where the constant factors involved are easily seen to be optimal. See (21) below for more details on (6). Another consequence of Theorem 1.1 is the Kemperman & Smit (1974, statement (d)) inequality

$$\frac{1}{2n-1} \sum_{i=1}^{n} a_i \quad \leq \quad f_{\mathrm{MG}}(a) \tag{7}$$

since (3) with $\{\frac{a_k}{2}\}_{k=1}^n$ instead of $\{\frac{a_k}{2}\}_{k=3}^n$ yields $f_{\mathrm{MG}}(a) \geq \max\{a_1, \frac{a_2}{2}, \ldots, \frac{a_n}{2}\}$ $\geq \frac{1}{2n-1} a_1 + \sum_{k=2}^n \frac{2}{2n-1} \frac{a_k}{2} = \text{L.H.S.}(7)$, where in the second step we have bounded a maximum from below by a convex combination. However, depending on $a \in [0, \infty[^n$, inequality (7) can be much worse but can only be slightly better than the left hand inequality in (5).

Theorem 1.1 and its proof remain valid if we replace $(S_i)_{i=1}^n$ by $(S_i)_{i=0}^n$ in the definition of MG and in the line following (19) and $\{2, \ldots, n\}$ by $\{1, \ldots, n\}$ in the line following (13). No such remark applies to Theorem 1.2, as becomes clear by considering the submartingale $(S_1, S_2) := (-1, 0)$.

We now present all other cases known to us where one of the functions $f_{r,\mathrm{A}}$ or $F_{r,\mathrm{A}}$ with $r \in [1, \infty[$ and $\mathrm{A} \in \{\mathrm{IC}, \mathrm{MG}, \mathrm{SMG}\}$ is obvious or has been determined in the literature. For comparison we also mention some corresponding results referring to one of the following assumptions:

IS: $X_1, \ldots, X_n$ are independent and symmetrically distributed,

IIDC: $X_1, \ldots, X_n$ are independent, identically distributed, and centred,

N: No assumption; i.e., $X_1, \ldots, X_n$ are arbitrary random variables.

We make no attempt to review moment inequalities optimal in senses weaker than ours, like bounds of optimal order or optimal constants in bounds of a special form, let alone related bounds involving moments of different orders or tail probabilities.

As preparation, let $r \in [1, \infty[$ and let us consider $n \in \mathbb{N}$, $a \in [0, \infty[^n$, $p \in ]0, 1]$, and independent

$$X_i \quad \sim \quad (1-p)\delta_0 + \frac{p}{2}\big(\delta_{-a_i^{1/r}/p} + \delta_{a_i^{1/r}/p}\big)$$

for $i \in \{1, \ldots, n\}$. Then $\mathbb{E}|X_i|^r = a_i$ for each $i$ and, by bounding the probability of the event $\{X_i X_j \neq 0 \text{ for some } i \neq j\}$, we have $\lim_{p \downarrow 0} \mathbb{E}|S_n|^r = \lim_{p \downarrow 0} \mathbb{E} \sum_{i=1}^n |X_i|^r = \sum_{i=1}^n a_i =: \|a\|_1$. This yields

$$f_{r,\mathrm{A}} \quad \leq \quad \|\cdot\|_1 \ \leq \ F_{r,\mathrm{A}} \tag{8}$$

whenever A is any of the six assumptions introduced above, if in the case $\mathrm{A} = \mathrm{IIDC}$ attention is restricted to arguments $a \in [0, \infty[^n$ with $a_1 = \ldots = a_n$.

Continuing now with the exponent $r = 1$ but turning to upper bounds, we have $F_{1,\mathrm{IC}} = F_{1,\mathrm{MG}} = F_{1,\mathrm{SMG}} = \|\cdot\|_1$ by the $L^1$ triangle inequality $\mathbb{E}|S_n| \leq \sum_{i=1}^n \mathbb{E}|X_i|$ and by the right hand inequality in (8) with $r = 1$.

Coming back to lower bounds for $r = 1$, we believe that $f_{1,\mathrm{IC}}$ is in general unknown, although it is an easy exercise to show that $f_{1,\mathrm{IC}}(a) = \max_{k=1}^n a_k$ whenever $n \in \mathbb{N}$ and $a \in [0,\infty[^n$ satisfy $a_k \geq \sum_{i\neq k} a_i$ for some $k$. Somewhat surprisingly, the corresponding problem under the assumption IIDC has already been solved in Mattner (2003), namely $f_{1,\mathrm{IIDC}}(a) = c_n a_1$ for $a \in [0,\infty[^n$ with $a_1 = \ldots = a_n$, where $c_n$ is $n$ times the maximal probability of a binomial distribution with parameters $n$ and $\lfloor n/2 \rfloor / n$, and thus $c_n \sim \sqrt{\frac{2n}{\pi}}$ for $n \to \infty$.

For $r = 2$, we obviously have $f_{2,\mathrm{IC}} = F_{2,\mathrm{IC}} = f_{2,\mathrm{MG}} = F_{2,\mathrm{MG}} = \|\cdot\|_1$.

