

ON LEHNER'S 'FREE' NONCOMMUTATIVE ANALOGUE OF DE FINETTI'S THEOREM

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ABSTRACT. Inspired by Lehner's results on exchangeability systems, we define 'weak conditional freeness' and 'conditional freeness' for stationary processes in an operator algebraic framework of noncommutative probability. We show that these two properties are equivalent, and thus the process embeds into a von Neumann algebraic amalgamated free product over the fixed point algebra of the stationary process.

Recently Lehner introduced 'weak freeness' for exchangeability systems within a cumulant approach to $*$ -algebraic noncommutative probability. A main result in [Leh06] is that an exchangeability system with weak freeness and certain other properties embeds into an amalgamated free product analogous to the classical de Finetti theorem. Here we investigate Lehner's approach from an operator algebraic point of view which is motivated by recent results on a noncommutative version of de Finetti's theorem [Kos08] and a certain 'braided' extension of it [GK]. For the classical de Finetti theorem, we refer the reader to Kallenberg's recent monograph [Kal05] on probabilistic symmetries and invariance principles.

Since tail events in probability theory lead to conditioning which goes beyond amalgamation, we define 'conditional freeness' and 'weak conditional freeness' as a slight generalization of Voiculescu's 'amalgamated freeness' [Voi85, VDN92] and Lehner's 'weak freeness' [Leh06], respectively. We investigate them for stationary processes in an operator algebraic setting of noncommutative probability (see [Kos08, GK] for details). Our main results are reformulated in terms of infinite minimal random sequences with stationarity:

- ◇ 'Weak conditional freeness' and 'conditional freeness' are equivalent, and the conditioning is with respect to the tail algebra of the random sequence (see Theorem 2.1).
- ◇ 'Weak freeness' and 'amalgamated freeness' are equivalent under a certain condition, and the amalgamation is with respect to the tail algebra of the random sequence (see Theorem 3.10).

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- ◊ Each of these four variations of Voiculescu’s central notion of freeness *implies* that the random sequence canonically embeds into a certain von Neumann algebra amalgamated free product and that it enjoys exchangeability (see Theorem 2.1, Theorem 3.10 and their corollaries).

Our results hint that, dropping the assumption of stationarity, a certain asymptotic version of weak conditional freeness seems perhaps already to be equivalent to conditional freeness, which would of course imply the distributional symmetry of exchangeability.

We summarize the content of this paper. Section 1 introduces ‘conditional freeness’ and ‘weak conditional freeness’ for stationary processes, and it provides a fixed point characterization theorem from [Kos08]. Section 2 contains our first main result, Theorem 2.1, on the equivalence of weak conditional freeness and conditional freeness. This result rests on an application of the mean ergodic theorem, also provided there. Finally, we relate our results in Section 3 to those obtained for exchangeability systems in [Leh06]. We will see that, up to a regularity condition on $*$ -algebraic probability spaces, an exchangeability system yields a stationary process in our sense. This observation is the starting point for Theorem 3.10, our second main result on the equivalence of amalgamated freeness and weak freeness.

Postscriptum. Recently a free version of de Finetti’s theorem has been found by Speicher and the author [KS].

1. PRELIMINARIES

We are interested in a W^* -probability space (\mathcal{M}, φ) consisting of a von Neumann algebra \mathcal{M} with separable predual and a faithful normal state φ on \mathcal{M} . A von Neumann subalgebra $\mathcal{M}_0 \subset \mathcal{M}$ is said to be φ -conditioned if the φ -preserving conditional expectation $E_0: \mathcal{M} \rightarrow \mathcal{M}_0$ exists. We say that an endomorphism $\alpha: \mathcal{M} \rightarrow \mathcal{M}$ is φ -conditioned if α is unital, φ -preserving and commutes with the modular automorphism group associated to (\mathcal{M}, φ) . Throughout this paper we will work in the GNS representation of (\mathcal{M}, φ) . Finally, $\bigvee_{i \in I} \mathcal{A}_i$ denotes the von Neumann algebra generated by the family $(\mathcal{A}_i)_{i \in I} \subset \mathcal{M}$.

Definition 1.1. A *stationary process* $\mathcal{M} \equiv (\mathcal{M}, \varphi, \alpha; \mathcal{M}_0)$ consists of a W^* -probability space (\mathcal{M}, φ) which is equipped with a φ -conditioned endomorphism α and a φ -conditioned von Neumann subalgebra $\mathcal{M}_0 \subset \mathcal{M}$. The *canonical filtration* $\mathcal{F}(\mathcal{M})$ of \mathcal{M} is the family $(\mathcal{M}_I)_{I \subset \mathbb{N}_0}$ of subalgebras $\mathcal{M}_I := \bigvee_{i \in I} \alpha^i(\mathcal{M}_0)$. We say that \mathcal{M} is *minimal* if $\mathcal{M} = \mathcal{M}_{\mathbb{N}_0}$.

