THE 2-LOG-CONVEXITY OF THE APÉRY NUMBERS

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(Communicated by Jim Haglund)

Abstract. We present an approach to proving the 2-log-convexity of sequences satisfying three-term recurrence relations. We show that the Apéry numbers, the Cohen-Rhin numbers, the Motzkin numbers, the Fine numbers, the Franel numbers of orders 3 and 4 and the large Schröder numbers are all 2-log-convex. Numerical evidence suggests that all these sequences are $k$-log-convex for any $k \geq 1$ possibly except for a constant number of terms at the beginning.

1. Introduction

In his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Apéry [2] introduced the numbers $A_n$ and $B_n$ as given by

\begin{align}
A_n &= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \\
B_n &= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.
\end{align}

The numbers $A_n$ and $B_n$ are often called the Apéry numbers. It has been shown by Apéry [2] that $A_n$ and $B_n$ satisfy the following three-term recurrence relations for $n \geq 2$:

\begin{align}
A_n &= \frac{34n^3 - 51n^2 + 27n - 5}{n^3} A_{n-1} - \frac{(n-1)^3}{n^3} A_{n-2}, \\
B_n &= \frac{11n^2 - 11n + 3}{n^2} B_{n-1} + \frac{(n-1)^2}{n^2} B_{n-2},
\end{align}

where $A_0 = 1, A_1 = 5, B_0 = 1, B_1 = 3$; see also [10, 13]. Congruences of the Apéry numbers have been investigated by Ahlgren, Ekhad, Ono, and Zeilberger [1], Beukers [3, 4], Chowla, Cowles and Cowles [5] and Gessel [9]. Note that the recurrence relations (1.3) and (1.4) can be derived by using Zeilberger’s algorithm [14].

Received by the editors December 4, 2009.

2010 Mathematics Subject Classification. Primary 05A20; Secondary 11B37, 11B83.

Key words and phrases. Apéry number, log-convexity, 2-log-convexity, infinite log-convexity.

The authors wish to thank the referee, Tomislav Došlić, and Tanguy Rivoal for helpful comments.

This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

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Cohen [6] and Rhin obtained the following recurrence relation of the numbers $U_n$ in connection with the rational approximation of $\zeta(4)$; see also [11]:

\begin{equation}
U_{n+1} = R(n)U_n + G(n)U_{n-1}, \quad n \geq 1,
\end{equation}

where $U_0 = 1$, $U_1 = 12$ and

\begin{equation*}
R(n) = \frac{3(2n+1)(3n^2 + 3n + 1)(15n^2 + 15n + 4)}{(n+1)^5}, \quad G(n) = \frac{3n^3(3n-1)(3n+1)}{(n+1)^5}.
\end{equation*}

Expressions of $U_n$ as double sums of products of binomial coefficients have been derived by Krattenthaler and Rivoal [11] and Zudilin [15, 16].

In this paper, we shall establish the 2-log-convexity of the sequences of the Apery numbers $A_n$, $B_n$, the Cohen-Rhin numbers $U_n$ and some other combinatorial sequences based on the three-term recurrence relations. Recall that an infinite positive sequence $\{a_n\}_{n=0}^\infty$ is said to be log-convex if for all $n \geq 1$,

\begin{equation}
a_n^2 \leq a_{n-1}a_{n+1}.
\end{equation}

We say that $\{a_n\}_{n=0}^\infty$ is 2-log-convex if $\{a_n\}_{n=0}^\infty$ is log-convex and for all $n \geq 1$,

\begin{equation}
(a_n a_{n+2} - a_{n+1}^2)^2 \leq (a_{n-1}a_{n+1} - a_n^2)(a_{n+1}a_{n+3} - a_{n+2}^2).
\end{equation}

Meanwhile, the sequence $\{a_n\}_{n=0}^\infty$ is called strictly log-convex (2-log-convex) if the inequality in (1.6) ((1.7)) is strict for all $n \geq 1$. Došljić [7] proved the log-convexity of $A_n$ by induction. In fact, using similar arguments one can show that $\{B_n\}_{n=0}^\infty$ and $\{U_n\}_{n=0}^\infty$ are log-convex.