Let $r \in ]1,2[$. Then von Bahr and Esseen (1965, Theorem 1) and the right hand inequality in (8) yield $F_{r,\mathrm{IS}} = \|\cdot\|_1$.

Let $r \in [3,\infty[$. Then $\mathbb{E}|X+Y|^r \geq \mathbb{E}|X|^r + \mathbb{E}|Y|^r$ whenever $X, Y$ are independent and centred $\mathbb{R}$-valued random variables, by Cox and Kemperman (1983, Theorem 2.6). This immediately yields the remarkable result $f_{r,\mathrm{IC}} = \|\cdot\|_1$, by induction and by the left hand inequality in (8). See Pinelis (2002, Theorem 6) for an extension of the Cox-Kemperman inequality to Hilbert space valued random variables.

Finally, computing $f_{r,\mathrm{N}}$ and $F_{r,\mathrm{N}}$ for $r \in [1,\infty[$ is equivalent to determining the best lower and upper bounds for the $L^r$ norm of a sum given the norms of the summands. This is a special case of an exercise in elementary normed vector space theory solved in Mattner (2008). The solution given there yields

$$\begin{aligned} f_{r,\mathrm{N}}(a) &= \max_{k=1}^n \Big( (a_k)^{\frac{1}{r}} - \sum_{i\neq k} (a_i)^{\frac{1}{r}} \Big)_+^r, \\ F_{r,\mathrm{N}}(a) &= \Big( \sum_{i=1}^n (a_i)^{\frac{1}{r}} \Big)^r \end{aligned}$$

for $n \in \mathbb{N}$ and $a \in [0,\infty[^n$.

## 2. Auxiliary facts from convex analysis

We assume as known some standard terminology and facts as given in Rockafellar (1970), but generalized in the obvious way from $\mathbb{R}^n$ to arbitrary and possibly infinite-dimensional vector spaces over $\mathbb{R}$. We state here two lemmas used in Section 3 below.

The first lemma says that partial infima of convex functions are convex. This will be applied below only in situations where all functions considered are finite-valued, but it seems simpler to state the general case.

**Lemma 2.1.** *Let $C$ and $D$ be convex subsets of vector spaces over $\mathbb{R}$ and let $E \subset C \times D$ and $f : E \to [-\infty,\infty]$ be convex. Then the function*

$$C \ni x \quad \mapsto \quad \inf\{f(x,y) : (x,y) \in E\} \in [-\infty,\infty]$$

*is convex.*

*Proof.* Easy and well-known; compare Rockafellar (1970, pp. 38-39). □

Our second lemma reduces the pointwise comparison of certain convex functions to a comparison at distinguished points. Let $E$ be a vector space over $\mathbb{R}$. Then the *dimension* of a convex set $C \subset E$ is, by definition, the dimension of the vector subspace $F \subset E$ obtained by translating the affine hull $A$ of $C$ towards the origin. If this dimension is finite, then $A$ is topologized by translating from $F$ its usual (unique Hausdorff topological vector space) topology, and the *relative boundary*

and the *relative interior* of $C$ are then the boundary and the interior of $C$ as a subset of the topological space $A$.

**Lemma 2.2.** *Let $C \subset E$ be convex, finite-dimensional, and compact. Let $f, g : C \to \mathbb{R}$ be functions with $f$ convex and $g = \sup_{i \in I} g_i$ being the pointwise supremum of a finite family of affine functions $g_i : C \to \mathbb{R}$. Assume that $f(x) \le g(x)$ holds for every $x$ satisfying one of the two conditions:*

(9) $\quad x \in$ relative boundary of $C$,

(10) $\quad x \in$ relative interior of $C$ and $\{g_i : i \in I,\, g_i(x) = g(x)\}$ contains at least $\dim C + 1$ affinely independent functions.

*Then $f \le g$ on $C$.*

We recall that a family $(g_i : i \in J)$ is *affinely independent* if $\sum_{i \in J} \alpha_i g_i = 0$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i \in J} \alpha_i = 0$ implies $\alpha_i = 0$ for $i \in J$.

For example, to prove $f(x) := x^2 \le |x| =: g(x)$ for $x \in [-1, 1] =: C$, it is enough to consider $x \in \{-1, 1, 0\}$, by Lemma 2.2 applied with $g_1(x) := -x$ and $g_2(x) := x$. Replacing here $x^2$ by $\frac{1}{2}(x^2 + 1)$ shows that the affine independence requirement in (10) cannot be strengthened to linear independence.

*Proof of Lemma* 2.2. We may exclude the trivial cases where $I$ is empty or $C$ has at most one element. Thus $k := \dim C \ge 1$. By a translation and by choosing a basis, we may further assume that $C \subset \mathbb{R}^k = E$. Then $C$ has nonempty interior $\operatorname{int} C$ as a subset of $\mathbb{R}^k$, and we can omit the adjective "relative" in conditions (9) and (10). Also, we can consider the affine functions $g_i$ as being defined on the entire space $\mathbb{R}^k$.