Notation 1.2. The *fixed point algebra* of α is denoted by \mathcal{M}^α , and E is the φ -preserving conditional expectation from \mathcal{M} onto \mathcal{M}^α . The *tail algebra* of \mathcal{M} is $\mathcal{M}^{\text{tail}} := \bigcap_{n \in \mathbb{N}_0} \alpha^n(\mathcal{M})$.

Remark 1.3. The φ -conditioning of \mathcal{M}_0 and α imply that $\mathcal{F}(\mathcal{M})$ is a family of φ -conditioned von Neumann subalgebras. In particular, the φ -conditioning of α ensures the existence of the φ -preserving conditional expectation E from \mathcal{M} onto \mathcal{M}^α . We will make use of this in the proof of Theorem 2.1.

Motivated by Lehner’s notion of ‘weak freeness’ and the author’s work on a noncommutative extended de Finetti theorem [Kos08, GK], we introduce ‘weak conditional freeness’ and ‘conditional freeness’.

Definition 1.4. Suppose \mathcal{M} is a minimal stationary process.

- (i) \mathcal{M} and its filtration $\mathcal{F}(\mathcal{M})$ satisfy *conditional freeness* if, for every $n \in \mathbb{N}$ and n -tuple $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}_0$,

$$E(x_1 x_2 x_3 \cdots x_n) = 0$$

whenever

$$x_j \in (\mathcal{M}^\alpha \vee \mathcal{M}_{I_{\mathbf{i}(j)}}) \cap \text{Ker } E$$

with mutually disjoint subsets $\{I_i \mid i \in \text{Ran } \mathbf{i}\}$ and $\mathbf{i}(1) \neq \mathbf{i}(2) \neq \cdots \neq \mathbf{i}(n)$.

- (ii) \mathcal{M} and its filtration $\mathcal{F}(\mathcal{M})$ satisfy *weak conditional freeness* if, for every $n \in \mathbb{N}$ and n -tuple $\mathbf{i}: \{1, 2, \dots, n\} \rightarrow \mathbb{N}_0$,

$$\varphi(x_1 x_2 x_3 \cdots x_n) = 0$$

whenever

$$x_j \in \mathcal{M}^\alpha \vee \mathcal{M}_{I_{\mathbf{i}(j)}} \quad \text{and} \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_j^* \alpha^k(x_j)) = 0$$

with mutually disjoint subsets $\{I_i \mid i \in \text{Ran } \mathbf{i}\}$ and $\mathbf{i}(1) \neq \mathbf{i}(2) \neq \cdots \neq \mathbf{i}(n)$.

Remark 1.5. Conditional freeness of $\mathcal{F}(\mathcal{M})$ is equivalent to Voiculescu's amalgamated freeness of the family $(\alpha^i(\mathcal{M}_0 \vee \mathcal{M}^\alpha))_{i \geq 0}$ in (\mathcal{M}, E) (compare [VDN92, Definition 3.8.2]). On the other hand, conditional freeness is a special case of conditional independence in [Kos08]. Note that \mathcal{M}^α may not be contained in \mathcal{M}_0 . The requirement $\mathcal{M}^\alpha \subset \mathcal{M}_0$ is very restrictive since $\mathcal{M}_I \cap \mathcal{M}^\alpha$ may be trivial for any finite set I . Such a situation occurs frequently for results of de Finetti type or, more generally, if tail events of random sequences are considered.

Remark 1.6. Our notion of 'conditional freeness' should not be confused with that given in [BLS96], a generalization of free products to algebras with two states.

Remark 1.7. 'Weak conditional freeness' formally simplifies to 'weak freeness' (see [Leh06, Definition 4.1]) if $\varphi(x_j^* \alpha^k(x_j)) = \varphi(x_j^* \alpha^{N_0}(x_j))$ whenever $k > N_0$ for some $N_0 \in \mathbb{N}$. But there is also a significant difference: weak conditional freeness is formulated with respect to a family of von Neumann subalgebras *saturated* by the fixed point algebra \mathcal{M}^α . In a *-algebraic or C*-algebraic approach one cannot expect that a nontrivial fixed point of α is in $\mathcal{M}^{\text{alg}} := \text{alg}\{\alpha^n(\mathcal{M}_0) \mid n \in \mathbb{N}_0\}$ or its norm closure; the situation $\mathcal{M}^\alpha \cap \overline{\mathcal{M}^{\text{alg}}}^{\|\cdot\|} = \mathbb{C} \cdot \mathbf{1}_{\mathcal{M}}$ may occur.

In Section 3 we will need a fixed point characterization result from [Kos08].

Definition 1.8. A stationary process \mathcal{M} is said to be *order \mathcal{N} -factorizable* if \mathcal{N} is a φ -conditioned von Neumann subalgebra of \mathcal{M} and $E_{\mathcal{N}}(xy) = E_{\mathcal{N}}(x)E_{\mathcal{N}}(y)$ for all $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ with $\max I < \min J$ or $\min I > \max J$.

Note that the inclusion $\mathcal{N} \subset \mathcal{M}_I \cap \mathcal{M}_J$ is *not* required in this definition.