This paper is organized as follows. In Section 2, we give a general framework to prove the 2-log-convexity of a sequence $\{S_n\}_{n=0}^\infty$ based on a lower bound $f_n$ and an upper bound $g_n$ for the ratio $S_n/S_{n-1}$, where the numbers $S_n$ satisfy a three-term recurrence relation. Section 3 demonstrates how to find the bounds $f_n$ and $g_n$. Section 4 is devoted to computations of the upper bounds for the ratios $A_n/A_{n-1}$, $B_n/B_{n-1}$ and $U_n/U_{n-1}$. In Section 5, we show that the sequences of $A_n$, $B_n$, $U_n$, the Motzkin numbers, the Fine numbers, the Franel numbers of orders 3 and 4 and the large Schröder numbers are all 2-log-convex. We conclude this paper with a conjecture on the infinite log-convexity in the spirit of the infinite log-concavity introduced by Moll [12].

2. A criterion

In this section, we present a criterion for the 2-log-convexity of a sequence $\{S_n\}_{n=0}^\infty$ satisfying a three-term recurrence relation. We need the assumption that the ratio $S_n/S_{n-1}$ has a lower bound $f_n$ and an upper bound $g_n$ along with an additional condition on $g_n$.

**Theorem 2.1.** Suppose $\{S_n\}_{n=0}^\infty$ is a positive log-convex sequence that satisfies the recurrence relation

\begin{equation}
S_n = b(n)S_{n-1} + c(n)S_{n-2}
\end{equation}

for $n \geq 2$. Let

\begin{equation*}
a_3(n) = 2b(n+2)b^2(n+1) + 2b(n+1)c(n+2) - b^3(n+1)
- b(n+1)b(n+2)b(n+3) - b(n+3)c(n+2) - c(n+3)b(n+1),
\end{equation*}

where $b(0) = 1$, $b(1) = 3$, $c(0) = 2$, $c(1) = 6$ and $f(n) = a_3(n)/a_2(n+1)$. Then $\{S_n\}_{n=0}^\infty$ is 2-log-convex.
\[ a_2(n) = 4b(n + 1)b(n + 2)c(n + 1) + 2c(n + 1)c(n + 2) + b^2(n + 1)b(n + 2)b(n + 3) \]
\[ + b(n + 1)b(n + 3)c(n + 2) + b^2(n + 1)c(n + 3) - 3c(n + 1)b^2(n + 1) \]
\[ - b(n + 3)b(n + 2)c(n + 1) - c(n + 3)c(n + 1) - b^2(n + 2)b^2(n + 1) \]
\[ - 2b(n + 2)b(n + 1)c(n + 2) - c^2(n + 2), \]
\[ a_1(n) = - c(n + 1)(2b(n + 2)c(n + 2) - 2b(n + 2)c(n + 1) \]
\[ - 2b(n + 3)b(n + 2)b(n + 1) - b(n + 3)c(n + 2) - 2c(n + 3)b(n + 1) \]
\[ + 3c(n + 1)b(n + 1) + 2b^2(n + 2)b(n + 1)), \]
\[ a_0(n) = - c^2(n + 1) (c(n + 1) - b(n + 2)b(n + 3) - c(n + 3) + b^2(n + 2)) \]
and
\[ \Delta(n) = 4a_3^2(n) - 12a_1(n)a_3(n). \] Assume that \( a_3(n) < 0 \) and \( \Delta(n) > 0 \) for all \( n \geq N \), where \( N \) is a positive integer. If there exist \( f_n \) and \( g_n \) such that for all \( n \geq N \),
\[ (C_1) \quad f_n \leq \frac{S_n}{S_{n-1}} < g_n, \]
\[ (C_2) \quad f_n \geq \frac{-2a_2(n) - \sqrt{\Delta(n)}}{6a_3(n)}, \]
\[ (C_3) \quad a_3(n)g_n^3 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n) > 0, \]
then \( \{S_n\}_{n=N}^{\infty} \) is strictly 2-log-convex; that is, for \( n \geq N \),
\[ (2.2) \quad (S_{n-1}S_{n+1} - S_n^2) (S_{n+1}S_{n+3} - S_{n+2}^2) > (S_nS_{n+2} - S_{n+1}^2)^2. \]