For every $x \in C$, we introduce the nonempty set

$$I_x \quad := \quad \{i \in I : g_i(x) = g(x)\}.$$

Let $x_0 \in C$. Then

$$x_0 \ \in \ C_0 \quad := \quad \{x \in C : g_i(x) = g(x) \text{ for } i \in I_{x_0}\}.$$

So it suffices to prove $f \le g$ on $C_0$. Now $C_0$ is easily seen to be convex and compact, $f$ is convex, and $g$ is affine on $C_0$ as $I_{x_0} \ne \emptyset$. Hence, by Rockafellar (1970, Corollary 18.5.1), it suffices to prove $f(x) \le g(x)$ whenever $x$ is an extreme point of $C_0$. We finish this proof by showing that every extreme point $x$ of $C_0$ satisfies (9) or (10):

Let $x \in C_0$ satisfy neither (9) nor (10). Then $x \in \operatorname{int} C$ and $\{g_i : i \in I_x\}$ contains at most $k$ affinely independent functions. Also, $g_i(x) < g(x)$ for $i \in I \setminus I_x$ and thus

$$x \ \in \ U \quad := \quad \big(\operatorname{int} C\big) \cap \{y \in \mathbb{R}^k : g_i(y) < g(y) \text{ for } i \in I \setminus I_x\}$$

and $U$ is open, due to the finiteness of $I$. Let us assume for notational convenience that $I_x = \{1, \ldots, n\}$. Assuming $n \ge 2$ until further notice, we have

$$\begin{aligned} x \ \in \ A \quad &:= \quad \{y \in \mathbb{R}^k : g_1(y) = \ldots = g_n(y)\} \\ &= \quad \{y \in \mathbb{R}^k : h_1(y) = 0, \ldots, h_{n-1}(y) = 0\} \\ &= \quad \{y \in \mathbb{R}^k : h_1(y) = 0, \ldots, h_\ell(y) = 0\} \end{aligned}$$

where $h_i := g_i - g_n$ for $i \in \{1, \ldots, n-1\}$ and, reordering if necessary, $(h_1, \ldots, h_\ell)$ is a maximal linearly independent subfamily of $(h_1, \ldots, h_{n-1})$. Then $(g_1, \ldots, g_\ell, g_n)$ is affinely independent, so $\ell + 1 \le k$. As the functions $h_i$ are affine and the set $A$ is nonempty, it follows that $\dim A \ge k - \ell \ge 1$. Thus $U \cap A$ contains a nondegenerate

line segment $S$ with midpoint $x$; this conclusion is also true if $n = 1$ and $A := \mathbb{R}^k$, and we no longer assume $n \geq 2$. For $y \in U \cap A$, we have $g_i(y) = g(y)$ for every $i \in I_x$. As $I_x \supset I_{x_0}$, we conclude that $U \cap A \subset C_0$ and hence $S \subset C_0$, so that $x$ is not an extreme point of $C_0$. □

## 3. Proofs of the main results

*Proof of Theorem* 1.1. In this proof we put

$$f(a) \;:=\; f_{\mathrm{MG}}(a) \;=\; \text{L.H.S.}(3) \quad \text{and} \quad g(a) \;:=\; \text{R.H.S.}(3)$$

for $n \in \mathbb{N}$ and $a \in [0, \infty[^n$.

Let $n \in \mathbb{N}$ and let $(S_k)_{k=1}^n = (\sum_{i=1}^k X_i)_{k=1}^n$ be a martingale. Then

$$\mathbb{E}|S_k| \;\leq\; \mathbb{E}|S_n| \qquad \text{for } k \in \{1, \ldots, n\}. \tag{11}$$

Applying first the $L^1$ triangle inequality and then (11) to each of the identities

$$\begin{aligned} X_k &= S_k - \sum_{i=1}^{k-1} X_i && \text{for } k \in \{1, \ldots, n\}, \\ X_k &= S_k - S_{k-1} && \text{for } k \in \{3, \ldots, n\} \end{aligned}$$

yields $\mathbb{E}|X_k| \leq \mathbb{E}|S_n| + \sum_{i=1}^{k-1} \mathbb{E}|X_i|$ and $\mathbb{E}|X_k| \leq 2\mathbb{E}|S_n|$, respectively, and hence

$$\mathbb{E}|S_n| \;\geq\; \max\Big\{\mathbb{E}|X_k| - \sum_{i=1}^{k-1} \mathbb{E}|X_i|\Big\}_{k=1}^n \cup \Big\{\frac{\mathbb{E}|X_k|}{2}\Big\}_{k=3}^n. \tag{12}$$

This proves $f \geq g$.

It remains to prove the reversed inequality $f \leq g$, and this will eventually be done by induction. For $n \in \mathbb{N}$, we let $f_n$ and $g_n$ denote the restrictions of $f$ and $g$ to $[0, \infty[^n$.