Theorem 1.9. *Suppose the minimal stationary process \mathcal{M} is order \mathcal{N} -factorizable for the φ -conditioned von Neumann subalgebra \mathcal{N} of \mathcal{M}^α . Then one has the equalities*

$$\mathcal{N} = \mathcal{M}^\alpha = \mathcal{M}^{\text{tail}}.$$

2. MAIN RESULT

We assume that the reader is familiar with von Neumann algebraic amalgamated free products. Their definition and the technical details of their construction can be found in [JPX07] (see also [VDN92] for an outline).

Theorem 2.1. *The following are equivalent for a minimal stationary process \mathcal{M} :*

- (a) $\mathcal{F}(\mathcal{M})$ satisfies weak conditional freeness;
- (b) $\mathcal{F}(\mathcal{M})$ satisfies conditional freeness;
- (c) $\mathcal{F}(\mathcal{M})$ embeds canonically into the von Neumann algebra amalgamated free product

$$(\widetilde{\mathcal{M}}, \widetilde{\varphi}) := \star_{n=0}^{\infty} \mathcal{M}^{\alpha} (\mathcal{M}_0 \vee \mathcal{M}^{\alpha}, \varphi|_{\mathcal{M}_0 \vee \mathcal{M}^{\alpha}}),$$

such that the endomorphism α of \mathcal{M} is turned into the unilateral shift $\widetilde{\alpha}$ on the amalgamated free product factors of $\widetilde{\mathcal{M}}$.

We record an immediate consequence before giving the proof. Let \mathbb{S}_{∞} denote the inductive limit of the symmetric groups \mathbb{S}_n .

Definition 2.2. A stationary process is said to be *exchangeable* if, for any $n \in \mathbb{N}_0$,

$$\varphi(\alpha^{i_1}(a_1)\alpha^{i_2}(a_2) \cdots \alpha^{i_n}(a_n)) = \varphi(\alpha^{\pi(i_1)}(a_1)\alpha^{\pi(i_2)}(a_2) \cdots \alpha^{\pi(i_n)}(a_n))$$

for all n -tuples $(i_1, \dots, i_n) \subset \mathbb{N}_0^n$ and $(a_1, \dots, a_n) \in \mathcal{M}_0^n$ and permutations $\pi \in \mathbb{S}_{\infty}$ on \mathbb{N}_0 .

Corollary 2.3. *A minimal stationary process \mathcal{M} with weak conditional freeness is exchangeable.*

Proof. Due to Theorem 2.1 we can assume that \mathcal{M} is already realized on the von Neumann algebra amalgamated free product over the fixed point algebra of α . So we can identify \mathcal{M}_0 with its embedding into the 0-th factor of the amalgamated free product. Then it is clear from the action of the shift endomorphism α that $\alpha^k(\mathcal{M}_0)$ is the embedding of \mathcal{M}_0 into the k -th factor of this product. Now it follows from the universal property of the amalgamated free product that

$$E(\alpha^{i_1}(a_1)\alpha^{i_2}(a_2) \cdots \alpha^{i_n}(a_n)) = E(\alpha^{\pi(i_1)}(a_1)\alpha^{\pi(i_2)}(a_2) \cdots \alpha^{\pi(i_n)}(a_n))$$

for all $n \in \mathbb{N}$, n -tuples $(i_1, \dots, i_n) \in \mathbb{N}_0^n$ and $(a_1, \dots, a_n) \in \mathcal{M}_0^n$. Since $\varphi \circ E = \varphi$, this entails the exchangeability of \mathcal{M} . □

We prepare the proof of Theorem 2.1 with an operator algebraic version of the von Neumann mean ergodic theorem.

Theorem 2.4. *Let \mathcal{M} be a stationary process. Then for each $x \in \mathcal{M}$,*

$$\text{SOT-} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \alpha^k(x) = E(x).$$

Proof. The strong operator topology (SOT) and the φ -topology generated by $x \mapsto \varphi(x^*x)^{1/2}$, $x \in \mathcal{M}$, coincide on norm bounded sets in \mathcal{M} . Thus this ergodic theorem is an immediate consequence of the usual mean ergodic theorem in Hilbert spaces (see [Pet83, Theorem 1.2] for example). □

Corollary 2.5. *Suppose \mathcal{M} is a minimal stationary process. Then for $x \in \mathcal{M}$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x^* \alpha^k(x)) = 0 \iff x \in \ker E.$$

Proof. This is immediate from Theorem 2.4, the faithfulness of φ and E , and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x^* \alpha^k(x)) = \varphi(x^* E(x)) = \varphi(E(x^*)E(x)) = 0.$$

□

Proof of Theorem 2.1. ‘(a) \Rightarrow (b)’: Let the tuple (x_1, x_2, \dots, x_n) satisfy the assertions of Definition 1.4(ii). Our goal is to show then that (ax_1, x_2, \dots, x_n) also satisfies them for any $a \in \mathcal{M}^\alpha$. Due to Corollary 2.5 it suffices to prove