**Proof.** By the recurrence relation (2.1), we have
\[ (S_{n-1}S_{n+1} - S_n^2) (S_{n+1}S_{n+3} - S_{n+2}^2) - (S_nS_{n+2} - S_{n+1}^2)^2 \]
\[ = S_{n+1} (2S_{n+1}S_{n+2} + S_{n-1}S_{n+1}S_{n+3} - S_{n+1} + S_{n+3}^2 - S_{n+2}^2 - S_{n-1}S_{n+2}) \]
\[ = S_{n+1} (a_3(n)S_n^3 + a_2(n)S_n^2S_{n-1} + a_1(n)S_nS_{n-1}^2 + a_0(n)S_{n-1}^3). \]
Since \( \{S_n\}_{n=0}^{\infty} \) is a positive sequence, in order to prove (2.2), it suffices to show that for all \( n \geq N \),
\[ (2.3) \quad a_3(n) \left( \frac{S_n}{S_{n-1}} \right)^3 + a_2(n) \left( \frac{S_n}{S_{n-1}} \right)^2 + a_1(n) \frac{S_n}{S_{n-1}} + a_0(n) > 0. \]
Consider the polynomial \( f(x) = a_3(n)x^3 + a_2(n)x^2 + a_1(n)x + a_0(n) \). Note that
\[ f'(x) = 3a_3(n)x^2 + 2a_2(n)x + a_1(n). \]
Since \( a_3(n) < 0 \) and \( \Delta(n) > 0 \) for all \( n \geq N \), we see that the quadratic function \( f'(x) \) is negative for \( x > \frac{-2a_2(n) - \sqrt{\Delta(n)}}{6a_3(n)} \). Thus, \( f(x) \) is strictly decreasing on the interval \( \left[ \frac{-2a_2(n) - \sqrt{\Delta(n)}}{6a_3(n)}, +\infty \right) \). From the assumption \( g_n > f_n \geq \frac{-2a_2(n) - \sqrt{\Delta(n)}}{6a_3(n)} \), it follows that \( f(x) \) is strictly decreasing on the interval \( [f_n, g_n] \). Since \( \frac{S_n}{S_{n-1}} \in [f_n, g_n] \),
it remains to show that \( f(g_n) > 0 \) for any \( n \geq N \), which is equivalent to condition \((C_3)\); that is,
\[
a_3(n)g_n^3 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n) > 0
\]
for any \( n \geq N \). This completes the proof. \( \square \)

3. A HEURISTIC APPROACH TO COMPUTING THE BOUNDS

In this section, we present a procedure to derive a lower bound \( f_n \) and an upper bound \( g_n \) for the ratio \( S_n/S_{n-1} \) based on a three-term recurrence relation of \( S_n \).

We first describe how to obtain an upper bound \( g_n \) as required in Theorem 2.1. As will be seen, this procedure is not guaranteed to give an upper bound \( g_n \), but it is practically valid for many cases.

Assume that \( \lim_{n \to \infty} b(n) = b \) and \( \lim_{n \to \infty} c(n) = c \), where \( b \) and \( c \) are two constants and \( b^2 + 4c > 0 \). All sequences considered in this paper satisfy this condition. Let
\[
x_0 = \frac{b + \sqrt{b^2 + 4c}}{2}.
\]
(3.1)

We begin with the case \( c(n) > 0 \) for \( n \geq N \), and we shall try to construct \( g_n \) which satisfies condition \((C_3)\) together with the inequality
\[
g_{n+1} - \left(b(n + 1) + \frac{c(n + 1)}{g_n}\right) > 0.
\]
(3.2)

In fact, condition \((3.2)\) is essential to finding an upper bound \( g_n \) for \( S_n/S_{n-1} \). As will be seen in the following lemma, if we find a function \( g_n \) satisfying \((3.2)\) and \( S_n/S_{n-1} < g_n \) for small \( n \), then we can deduce that \( g_n \) is an upper bound for \( S_n/S_{n-1} \) for any \( n \).