A key observation is that each $f_n$ is convex. To see this, let us consider the canonical process $X = (X_1, \ldots, X_n) := \mathrm{id}_{\mathbb{R}^n}$ and the set of laws

$$\mathcal{P} \;:=\; \{P \in \mathrm{Prob}(\mathbb{R}^n) : X \text{ satisfies MG under } P\}.$$

The latter is convex, since $P \in \mathrm{Prob}(\mathbb{R}^n)$ belongs to $\mathcal{P}$ iff it satisfies the linear constraint

$$\int_{\mathbb{R}^n} X_i\, h(X_1, \ldots, X_{i-1})\, \mathrm{d}P \;=\; 0 \tag{13}$$

for each $i \in \{2, \ldots, n\}$ and each measurable indicator $h : \mathbb{R}^{i-1} \to \{0, 1\}$. Recalling the notation $S_n := \sum_{i=1}^n X_i$, we have

$$f_n(a) \;=\; \inf\Big\{\int_{\mathbb{R}^n} |S_n|\, \mathrm{d}P \,:\, P \in \mathcal{P},\, \int_{\mathbb{R}^n} |X_i|\, \mathrm{d}P = a_i \text{ for each } i\Big\} \tag{14}$$

for $a \in [0, \infty[^n$. Thus an application of Lemma 2.1, with $C = [0, \infty[^n$, $D = \mathcal{P}$, $E = \{(a, P) \in C \times D \,:\, \int_{\mathbb{R}^n} |X_i|\, \mathrm{d}P = a_i \text{ for each } i\}$ and $f = ((a, P) \mapsto \int_{\mathbb{R}^n} |S_n|\, \mathrm{d}P)$, yields the claimed convexity of $f_n$.

Next, the functions $f_n$ and $g_n$ are homogeneous: Since constant multiples of martingales are martingales, we have $f_n(\lambda a_1, \ldots, \lambda a_n) = \lambda f_n(a_1, \ldots, a_n)$ for $\lambda \in [0, \infty[$ and $a \in [0, \infty[^n$, and the same scaling relations obviously hold for the functions $g_n$.

By the homogeneity just observed, it suffices to prove $f_n \leq g_n$ on the simplex $C_n := \{a \in [0, \infty[^n : \sum_{i=1}^n a_i = 1\}$, for each $n \in \mathbb{N}$. The case $n = 1$ is trivial. So

let us assume that $n \in \mathbb{N}$ with $n \geq 2$ and that $f_{n-1} \leq g_{n-1}$ on $C_{n-1}$. To prove $f_n \leq g_n$ on $C_n$, we will apply Lemma 2.2 with $C = C_n$, $\dim C = n-1$, $f = f_n$, $g = g_n$, and $(g_i : i \in I) = (g_k^\nu : \nu \in \{1,2\},\, k \in K_\nu)$ with

$$\begin{aligned} g_k^1(a) &:= a_k - \sum_{i=1}^{k-1} a_i && \text{for } k \in K_1 := \{1,\dots,n\}, \\ g_k^2(a) &:= \frac{a_k}{2} && \text{for } k \in K_2 := \{3,\dots,n\} \end{aligned}$$

for $a \in [0,\infty[^n$.

Suppose first that $a \in C_n$ belongs to the relative boundary of $C_n$. Then for some $j \in \{1,\dots,n\}$ we have $a_j = 0$ and hence

$$\begin{aligned} f_n(a) &= f_{n-1}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n) && \text{[obvious by definition of } f_n] \\ &\leq g_{n-1}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n) && \text{[by induction hypothesis]} \\ &= g_n(a) && \text{[obvious by definition of } g_n]. \end{aligned}$$

Now suppose that $a \in C_n$ is as in (10); that is, $a$ belongs to the relative interior of $C_n$ and with

$$K_\nu(a) := \{k \in K_\nu : g_k^\nu(a) = g_n(a)\} \qquad \text{for } \nu \in \{1,2\} \tag{15}$$

we have at least $\dim C_n + 1 = n$ affinely independent functions in $\{g_k^\nu : \nu \in \{1,2\},\, k \in K_\nu(a)\}$, so that in particular

$$\sharp K_1(a) + \sharp K_2(a) \geq n \geq 2 \tag{16}$$

and $a_i > 0$ for every $i \in \{1,\dots,n\}$. We now must have

$$K_1(a) = \{1,2\}, \qquad K_2(a) = \{3,\dots,n\}, \tag{17}$$

for otherwise, in view of (16) and $\sharp K_2 = n-2$, there would exist $k, \ell \in K_1(a)$ with $k < \ell$ and $\ell \geq 3$, so that

$$a_\ell - \sum_{i=1}^{\ell-1} a_i = g_\ell^1(a) = g_n(a) = g_k^1(a) = a_k - \sum_{i=1}^{k-1} a_i = 2a_k - \sum_{i=1}^{k} a_i,$$

yielding

$$a_\ell = 2a_k + \sum_{i=k+1}^{\ell-1} a_i > 2\Big(a_k - \sum_{i=1}^{k-1} a_i\Big),$$

since one of the two sums above is nonempty, and hence the contradiction

$$g_n(a) = g_k^1(a) = a_k - \sum_{i=1}^{k-1} a_i < \frac{a_\ell}{2} = g_\ell^2(a) \leq g_n(a).$$