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_1^* \alpha^k(x_1)) = 0 \implies \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_1^* a^* \alpha^k(ax_1)) = 0.$$

Indeed the mean ergodic theorem, the Kadison-Schwarz inequality and properties of conditional expectations yield

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_1^* a^* \alpha^k(ax_1)) &= \varphi(x_1^* a^* E(ax_1)) \\ &= \varphi(E(x_1^*) a^* a E(x_1)) \\ &\leq \|a\|^2 \varphi(E(x_1^*) E(x_1)) \\ &= \|a\|^2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_1^* \alpha^k(x_1)). \end{aligned}$$

By our initial assumption, (x_1, \dots, x_n) satisfies $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_1^* \alpha^k(x_1)) = 0$. We conclude from above estimates that, for any $a \in \mathcal{M}^\alpha$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_1^* a^* \alpha^k(ax_1)) = 0.$$

Altogether we have shown that the tuple (ax_1, x_2, \dots, x_n) satisfies all assertions of Definition 1.4(ii) if the tuple (x_1, x_2, \dots, x_n) does as well. Thus, by weak conditional freeness, $\varphi(x_1 x_2 x_3 \cdots x_n) = 0$ implies $\varphi(ax_1 x_2 x_3 \cdots x_n) = 0$ for every $a \in \mathcal{M}^\alpha$. But this entails $E(x_1 x_2 x_3 \cdots x_n) = 0$ by routine arguments.

We note for the proof of ‘(b) \Rightarrow (c)’ that conditional freeness of $\mathcal{F}(\mathcal{M})$ implies the amalgamated freeness of the family $(\mathcal{M}_{\{i\}} \vee \mathcal{M}^\alpha)_{i \in \mathbb{N}_0}$ in (\mathcal{M}, E) in the category of C^* -algebraic probability spaces. Thus Voiculescu’s construction of the amalgamated free product of C^* -algebras applies [Voi85]. Since the definition of \mathcal{M} ensures the existence of φ -preserving conditional expectations (see Remark 1.3), all assumptions of [JPX07, Section 1] which are needed for the construction of the von Neumann amalgamated free product $(\widetilde{\mathcal{M}}, \widetilde{\varphi})$ are satisfied. Finally, one verifies that α becomes the shift $\widetilde{\alpha}$ on the amalgamated free product factors of $\widetilde{\mathcal{M}}$.

The remaining implication ‘(c) \Rightarrow (a)’ follows from the fact that the shift $\widetilde{\alpha}$ has the von Neumann subalgebra $\bigstar_{n=0}^\infty \mathcal{M}^\alpha$ of $\widetilde{\mathcal{M}}$ as a fixed point algebra. Now another application of the mean ergodic theorem in the style of Corollary 2.5 shows the weak

conditional freeness of the minimal stationary process $\widetilde{\mathcal{M}} \equiv (\widetilde{\mathcal{M}}, \widetilde{\varphi}, \widetilde{\alpha}, \widetilde{\mathcal{M}}_0)$, where $\widetilde{\mathcal{M}}_0$ is the canonical embedding of \mathcal{M}_0 . □

3. APPLICATION TO LEHNER'S EXCHANGEABILITY SYSTEMS

This section is devoted to the discussion under which conditions an exchangeability system (defined in [Leh06]) can lead to a stationary process \mathcal{M} in the sense of Definition 1.1. Also we will show that under some mild regularity and modular conditions, Lehner's notion of weak freeness is equivalent to amalgamated freeness (over the fixed point algebra of the induced stationary process).

Definition 3.1. A **-probability space* $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ consists of a unital *-algebra over \mathbb{C} and a unital positive linear functional $\varphi^{\text{alg}}: \mathcal{A}^{\text{alg}} \rightarrow \mathbb{C}$. We say that $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ is a *regular *-probability space* if the following three conditions are satisfied:

- (i) *Countability (C)*: \mathcal{A}^{alg} contains a τ -dense sequence where the τ -topology is generated by the seminorms $x \mapsto (\varphi^{\text{alg}}(x^*x))^{1/2}$.
- (ii) *Left regularity (LR)*: For every $a \in \mathcal{A}^{\text{alg}}$ there exists a constant C_a such that

$$\varphi^{\text{alg}}(x^*a^*ax) < C_a\varphi^{\text{alg}}(x^*x).$$

- (iii) *Right regularity (RR)*: For every $a \in \mathcal{A}^{\text{alg}}$ there exists a constant D_a such that

$$\varphi^{\text{alg}}(ax^*xa^*) < D_a\varphi^{\text{alg}}(x^*x).$$

The above notions of left and right regularity are motivated from an inspection of the usual GNS construction for C*-algebras and from the definition of left Hilbert algebras in Tomita-Takesaki theory (see [Tak03b] or [Con94]). But here we will not dwell any longer on this connection to modular theory. Instead we concentrate on the fact that (LG) and (RG) allow a GNS construction such that the GNS vector is both cyclic and separating for the representation.

Remarks 3.2. (1) Condition (C) will ensure the separability of the GNS Hilbert space.