**Lemma 3.1.** Let \( S_n \) be the sequence defined by the recurrence relation \((2.1)\). Assume that \( N \) is a positive integer such that \( c(n) < 0 \) for \( n \geq N \). If \( \frac{S_n}{S_{n-1}} \leq g_N \) and condition \((3.2)\) holds for \( n \geq N \), then we have, for \( n \geq N \),
\[
\frac{S_n}{S_{n-1}} \leq g_n.
\]
(3.3)

**Proof.** We use induction on \( n \). Obviously, the lemma holds for \( n = N \). We assume that it is true for \( n = m \geq N \), that is, \( \frac{S_m}{S_{m-1}} < g_m \). Since \( c(m) < 0 \) for \( m \geq N \), we see that
\[
c(m + 1)S_{m-1} < \frac{c(m + 1)}{g_m}.
\]
(3.4)

We now consider the case \( n = m + 1 \). From \((2.1)\) and \((3.4)\) it follows that
\[
\frac{S_{m+1}}{S_m} = b(m + 1) + c(m + 1)S_{m-1} \leq b(m + 1) + \frac{c(m + 1)}{g_m}.
\]
(3.5)

From \((3.2)\) and \((3.5)\) we deduce that, for \( m \geq N \),
\[
g_{m+1} - \frac{S_{m+1}}{S_m} \geq g_{m+1} - \left(b(m + 1) + \frac{c(m + 1)}{g_m}\right) > 0,
\]
which is the statement of the lemma for \( n = m + 1 \). This completes the proof. \( \square \)
Now we present a heuristic procedure to find the desired upper bound \( g_n \). Let \( g_n = x_0 \) as given by (3.1). If \( g_n \) satisfies conditions \( (C_3) \) and \( (3.2) \), then \( g_n \) is the desired choice. Otherwise, let \( g_n = x_0 + \frac{a_1}{n} \). Substitute \( g_n \) into (3.2) and let \( Y(n) \) denote the numerator of the left hand side of (3.2), which is often a polynomial in \( n \) and \( x \). Setting the coefficient of the highest degree in \( \frac{a_1}{n} \) of \( Y(n) \) to be 0, we obtain an equation in \( x \). If \( x_1 \) is the unique solution of this equation, then we set \( g_n = x_0 + \frac{a_1}{n} \). If \( g_n = x_0 + \frac{a_1}{n} \) satisfies conditions \( (C_3) \) and \( (3.2) \), then \( g_n \) is the desired choice. Otherwise, set \( g_n = x_0 + \frac{a_1}{n} + \frac{a_2}{n^2} \) and repeat the above process to find a solution \( x_2 \) of the equation. By iteration, we may find \( x_0, x_1, \ldots, x_i \) such that \( g_n = x_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_i}{n^i} \) satisfies conditions \( (C_i) \) and \( (3.2) \).

For example, let \( S_n = A_n \), where \( A_n \) is the Apéry number defined by (3.1). Since \( \lim_{n \to \infty} b(n) = 34 \) and \( \lim_{n \to \infty} c(n) = -1 \), by the definition of \( A_n \), we have \( x_0 = 17 + 12\sqrt{2} \). Since \( g_n = 17 + 12\sqrt{2} \) does not satisfy condition \( (C_3) \) in Theorem 2.1 we further consider \( g_n = 17 + 12\sqrt{2} + \frac{1}{n} \). Let \( Y(n) \) denote the numerator of the left hand side of (3.2). It is easy to see that \( Y(n) \) is a cubic polynomial in \( n \) with the leading coefficient equal to

\[
E_1 = -(17\sqrt{2} - 24)(48x + 864\sqrt{2} + 1224).
\]

Setting \( E_1 = 0 \) gives \( x_1 = -\frac{51}{2} - 18\sqrt{2} \). Again, \( g_n = x_0 + \frac{a_1}{n} \) does not satisfy (3.2).