By (17) we have $g_n(a) = a_1 = a_2 - a_1 = \frac{a_3}{2} = \ldots = \frac{a_n}{2}$. and thus

$$a_1 = \frac{a_2}{2} = \ldots = \frac{a_n}{2}. \tag{18}$$

To prove $f_n(a) \leq g_n(a)$ in this case, let us consider $p \in ]0,1[$ and independent random variables $Y_1,\dots,Y_n$ with

$$Y_1 \sim \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1, \qquad Y_i \sim (1-p)\delta_0 + p\delta_{1/p} \qquad \text{for } i \in \{2,\dots,n\},$$

and let us put

$$S_0 := 0, \quad S_k := \prod_{i=1}^{k} Y_i, \quad X_k := S_k - S_{k-1} \quad \text{for } k \in \{1, \dots, n\}. \tag{19}$$

Then, since $\mathbb{E}Y_i = 1$ for $i \geq 2$, the process $(S_i)_{i=1}^n$ is a martingale. We have $\mathbb{E}|S_i| = 1$ for $i \in \{1, \dots, n\}$ and hence $\mathbb{E}|X_1| = 1$ and

$$\mathbb{E}|X_i| \;=\; \mathbb{E}|Y_i - 1|\mathbb{E}|S_{i-1}| \;=\; \big((1-p) + p(\frac{1}{p} - 1)\big) \cdot 1 = 2 - 2p$$

for $i \in \{2, \dots, n\}$. Thus

$$f_n(1, 2-2p, \dots, 2-2p) \;\leq\; 1.$$

By the continuity of the convex function $f_n$ on the open set $]0, \infty[^n$ (compare Rockafellar (1970, p. 82)), we can let $p \to 0$ to deduce $f_n(1, 2, \dots, 2) \leq 1$. Hence, by the homogeneity of $f_n$, we get

$$f_n(a) \;\leq\; a_1 \;=\; g_n(a)$$

for the $a$ satisfying (18).

By the above, an application of Lemma 2.2 yields $f_n \leq g_n$ on $C_n$, which completes our inductive proof of $f \leq g$. □

*Proof of Theorem* 1.2. This is parallel to but more complicated than the previous proof we assume the reader has studied. We may and do replace "submartingale or supermartingale" in the definition of SMG by "submartingale". We then put

$$f(a) \;:=\; f_{\mathrm{SMG}}(a) \;=\; \text{L.H.S.}(4) \quad \text{and} \quad g(a) \;:=\; \text{R.H.S.}(4)$$

for $n \in \mathbb{N}$ and $a \in [0, \infty[^n$.

Let $n \in \mathbb{N}$ and let $(S_k)_{k=0}^n = (\sum_{i=1}^k X_i)_{k=0}^n$ be a submartingale. Then

$$\mathbb{E}|S_k| \;\leq\; 2\mathbb{E}|S_n| \qquad \text{for } k \in \{1, \dots, n-1\} \tag{20}$$

by Doob (1953, page 311, Theorem 3.1 (ii)) or by noting that $(0, S_k, S_n)$ is a submartingale; that is, $\mathbb{E}S_k \geq 0$ and $\mathbb{E}\big(S_n \big| S_k\big) \geq S_k$, and so $\mathbb{E}|S_k| \leq \mathbb{E}|S_k| + \mathbb{E}S_k = 2\mathbb{E}(S_k)_+ \leq 2\mathbb{E}(\mathbb{E}\big(S_n \big| S_k\big))_+ \leq 2\mathbb{E}\mathbb{E}\big((S_n)_+ \big| S_k\big) \leq 2\mathbb{E}|S_n|$.

Now applying first the $L^1$ triangle inequality to each of the identities

$$\begin{aligned}
X_k &= S_n - \sum_{i \neq k} X_i && \text{for } k \in \{1, \dots, n\},\\
X_k &= S_k - \sum_{i < k} X_i && \text{for } k \in \{1, \dots, n-1\},\\
X_k &= S_n - S_{k-1} - \sum_{i > k} X_i && \text{for } k \in \{3, \dots, n\},\\
X_k &= S_k - S_{k-1} && \text{for } k \in \{3, \dots, n-1\}
\end{aligned}$$

and then (20) to the resulting inequalities in the last three groups yields $\mathbb{E}|X_k| \leq \mathbb{E}|S_n| + \sum_{i \neq k} \mathbb{E}|X_i|$, $\mathbb{E}|X_k| \leq 2\mathbb{E}|S_n| + \sum_{i<k} \mathbb{E}|X_i|$, $\mathbb{E}|X_k| \leq 3\mathbb{E}|S_n| + \sum_{i>k} \mathbb{E}|X_i|$, and $\mathbb{E}|X_k| \leq 4\mathbb{E}|S_n|$, respectively, and hence the analogue to (12) proving $f \geq g$ in the present case.

To prove $f \leq g$, let $f_n$ and $g_n$ denote the restrictions of $f$ and $g$ to $[0, \infty[^n$.

We have

$$g_n(a) \;\geq\; \max\Big\{\frac{a_1}{2}\Big\} \cup \Big\{\frac{a_k}{4}\Big\}_{k=2}^{n-1} \cup \Big\{\frac{a_n}{3}\Big\} \;\geq\; \frac{1}{4}\max_{k=1}^{n} a_k \tag{21}$$

for $n \in \mathbb{N}$ and $a \in [0,\infty[^n$, trivially if $n = 1$ and otherwise since $g_n(a) \geq \frac{a_1}{2}$, $g_n(a) \geq \max\{\frac{a_1}{2}, \frac{a_2-a_1}{2}\} \geq \frac{a_2}{4}$ if $n \geq 3$, $g_n(a) \geq \frac{a_k}{4}$ if $k \in \{3,\dots,n-1\}$, $g_2(a) \geq \max\{a_2 - a_1, \frac{a_1}{2}\} \geq \frac{a_2}{3}$, and $g_n(a) \geq \frac{a_n}{3}$ if $n \geq 3$.