(2) Suppose \mathcal{A}^{alg} is a pre-C*-algebra. Then condition (LR) is automatically satisfied. If φ^{alg} is a faithful state on the pre-C*-algebra \mathcal{A}^{alg} , then both conditions (LR) and (RR) are superfluous, as becomes evident by the usual GNS construction.

(3) Suppose the state φ^{alg} on the *-algebra \mathcal{A}^{alg} is tracial. Then the conditions (LR) and (RR) are equivalent with constant $D_a = C_a$. This is easily concluded from the estimate

$$\varphi^{\text{alg}}(ax^*xa^*) = \varphi^{\text{alg}}(xa^*ax^*) < C_a\varphi^{\text{alg}}(xx^*) = C_a\varphi^{\text{alg}}(x^*x).$$

It is well known that condition (LR) allows us to extend the procedure of the GNS construction from C*-algebras to *-algebras over \mathbb{C} . Let us make this more precise for the convenience of the reader.

Proposition 3.3. *Let $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ be a *-probability space with (LR). Then there exists a representation Π of \mathcal{A}^{alg} on a Hilbert space \mathcal{H} and a vector $\xi \in \mathcal{H}$ which is cyclic for Π such that $\varphi^{\text{alg}}(x) = \langle \xi, \Pi(x)\xi \rangle$ for all $x \in \mathcal{A}^{\text{alg}}$.*

Proof. Let $\mathcal{N} := \{x \in \mathcal{A}^{\text{alg}} \mid \varphi^{\text{alg}}(x^*x) = 0\}$. Now (LR) implies that the left multiplication $\mathcal{N} \ni x \mapsto ax$ is bounded on \mathcal{N} for each $a \in \mathcal{A}^{\text{alg}}$. Thus the proof of the usual GNS construction translates literally. □

Similarly, condition (RR) ensures that there exists an antirepresentation Π' of \mathcal{A}^{alg} on a Hilbert space \mathcal{H} and a vector $\xi \in \mathcal{H}$ which is cyclic for Π' such that $\varphi^{\text{alg}}(x^*) = \langle \xi, \Pi'(x)\xi \rangle$ for all $x \in \mathcal{A}^{\text{alg}}$. If both (LR) and (RR) are valid, then Π and Π' are of course realized on the same Hilbert space \mathcal{H} with the same cyclic vector ξ . It is easy to see that $\Pi'(\mathcal{A}^{\text{alg}})$ is contained in the commutant of $\Pi(\mathcal{A}^{\text{alg}})$, and vice versa.

The regularity conditions (LR) and (RR) allow us to extend φ^{alg} to a faithful normal state on the von Neumann algebra $\Pi(\mathcal{A}^{\text{alg}})''$. To be more precise, suppose $\varphi^{\text{alg}}(x^*x) = 0$ for some $x \in \mathcal{A}^{\text{alg}}$. Then we conclude from (RR) that $\varphi^{\text{alg}}(ax^*xa^*) = 0$, and further from (LR) that $\langle \Pi(x)\Pi(a^*)\xi, \Pi(x)\Pi(a^*)\xi \rangle = 0$ for all $a \in \mathcal{A}^{\text{alg}}$. But ξ is cyclic for Π and thus $\Pi(x) = 0$. Consequently, $\varphi := \langle \xi, \bullet \xi \rangle$ defines a faithful normal state on the von Neumann algebra $\mathcal{M} := \Pi(\mathcal{A}^{\text{alg}})''$. Now the faithfulness of φ ensures the σ -finiteness of \mathcal{M} ([Tak03a, Chapter II, Prop. 3.19]). Since \mathcal{A}^{alg} is countable generated by condition (C), so \mathcal{M} is countable generated; i.e. \mathcal{M} contains a SOT-dense sequence. But this implies the separability of the predual of \mathcal{M} , as it is required in our notion for a W^* -probability space. Altogether we arrive at the following conclusion.

Lemma 3.4. *Let $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ be a regular $*$ -probability space with the GNS triple (Π, \mathcal{H}, ξ) . Then a W^* -probability space (\mathcal{M}, φ) is defined by the double commutant $\mathcal{M} := \Pi(\mathcal{A}^{\text{alg}})''$ in $\mathcal{B}(\mathcal{H})$ and $\mathcal{M} \ni x \mapsto \varphi(x) := \langle \xi, x\xi \rangle$.*

Note in above lemma that the state φ^{alg} is not required to be faithful; it only matters that φ^{alg} induces a faithful state for the representation.

We are ready to construct a stationary process \mathcal{M} starting from a $*$ -algebraic setting of infinite random sequences.

Definition 3.5. Let $\mathcal{A}_0^{\text{alg}}$ be a complex unital $*$ -algebra and $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ a (regular) $*$ -probability space. A family of unital $*$ -algebra homomorphisms $(\iota_n^{\text{alg}})_{n \geq 0} : \mathcal{A}_0^{\text{alg}} \rightarrow \mathcal{A}^{\text{alg}}$ is called a *family of (regular) $*$ -algebraic random variables*.