So we continue to consider \( g_n = x_0 + \frac{a_1}{n} + \frac{a_2}{n^2} \) and we find that \( x_2 = \frac{609}{64} \sqrt{2} + \frac{27}{4} \). Now, \( g_n = x_0 + \frac{a_1}{n} + \frac{a_2}{n^2} \) does not satisfy condition \( (C_3) \). Repeating the above procedure, we find that \( x_3 = -\frac{225}{128} \sqrt{2} - \frac{645}{256} \) and \( g_n = x_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} \) satisfies (3.2) and condition \( (C_3) \).

For the case \( c(n) > 0 \), we aim to construct an upper bound \( g_n \) which satisfies condition \( (C_3) \) and the inequality

\[
(3.6) \quad g_n - \left( b(n) + \frac{c(n)}{b(n-1) + c(n-1)/g_{n-2}} \right) > 0.
\]

Similarly, if we find a function \( g_n \) satisfying (3.6) and \( S_n/S_{n-1} < g_n \) for certain \( n \), then we can deduce that \( g_n \) is an upper bound for any \( n \). To be precise, we have the following lemma.

**Lemma 3.2.** Let \( S_n \) be defined by (2.1). If there exists a positive integer \( N \) such that the inequality (3.6) holds, \( S_n/S_{n-1} \leq g_N \), \( S_{n+1}/S_n \leq g_{N+1} \) and \( c(n) > 0 \) for \( n \geq N \), then we have for \( n \geq N \),

\[
(3.7) \quad \frac{S_n}{S_{n-1}} \leq g_n.
\]

**Proof.** We conduct induction on \( n \). Clearly, the lemma holds for \( n = N \) and \( n = N + 1 \). Assume that it is true for \( n = m - 2 \geq N \); that is,

\[
(3.8) \quad \frac{S_{m-2}}{S_{m-3}} \leq g_{m-2}.
\]

We shall show that the lemma is true for \( n = m \); that is,

\[
(3.9) \quad \frac{S_m}{S_{m-1}} \leq g_m.
\]
Since \( c(n) > 0 \) for \( n \geq N \), from (2.1) and (3.8) it follows that
\[
\begin{align*}
S_m & \leq b(m) + \frac{c(m)}{b(m-1) + \frac{c(m-1)}{g_{m-2}}},
\end{align*}
\]

In view of (3.6) and (3.10), we find that
\[
\begin{align*}
g_m - \frac{S_m}{S_{m-1}} & \geq g_m - \left( b(m) + \frac{c(m)}{b(m-1) + \frac{c(m-1)}{g_{m-2}}}, \right) > 0,
\end{align*}
\]
which yields (3.9). This completes the proof.

Now we can use the same approach as in the case \( c(n) < 0 \) to find an upper bound \( g_n \). Moreover, if we have obtained an approximation \( g_n \) that does not simultaneously satisfy (3.2), (3.6) and condition (C3), instead of going further to update the estimation of \( g_n \), we may try to adjust some coefficients to find a desired bound. For example, let
\[
g_n = \frac{11}{2} + \frac{5\sqrt{5}}{2} - \left( \frac{11}{2} + \frac{5\sqrt{5}}{2} \right) \frac{1}{n}
\]
\[
+ \left( \frac{7}{10} + \frac{3}{2} \right) \frac{1}{n^2} + \frac{1}{25n^3} + \left( \frac{1}{50} + \frac{23}{1250} \right) \frac{1}{n^4}.
\]

Here \( g_n \) satisfies condition (C3) in Theorem 2.1 but it fails to satisfy (3.6). If we replace the coefficient \( \frac{1}{50} \) in (3.11) by \( \frac{1}{25} \), then the adjusted bound \( g'_n \) satisfies both conditions (C3) and (3.6).

To conclude this section, we need to mention that it is much easier to find a lower bound \( f_n \) for the ratio \( S_n/S_{n-1} \). In many cases, we have \( f(n) = b(n) \) when \( b(n) \) and \( c(n) \) are positive for \( n \geq N \) and \( f_n = b(n) + c(n) \) when \( c(n) \) is negative for \( n \geq N \) and \( S_n \geq S_{n-1} \).