Each $f_n$ is convex as in the previous proof, where we only have to replace the equality sign in (13) by $\leq$ and $\{2,\dots,n\}$ by $\{1,\dots,n\}$, and the functions $f_n$ and $g_n$ are homogeneous. Proceeding again by induction, we apply Lemma 2.2 as above, with $(g_i : i \in I) = (g_k^\nu : \nu \in \{1,2,3,4\},\, k \in K_\nu)$ where

$$\begin{aligned}
g_k^1(a) &:= a_k - \sum_{i\neq k} a_i && \text{for } k \in K_1 := \{1,\dots,n\},\\
g_k^2(a) &:= \frac{1}{2}\Big(a_k - \sum_{i<k} a_i\Big) && \text{for } k \in K_2 := \{1,\dots,n-1\},\\
g_k^3(a) &:= \frac{1}{3}\Big(a_k - \sum_{i>k} a_i\Big) && \text{for } k \in K_3 := \{3,\dots,n\},\\
g_k^4(a) &:= \frac{a_k}{4} && \text{for } k \in K_4 := \{3,\dots,n-1\}
\end{aligned}$$

for $a \in [0,\infty[^n$. Trivially, $f_1 \leq g_1$. So let $n \in \mathbb{N}$ with $n \geq 2$. As above, the induction hypothesis $g_{n-1} \leq f_{n-1}$ yields $f_n(a) \leq g_n(a)$ for $a$ belonging to the relative boundary of $C_n$. Hence, defining $K_\nu(a)$ as in (15) but now with $\nu \in \{1,2,3,4\}$, we assume for the rest of this proof that $a \in ]0,\infty[^n$ and that we have at least $\dim C_n + 1 = n$ affinely independent functions in $\{g_k^\nu : \nu \in \{1,2,3,4\},\, k \in K_\nu(a)\}$, so that in particular

$$\sharp K_1(a) + \sharp K_2(a) + \sharp K_3(a) + \sharp K_4(a) \;\geq\; n \;\geq\; 2. \tag{22}$$

We will now prove in twelve steps that one of the two conditions

$$g_n(a) \;=\; \frac{a_1}{2} \;=\; \frac{a_2}{4} \;=\; \dots \;=\; \frac{a_{n-1}}{4} \;=\; \frac{a_n}{3}, \tag{23}$$

$$g_n(a) \;=\; \frac{a_1}{2} \;=\; \frac{a_2}{4} \;=\; \dots \;=\; \frac{a_{n-1}}{4} \;=\; a_n \tag{24}$$

holds. (For $n = 2$, condition (23) reads $g_n(a) = \frac{a_1}{2} = \frac{a_2}{3}$. Similarly for (24).)

*Step* 1. $g_n(a) > 0$.

*Proof.* $g_n(a) \geq g_1^2(a) = \frac{a_1}{2} > 0$. □

*Step* 2. $\sharp K_1(a) \leq 1$.

*Proof.* Otherwise there are $k, \ell \in \{1,\dots,n\}$ with $k \neq \ell$ and $g_k^1(a) = g_\ell^1(a) = g_n(a)$. The first equation reads

$$a_k - \sum_{i\neq k} a_i \;=\; a_\ell - \sum_{i\neq \ell} a_i$$

and yields $a_k - a_\ell = a_\ell - a_k$; hence $a_k - a_\ell = 0$, contradicting Step 1 through

$$g_n(a) \;=\; g_k^1(a) \;=\; a_k - \sum_{i\neq k} a_i \;\leq\; a_k - a_\ell \;=\; 0.$$ □

*Step* 3. $\sharp K_3(a) \leq 1$.

*Proof.* Otherwise there are $k, \ell \in \{3, \dots, n\}$ with $k < \ell$ and $g_k^3(a) = g_\ell^3(a) = g_n(a)$. Hence $a_k - \sum_{i>k} a_i = a_\ell - \sum_{i>\ell} a_i$ and thus

$$a_k \;=\; a_\ell + \sum_{i=k+1}^{\ell} a_i \;\geq\; 2a_\ell,$$

yielding the contradiction

$$g_n(a) \;=\; g_\ell^3(a) \;=\; \frac{1}{3}\Big(a_\ell - \sum_{i>\ell} a_i\Big) \;\leq\; \frac{a_\ell}{3} \;<\; \frac{a_\ell}{2} \;\leq\; \frac{a_k}{4} \;=\; g_k^4(a) \;\leq\; g_n(a). \qquad \square$$

*Step* 4. $\sharp K_2(a) \geq 1$.

*Proof.* Otherwise, by Steps 2 and 3, we would have L.H.S.(22) $\leq 1 + 0 + \min\{1, n-2\} + \max\{0, n-3\} = n-1$. $\square$

*Step* 5. *If* $\sharp K_2(a) \geq 2$, *then* $K_2(a) = \{1, 2\}$.