Let (Π, \mathcal{H}, ξ) denote the GNS triple of the regular $*$ -probability space $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ and (\mathcal{M}, φ) the associated W^* -probability space according to Lemma 3.4. Given a family of regular $*$ -algebraic random variables $(\iota_n)_{n \geq 0}$, we let $\mathcal{M}_I := \bigvee_{i \in I} \Pi \circ \iota_i^{\text{alg}}(\mathcal{A}_0^{\text{alg}})$ and write \mathcal{M}_k if $I = \{k\}$. This gives us the filtration $(\mathcal{M}_I)_{I \subset \mathbb{N}_0}$. Furthermore we may assume without loss of generality the minimality of this filtration.

Now suppose for a moment that $(\iota_n^{\text{alg}})_{n \geq 0}$ is stationary; i.e. for every $N > 0$,

$$\varphi^{\text{alg}}(\iota_{\mathbf{i}(1)}^{\text{alg}}(a_1) \cdots \iota_{\mathbf{i}(n)}^{\text{alg}}(a_n)) = \varphi^{\text{alg}}(\iota_{\mathbf{i}(1)+N}^{\text{alg}}(a_1) \cdots \iota_{\mathbf{i}(n)+N}^{\text{alg}}(a_n))$$

for all n -tuples $\mathbf{i} : \{1, \dots, n\} \rightarrow \mathbb{N}_0$ and $(a_1, \dots, a_n) \in (\mathcal{A}_0^{\text{alg}})^n$. It follows from the asserted stationarity and minimality that there exists a unital φ -preserving endomorphism α of \mathcal{M} such that $\alpha^k(\mathcal{M}_0) = \mathcal{M}_k$. Consequently, the quadruple $(\mathcal{M}, \varphi, \alpha; \mathcal{M}_0)$ defines a (minimal) stationary process \mathcal{M} , provided two additional *modular conditions* are satisfied:

1. \mathcal{M}_0 is φ -conditioned;
2. α is φ -conditioned.

We will show next that these two modular conditions can always be ensured in the context of exchangeability. Similar to Definition 2.2, a family of regular $*$ -algebraic

random variables $(\iota_n^{\text{alg}})_{n \in \mathbb{N}_0}$ is said to be *exchangeable* if, for any $n \in \mathbb{N}_0$,

$$\varphi^{\text{alg}}(\iota_{i_1}^{\text{alg}}(a_1)\iota_{i_2}^{\text{alg}}(a_2)\cdots\iota_{i_n}^{\text{alg}}(a_n)) = \varphi(\iota_{\pi(i_1)}^{\text{alg}}(a_1)\iota_{\pi(i_2)}^{\text{alg}}(a_2)\cdots\iota_{\pi(i_n)}^{\text{alg}}(a_n))$$

for all n -tuples $(i_1, \dots, i_n) \subset \mathbb{N}_0^n$, $(a_1, \dots, a_n) \in (\mathcal{A}_0^{\text{alg}})^n$ and $\pi \in \mathbb{S}_\infty$.

Lemma 3.6. *Let $(\iota_n^{\text{alg}})_{n \geq 0}$ be an exchangeable family of regular $*$ -algebraic random variables. Then the GNS construction yields an exchangeable stationary process \mathcal{M} in the sense of Definition 2.2.*

Proof. We assume without loss of generality that \mathcal{A}^{alg} is generated by the ranges of $(\iota_n^{\text{alg}})_{n \geq 0}$. As before, let (Π, \mathcal{H}, ξ) denote the GNS triple of $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$ and let (\mathcal{M}, φ) be the W^* -probability space according to Lemma 3.4. Now exchangeability implies that, for $\pi \in \mathbb{S}_\infty$, the complex linear extension of the map

$$\Pi(\iota_{i_1}^{\text{alg}}(a_1)\iota_{i_2}^{\text{alg}}(a_2)\cdots\iota_{i_n}^{\text{alg}}(a_n)) \mapsto \Pi(\iota_{\pi(i_1)}^{\text{alg}}(a_1)\iota_{\pi(i_2)}^{\text{alg}}(a_2)\cdots\iota_{\pi(i_n)}^{\text{alg}}(a_n))$$