4. Upper bounds for \( A_n/A_{n-1}, B_n/B_{n-1} \) and \( U_n/U_{n-1} \)

In this section, we shall use the heuristic approach described in the previous section to find upper bounds for the ratios \( A_n/A_{n-1}, B_n/B_{n-1} \) and \( U_n/U_{n-1} \).

Lemma 4.1. Let
\[
P(n) = 17 + 12\sqrt{2} - \left( \frac{51}{2} + 18\sqrt{2} \right) \frac{1}{n}
\]
\[
+ \left( \frac{27}{2} + \frac{609}{64} \sqrt{2} \right) \frac{1}{n^2} - \left( \frac{645}{256} + \frac{225\sqrt{2}}{128} \right) \frac{1}{n^3}.
\]

For \( n \geq 2 \), we have \( \frac{A_n}{A_{n-1}} < P(n) \).
Proof. For the Apéry numbers $A_n$, we use Lemma 3.1 by setting $N = 2$ and $g_n = P(n)$. Evidently, $\frac{A_2}{A_1} < P(2)$. Also, it is easily checked that

$$P(n+1) - \left(\frac{(2n+1)(17n^2 + 17n + 5)}{(n+1)^3} - \frac{n^3}{(n+1)^3}P(n)\right)$$

$$= \frac{9(17 - 12\sqrt{2})(5664n^2 - 3560\sqrt{2}n + 1225)}{256(256n^3 - 384n^2 - 60\sqrt{2}n + 288n + 90\sqrt{2} - 165)(n+1)^3},$$

which is positive for $n \geq 2$. By Lemma 3.1 we see that $P(n)$ is an upper bound for $A_n/A_{n-1}$ when $n \geq 2$. This completes the proof. □

Lemma 4.2. Let

$$(4.2) \quad T(n) = \frac{11}{2} + \frac{5\sqrt{5}}{2} - \left(\frac{11}{2} + \frac{5\sqrt{5}}{2}\right)\frac{1}{n}$$

$$+ \left(\frac{7}{10}\sqrt{5} + \frac{3}{2}\right)\frac{1}{n^2} + \frac{1}{25n^3} + \left(\frac{1}{25} + \frac{23\sqrt{5}}{1250}\right)\frac{1}{n^4}.$$ 

For $n \geq 20$, we have $\frac{B_n}{B_{n-1}} < T(n)$.

Proof. Set $N = 20$ and $g_n = T(n)$ in Lemma 3.2. It is easy to check that $\frac{B_{20}}{B_{19}} < T(20)$ and $\frac{B_{21}}{B_{20}} < T(21)$. Moreover, it is not difficult to verify that

$$T(n) - \left(\frac{11n^2 - 11n + 3}{n^2} + \frac{(n-1)^2}{n^2} \left(\frac{11n^2 - 33n + 25}{(n-1)^2} \frac{1}{n^2} + \frac{(n-2)^2}{(n-1)^2} \frac{1}{n^2} T(n-2)\right)\right)$$

$$= \frac{(123\sqrt{5} - 275)J(n)}{1250n^4K(n)},$$

where $J(n)$ and $K(n)$ are given by

$J(n) = 1718750n^6 - 4656250\sqrt{5}n^5 - 18026250n^4 + 98010000n^3$ + $38885750\sqrt{5}n^4 - 136205250\sqrt{5}n^3 - 310595950n^2$ + $248642319\sqrt{5}n^2 + 557184100n^2 - 233557457\sqrt{5}n - 522290000n$ + $199152500 + 89063225\sqrt{5},$

$K(n) = 2500n^6 - 30000n^5 + 150000n^4 - 500\sqrt{5}n^4 - 401100n^3 + 4500\sqrt{5}n^3$ + $642325n^2 - 30881\sqrt{5}n^2 - 619575n - 78143\sqrt{5}n + 120525\sqrt{5} + 278125.$

It follows that $J(n)$ and $K(n)$ are positive for $n \geq 20$. Hence we have

$$(4.3) \quad \frac{11n^2 - 11n + 3}{n^2} + \frac{(n-1)^2}{n^2} \left(\frac{11n^2 - 33n + 25}{(n-1)^2} \frac{1}{n^2} + \frac{(n-2)^2}{(n-1)^2} \frac{1}{n^2} T(n-2)\right) < T(n).$$

In view of Lemma 3.2 we deduce that $T(n)$ is an upper bound for $B_n/B_{n-1}$ when $n \geq 20$. □
Using the same procedure, we find the following upper bound for $U_n/U_{n-1}$. The proof is omitted.