*Proof.* Otherwise there are $k, \ell \in \{1, \dots, n-1\}$ with $k < \ell$, $\ell \geq 3$, and $g_k^2(a) = g_\ell^2(a) = g_n(a)$. Hence $a_\ell - \sum_{i<\ell} a_i = a_k - \sum_{i<k} a_i = 2a_k - \sum_{i\leq k} a_i$, yielding

$$a_\ell \;=\; 2a_k + \sum_{i=k+1}^{\ell-1} a_i \;>\; 2\Big(a_k - \sum_{i=1}^{k-1} a_i\Big),$$

since one of the two sums above is nonempty, and hence the contradiction

$$g_n(a) = g_k^2(a) \;=\; \frac{1}{2}\Big(a_k - \sum_{i=1}^{k-1} a_i\Big) \;<\; \frac{a_\ell}{4} \;=\; g_\ell^4(a) \;\leq\; g_n(a). \qquad \square$$

*Step* 6. *If* $K_2(a) = \{1, 2\}$, *then* $K_1(a) = \emptyset$ *or both* $n = 3$ *and* $K_1(a) = \{2\}$.

*Proof.* Let $K_2(a) = \{1, 2\}$. Then $n \geq 3$. For $1 \in K_1(a)$ we get

$$g_n(a) \;=\; g_1^1(a) \;=\; a_1 - \sum_{i\neq 1} a_i \;\leq\; a_1 - a_2 \;=\; -2g_2^2(a) \;=\; -2g_n(a),$$

contradicting Step 1. For $3 \leq k \in K_1(a)$ we get

$$\begin{aligned} g_n(a) \;=\; a_k - \sum_{i\neq k} a_i \;&\leq\; a_k - a_1 - a_2 \\ &=\; a_k - 4g_1^2(a) - 2g_2^2(a) \;=\; a_k - 6g_n(a), \end{aligned}$$

contradicting (21) and $a_k > 0$. If $K_1(a) = \{2\}$, then

$$g_n(a) \;=\; g_2^2(a) \;=\; g_2^1(a) \;=\; 2g_2^2(a) - \sum_{i>2} a_i \;=\; 2g_n(a) - \sum_{i>2} a_i$$

yields $a_k \leq \sum_{i>2} a_i = g_n(a)$ for $k \geq 3$, and hence $K_3(a) = K_4(a) = \emptyset$, implying $n = 3$ in view of (22). $\square$

*Step* 7. $K_4(a) = \{3, \dots, n-1\}$.

*Proof.* This is trivial if $n \leq 3$. For $n \geq 4$, inequality (22) and Steps 2, 3, 5 and 6 yield $n \leq 3 + \sharp K_4(a)$ and hence the claim. $\square$

*Step* 8. $K_3(a) \subset \{n-1, n\}$.

*Proof.* If $k \in \{3, \dots, n-2\}$, then Step 7 implies $k, k+1 \in K_4(a)$ and thus $g_k^3(a) < \frac{1}{3}(a_k - a_{k+1}) = \frac{4}{3}(g_k^4(a) - g_{k+1}^4(a)) = 0$; hence $k \notin K_3(a)$. $\square$

*Step* 9. *If* $n \geq 4$, *then* $K_1(a) = \emptyset$.

*Proof.* Let $n \geq 4$ and $k \in K_1(a)$. Then $a_k = \max_{i=1}^n a_i$. If $\ell \in \{3, \dots, n-1\}$, then Step 7 yields $4g_n(a) = a_\ell$, and, using (21), we get $a_k \leq a_\ell$ and thus $k = \ell$, for otherwise $g_k^1(a) \leq a_k - a_\ell \leq 0$. For $n \geq 5$ we thus get a contradiction by considering $\ell = 3$ and $\ell = 4$.

So let $n = 4$. Then $K_1(a) = \{3\}$; hence $\sharp K_2(a) = 1$ by Steps 4 to 6, $K_3(a) \subset \{3, 4\}$ by Step 8, and $K_4(a) = \{3\}$ by Step 7. If $4 \in K_3(a)$, then $g_4^3(a) = g_n(a) = g_3^4(a)$ yields $a_4 = \frac{3}{4}a_3$ and hence the contradiction

$$g_n(a) \;=\; g_3^1(a) \;<\; a_3 - a_4 \;=\; \frac{a_3}{4} \;=\; g_n(a).$$

Thus, in view of (22), we must have $K_3(a) = \{3\}$. Hence $g_3^1(a) = g_3^3(a)$ and thus $a_1 + a_2 = \frac{2}{3}(a_3 - a_4)$, so that for $\ell \in \{1, 2\}$ we get $g_\ell^2(a) < \frac{a_1+a_2}{2} = g_3^3(a) = g_n(a)$. Hence $K_2(a) = \{3\}$. So $G := \{g_k^\nu : \nu \in \{1, 2, 3, 4\}, k \in K_\nu(a)\} = \{g_3^1, g_3^2, g_3^3, g_3^4\}$, but the identity $g_3^1 - 2g_3^2 - 3g_3^3 + 4g_3^4 = 0$ shows that $G$ does not contain $n = 4$ affinely independent functions, contrary to our assumption preceding (22). □

*Step* 10. *If* $n \geq 4$, *then* (23) *or* (24).