defines a $*$ -automorphism γ_π of $\Pi(\mathcal{A}^{\text{alg}})$ such that $\varphi \circ \gamma_\pi(x) = \varphi(x)$ for any $x \in \Pi(\mathcal{A}^{\text{alg}})$. Consequently, γ_π extends to a φ -preserving automorphism \mathcal{M} which we denote by the same symbol. Now consider $\mathcal{M}_k := \bigvee_{t \in \mathbb{R}} \sigma_t^\varphi(\Pi(\iota_k^{\text{alg}}(\mathcal{A}^{\text{alg}})))$, the von Neumann subalgebra generated by the represented range of a random variable under the action of the modular automorphism group $(\sigma_t^\varphi)_{t \in \mathbb{R}}$ associated to (\mathcal{M}, φ) . By construction, each \mathcal{M}_k is globally σ_t^φ -invariant and thus φ -conditioned. Since $(\sigma_t^\varphi)_{t \in \mathbb{R}}$ and a φ -preserving automorphism of \mathcal{M} commute, we conclude that $\gamma_\pi(\mathcal{M}_k) = \mathcal{M}_{\pi(k)}$ for $\pi \in \mathbb{S}_\infty$ and $k \in \mathbb{N}_0$. Let γ_i be the automorphism induced by the transposition $(i-1, i) \mapsto (i, i-1)$, with $i \in \mathbb{N}$. Note that, by our assumption of minimality, the $*$ -algebra $\bigcup_{k \geq 0} \mathcal{M}_k$ is weak $*$ -dense in \mathcal{M} . Since γ_i acts trivially on any \mathcal{M}_k with $k < i+1$, we conclude that $\alpha := \text{SOT-}\lim_{n \rightarrow \infty} \gamma_1 \gamma_2 \cdots \gamma_n$ defines an endomorphism of \mathcal{M} . It is evident from its definition that α is φ -conditioned and $\mathcal{M}_k = \alpha^k(\mathcal{M}_0)$. Consequently $\mathcal{M} := (\mathcal{M}, \varphi, \alpha; \mathcal{M}_0)$ is a minimal stationary process.

Finally, we need to check that \mathcal{M} is exchangeable in the sense of Definition 2.2. For this purpose, let ι_0 denote the trivial embedding of \mathcal{M}_0 into \mathcal{M} and consider $\iota_n := \alpha^n \iota_0$ for $n \in \mathbb{N}$. We observe that $(\iota_n)_{n \geq 0}$ satisfies all conditions of [GK, Theorem 1.9(b)]. But these conditions characterize exchangeability. \square

The remainder of this section assumes some familiarity with [Leh06].

Corollary 3.7. *An exchangeability system \mathcal{E} in [Leh06] yields an exchangeable stationary process \mathcal{M} if \mathcal{E} can be realized on a regular $*$ -probability space $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$.*

Proof. This is immediate from Definition 3.5, Lemma 3.6 and the definition of an exchangeability system in [Leh06]. \square

The main results in [Leh06], Theorem 1.18 and Theorem 4.3, consider an exchangeability system realized on $(\mathcal{A}^{\text{alg}}, \varphi^{\text{alg}})$, where \mathcal{A}^{alg} is a pre-C $*$ -algebra and φ^{alg} a faithful state on \mathcal{A}^{alg} . Such a pair is a regular $*$ -probability space provided the very mild countability condition (C) is met. (Actually this condition can be removed; it only serves the purpose of ensuring that the representation acts on a separable Hilbert space.)

Corollary 3.7 motivates us to consider from now on a minimal stationary process $\mathcal{M} = (\mathcal{M}, \varphi, \alpha; \mathcal{M}_0)$ as the starting point for the further discussion of Lehner’s weak freeness condition.

Definition 3.8 ([Leh06]). Suppose \mathcal{M} is a minimal stationary process with a weak*-dense *-algebra $\mathcal{M}_0^{\text{alg}} \subset \mathcal{M}_0$, and let $\mathcal{M}_I^{\text{alg}} := \text{alg}\{\alpha^i(\mathcal{M}_0^{\text{alg}}) \mid i \in I\}$. We say that \mathcal{M} and $\mathcal{F}(\mathcal{M})$ satisfy (algebraic) weak freeness if

$$\varphi(x_1 x_2 \cdots x_n) = 0$$

whenever $\varphi(x_j^* \alpha^{N_j}(x_j)) = 0$ for $x_j \in \mathcal{M}_{I_j}^{\text{alg}}$ and $N_j > \min\{N \mid x_j \in \mathcal{M}_{\{0, \dots, N\}}^{\text{alg}}\}$ with mutually disjoint subsets $\{I_i \mid i \in \text{Ran } \mathbf{i}\}$ and $\mathbf{i}(1) \neq \mathbf{i}(2) \neq \dots \neq \mathbf{i}(n)$.

[Leh06, Definition 4.1] introduces weak freeness in the context of exchangeability systems. The above definition relies only on stationarity and is thus a proper generalization of Lehner's notion if the underlying *-probability space is regular.

Remark 3.9. The condition $N_j > \min\{N \mid x_j \in \mathcal{M}_{\{0, \dots, N\}}^{\text{alg}}\}$ simplifies to $N_j > \max I_j$ if the index set I_j is bounded. Note that, for unbounded I_j , the choice of N_j depends on x_j .

We generalize this observation to our second main result which improves the main results in [Leh06] for a large class of exchangeability systems.