**Lemma 4.3.** Let

$$Q(n) = 135 + 78\sqrt{3} - \left(\frac{675}{2} + 195\sqrt{3}\right) \frac{1}{n} + \left(\frac{9737}{48}\sqrt{3} + 351\right) \frac{1}{n^2}$$

$$- \left(\frac{3497}{32}\sqrt{3} + \frac{6045}{32}\right) \frac{1}{n^3} + \left(\frac{841763}{27648}\sqrt{3} + \frac{2701}{32}\right) \frac{1}{n^4}.$$

For $n \geq 100$, we have $\frac{U_n}{U_{n-1}} < Q(n)$.

5. The 2-log-convexity

Based on the criterion given in Theorem 2.1 and the upper bounds obtained in the previous section, we shall give the proofs of the 2-log-convexity of the sequences of Apéry numbers and other aforementioned combinatorial numbers.

**Theorem 5.1.** The sequence $\{A_n\}_{n=0}^\infty$ is strictly 2-log-convex.

**Proof.** We first consider the case $n \geq 2$. To apply Theorem 2.1 let

$$b(n) = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} \quad \text{and} \quad c(n) = -\frac{(n - 1)^3}{n^3}.$$

It is straightforward to check that $a_3(n) < 0$ and $\Delta(n) > 0$ for $n \geq 2$. Since

$$\left(\frac{n - 1}{n}\right)^2 \left(\frac{n - 1 + k}{k}\right)^2 \geq \left(\frac{n - 2}{n}\right)^2 \left(\frac{n - 2 + k}{k}\right)^2,$$

we have $A_{n-1} \geq A_{n-2}$. Let

$$f_n = \frac{33n^3 - 48n^2 + 24n - 4}{n^3}.$$

Thus, by the recurrence relation (1.3), we see that

$$\frac{A_n}{A_{n-1}} = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} - \frac{(n - 1)^3}{n^3} \frac{A_{n-2}}{A_{n-1}}$$

$$\geq \frac{34n^3 - 51n^2 + 27n - 5 - (n - 1)^3}{n^3} = f_n.$$

Set $g_n = P(n)$, where $P(n)$ is given by (1.1). We proceed to verify conditions (C1), (C2) and (C3) in Theorem 2.1. By (5.1) and Lemma 4.1, we find that $f_n \leq \frac{A_n}{A_{n-1}} < g_n$, which is condition (C1). Define $R_1(n) = 6a_3(n)f_n + 2a_2(n)$. It is easily checked that $R_1(n) = -\frac{H_1(n)}{L_1(n)}$, where $H_1(n)$ and $L_1(n)$ are polynomials in $n$ and the leading coefficients of $H_1(n)$ and $L_1(n)$ are positive. Hence we deduce that $R_1(n) < 0$ for $n \geq 2$. Similarly, define $R_2(n) = \Delta(n) - R_1^2(n)$, which can be rewritten as $-\frac{H_2(n)}{L_2(n)}$, where $H_2(n)$ and $L_2(n)$ are polynomials in $n$ and the leading coefficients of $H_2(n)$ and $L_2(n)$ are positive. Consequently, we deduce that $R_2(n) < 0$ for $n \geq 2$. It follows that for $n \geq 2$,

$$6a_3(n)f_n + 2a_2(n) < -\sqrt{\Delta(n)},$$
which is equivalent to the following inequality for $n \geq 2$:

$$f_n > \frac{-2a_2(n) - \sqrt{\Delta(n)}}{6a_3(n)}.$$ 

This is exactly condition $(C_2)$. Finally, it remains to verify condition $(C_3)$. To this end, we find that

$$a_3(n)g_n^2 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n)$$

$$= 9 \left(30733178557 + 21731638968 \sqrt{2}\right) \frac{H_3(n)}{L_3(n)},$$

where $H_3(n)$ and $L_3(n)$ are polynomials in $n$. Observe that the leading coefficients of $H_3(n)$ and $L_3(n)$ are both positive. This implies that the right hand side of (5.2) is positive for $n \geq 2$. Now we are left with the case $n = 1$, that is,

$$(A_0A_2 - A_1^2)(A_2A_4 - A_3^2) > (A_1A_3 - A_2^2)^2,$$

which can be easily checked. This completes the proof.

\end{proof}

**Theorem 5.2.** The sequence $\{B_n\}_{n=0}^{\infty}$ is strictly 2-log-convex.

\begin{proof}
For $n \geq 20$, apply Theorem 2.1 with

$$f_n = \frac{11n^2 - 11n + 3}{n^2},$$

and $g_n = T(n)$, where $T(n)$ is given by (1.2). Using the argument in the proof of Theorem 5.1, we find that $f_n$ and $g_n$ satisfy all the conditions in Theorem 2.1. Finally, it is easy to verify that for $1 \leq n \leq 19$,

$$(B_{n-1}B_{n+1} - B_n^2)(B_{n+1}B_{n+3} - B_{n+2}^2) > (B_nB_{n+2} - B_{n+1}^2)^2.$$ 

This completes the proof.

\end{proof}

**Theorem 5.3.** The sequence $\{U_n\}_{n=0}^{\infty}$ is strictly 2-log-convex.

The above theorem follows from Theorem 2.1 by setting

$$f_n = \frac{3(2n-1)(3n^2 - 3n + 1)(15n^2 - 15n + 4)}{n^5},$$

and setting $g_n = Q(n)$, where $Q(n)$ is given by (4.4). The proof is similar to that of Theorem 5.1 and is omitted.

Došlić [7, 8] has proved the log-convexity of several well-known sequences of combinatorial numbers such as the Motzkin numbers $M_n$, the Fine numbers $F_n$, the Franel numbers $F_n^{(3)}$ and $F_n^{(4)}$ of orders 3 and 4, and the large Schröder numbers $s_n$. Based on the recurrence relations satisfied by these numbers, we utilize Theorem 2.1 to deduce that these sequences are all strictly 2-log-convex possibly except for a fixed number of terms at the beginning.

We conclude this paper with a conjecture concerning the infinite log-convexity of the Apéry numbers. The notion of infinite log-convexity is analogous to that of infinite log-concavity introduced by Moll [12]. Given a sequence $A = \{a_i\}_{0 \leq i \leq \infty}$, define the operator $\mathcal{L}$ by

$$\mathcal{L}(A) = \{b_i\}_{0 \leq i \leq \infty},$$

where $b_i = a_{i-1}a_{i+1} - a_i^2$ for $i \geq 1$. We say that $\{a_i\}_{0 \leq i \leq \infty}$ is $k$-log-convex if $\mathcal{L}^j(\{a_i\}_{0 \leq i \leq \infty})$ is log-convex for $j = 0, 1, \ldots, k - 1$, and that $\{a_i\}_{0 \leq i \leq \infty}$ is $\infty$-log-convex if $\mathcal{L}^k(\{a_i\}_{0 \leq i \leq \infty})$ is log-convex for any $k \geq 0$. 

Conjecture 5.4. The sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, $\{U_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ are infinitely log-convex. The sequences $\{M_n\}_{n=0}^{\infty}$, $\{F_n\}_{n=0}^{\infty}$, $\{F_n^{(3)}\}_{n=0}^{\infty}$ and $\{F_n^{(4)}\}_{n=0}^{\infty}$ are $k$-log-convex for any $k \geq 1$ except for a constant number (depending on $k$) of terms at the beginning.

References

2. R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$. Astérisque 61 (1979) 11–13.