*Proof.* The assumption together with (22) and Steps 3, 5, and 7 to 9 imply $K_1(a) = \emptyset$, $K_2(a) = \{1, 2\}$, $K_4(a) = \{3, \dots, n-1\}$, and $K_3(a) = \{n-1\}$ or $K_3(a) = \{n\}$. The second possibility yields (23); the first yields (24). □

*Step* 11. *If* $n = 3$, *then* (23) *or* (24).

*Proof.* Let $n = 3$. Then $K_4(a) = \emptyset$, $K_3(a) \subset \{3\}$ and, using Step 14, $\emptyset \neq K_2(a) \subset \{1, 2\}$. If $K_3(a) = \emptyset$, then (22) and Steps 2, 5 and 6 yield $K_1(a) = \{2\}$ and $K_2(A) = \{1, 2\}$, and hence (24). So let $K_3(a) = \{3\}$. Then $K_1(a) = \emptyset$, for otherwise either $3 \in K_1(a)$ yielding $\frac{a_3}{3} = g_n(a) = a_3 - (a_1 + a_2)$ and hence the contradiction $g_n(a) = \frac{a_3}{3} = \frac{a_1+a_2}{2} > \max\{\frac{a_1}{2}, \frac{a_2-a_2}{2}\} = g_n(a)$ or there is a $k \in \{1, 2\} \cap K_1(a)$ yielding $\frac{a_3}{3} = g_n(a) = |a_2 - a_1| - a_3$ and hence the contradiction $g_n(a) = \frac{a_3}{3} = \frac{1}{4}|a_2 - a_1| \leq \frac{1}{2}\max\{\frac{a_1}{2}, \frac{a_1+a_2}{2}\} = \frac{1}{2}g_n(a)$. Thus (22) yields $K_2(a) = \{1, 2\}$ and (23) follows. □

*Step* 12. *If* $n = 2$, *then* (23) *or* (24).

*Proof.* Let $n = 2$. Then $K_3(a) = K_4(a) = \emptyset$, and Step 4 yields $K_2(a) = \{1\}$. Thus (22) and Step 2 yield either $K_1(a) = \{1\}$ and hence (24) or $K_1(a) = \{2\}$ and hence (23). □

This completes our proof that (23) or (24) holds. To prove $f_n(a) \leq g_n(a)$ in each of these cases, we will consider simple modifications of the martingales used in the proof of Theorem 1.1. Let $p$, $Y_1, \dots, Y_n$, $S_0, \dots, S_{n-1}$ and $X_1, \dots, X_{n-1}$ be as in (19) and its preceding four lines.

In case of (23) we define

$$S_n := \big(S_{n-1}Y_n\big)_+ \quad \text{and} \quad X_n := S_n - S_{n-1} = (Y_n - 1)(S_{n-1})_+ + (S_{n-1})_-.$$

Then $(S_i)_{i=0}^n$ is a submartingale, and we have $\mathbb{E}|S_n| = \mathbb{E}Y_n\mathbb{E}(S_{n-1})_+ = \frac{1}{2}$, $\mathbb{E}|X_1| = 1$, $\mathbb{E}|X_i| = 2 - 2p$ for $i \in \{2, \dots, n-1\}$, and

$$\mathbb{E}|X_n| \;=\; \mathbb{E}|Y_n - 1|\,\mathbb{E}(S_{n-1})_+ + \mathbb{E}(S_{n-1})_- \;=\; (2 - 2p)\frac{1}{2} + \frac{1}{2} \;=\; \frac{3}{2} - p.$$

Thus

$$f_n(1, 2-2p, \ldots, 2-2p, \frac{3}{2}-p) \quad \leq \quad \frac{1}{2}$$

and hence $f_n(1, 2, \ldots, 2, \frac{3}{2}) \leq \frac{1}{2}$, yielding $f_n(a) \leq \frac{a_1}{2} = g_n(a)$.

In case of (24) we define

$$S_n := \big(S_{n-1}\big)_+ \quad \text{and} \quad X_n := S_n - S_{n-1} = \big(S_{n-1}\big)_-.$$

Then $(S_i)_{i=0}^n$ is a submartingale with $\mathbb{E}|S_n| = \frac{1}{2}\mathbb{E}|S_{n-1}| = \frac{1}{2}$, $\mathbb{E}|X_1| = 1$, $\mathbb{E}|X_i| = 2-2p$ for $i \in \{2, \ldots, n-1\}$, and $\mathbb{E}|X_n| = \frac{1}{2}\mathbb{E}|S_{n-1}| = \frac{1}{2}$. Thus

$$f_n(1, 2-2p, \ldots, 2-2p, \frac{1}{2}) \quad \leq \quad \frac{1}{2}$$

and hence $f_n(1, 2, \ldots, 2, \frac{1}{2}) \leq \frac{1}{2}$, again yielding $f_n(a) \leq \frac{a_1}{2} = g_n(a)$. □

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