Theorem 3.10. *Let \mathcal{M} be a minimal stationary process such that*

$$(*) \quad \mathcal{M}^\alpha \cap \mathcal{M}_0^{\text{alg}} = \mathcal{M}^\alpha \cap \mathcal{M}_{\mathbb{N}_0}^{\text{alg}}$$

for some weak*-dense *-algebra $\mathcal{M}_0^{\text{alg}}$ in \mathcal{M}_0 . Then the following are equivalent:

- (a) $\mathcal{F}(\mathcal{M})$ satisfies weak freeness;
- (b) $\mathcal{F}(\mathcal{M})$ satisfies amalgamated freeness in (\mathcal{M}, E) ;
- (c) $\mathcal{F}(\mathcal{M})$ embeds canonically into the von Neumann algebra amalgamated free product

$$(\widetilde{\mathcal{M}}, \widetilde{\varphi}) := \bigstar_{n=0}^{\infty} \mathcal{M}^\alpha(\mathcal{M}_0, \varphi|_{\mathcal{M}_0}),$$

such that the endomorphism α of \mathcal{M} is turned into the unilateral shift $\widetilde{\alpha}$ on the amalgamated free product factors of $\widetilde{\mathcal{M}}$.

An immediate consequence is that the assumption of 'exchangeability' in [Leh06, Theorem 4.3] is turned into a conclusion.

Corollary 3.11. *A minimal stationary process with weak freeness is exchangeable.*

Proof. Repeat the proof of Corollary 2.3. □

Proof of Theorem 3.10. '(a) \Rightarrow (b)': Suppose \mathcal{M} satisfies weak freeness. We conclude for x_j (as stated in Definition 3.8) that

$$\varphi(x_j^* \alpha^N(x_j)) = \varphi(x_j^* \alpha^{N+1}(x_j)) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi(x_j^* \alpha^n(x_j))$$

and, by Corollary 2.5,

$$\varphi(x_j^* \alpha^N(x_j)) = 0 \iff E(x_j) = 0.$$

We identify in a second step which elements $x \in \mathcal{M}_{\mathbb{N}_0}^{\text{alg}}$ satisfy $E(x) = 0$. For this purpose let \mathcal{N} and \mathcal{N}_0 be the von Neumann algebras generated by the orbit of

$$\mathcal{N}^{\text{alg}} := \mathcal{M}^\alpha \cap \mathcal{M}_{\mathbb{N}_0}^{\text{alg}}, \quad \text{resp.} \quad \mathcal{N}_0^{\text{alg}} := \mathcal{M}^\alpha \cap \mathcal{M}_0^{\text{alg}},$$

under the action of the modular automorphism group associated to (\mathcal{M}, φ) . By construction and Takesaki's theorem, \mathcal{N} and \mathcal{N}_0 are φ -conditioned, and we let $E_{\mathcal{N}}$, resp. $E_{\mathcal{N}_0}$, denote the corresponding conditional expectations from \mathcal{M} onto \mathcal{N} , resp. \mathcal{N}_0 . Note that $\mathcal{N} \subset \mathcal{M}^\alpha$ implies $E_{\mathcal{N}}E = E_{\mathcal{N}}$. Furthermore we know $\mathcal{N}_0 \subset \mathcal{M}_0$ (we do not know if $\mathcal{N} \subset \mathcal{M}_0$).

Suppose $y \in \mathcal{M}_{\mathcal{N}_0}^{\text{alg}}$ and let $x := y - E(y)$. It is easy to see that $x \in \mathcal{M}_{\mathcal{N}_0}^{\text{alg}}$ if and only if $E(y) \in \mathcal{M}_{\mathcal{N}_0}^{\text{alg}}$. Thus $E(y) \in \mathcal{N}^{\text{alg}} \subset \mathcal{N}$ and $E(y) = E_{\mathcal{N}}E(y) = E_{\mathcal{N}}(y)$. At this point we make use of the assumption $(*)$ which ensures $\mathcal{N}^{\text{alg}} = \mathcal{N}_0^{\text{alg}}$, and consequently $\mathcal{N}_0 = \mathcal{N}$. Thus $x \in \mathcal{M}_{\mathcal{N}_0}^{\text{alg}}$ satisfies $E(x) = 0$ if and only if $x = y - E_{\mathcal{N}_0}(y)$ for some $y \in \mathcal{M}_{\mathcal{N}_0}^{\text{alg}}$.

We conclude from this that the filtration $\mathcal{F}(\mathcal{M})$ satisfies amalgamated freeness in $(\mathcal{M}, E_{\mathcal{N}_0})$. Suppose that we can prove that $\mathcal{N}_0 \subset \mathcal{M}^\alpha$ already implies $\mathcal{N}_0 = \mathcal{M}^\alpha$. Then Theorem 2.1 applies, and we are done. But the implication that $\mathcal{N} \subset \mathcal{M}^\alpha$ forces $\mathcal{N}_0 = \mathcal{M}^\alpha$ is the content of the fixed point characterization result, Theorem 1.9, as soon as we can ensure that amalgamated freeness in $(\mathcal{M}, E_{\mathcal{N}_0})$ implies order \mathcal{N}_0 -factorizability (see Definition 1.8). This is easily verified, and thus the proof is completed. \square

Remark 3.12. We do not know at the time of this writing if the assertion $(*)$ is superfluous in Theorem 3.10. Can it be that the $*$ -algebra generated by $\mathcal{M}_0^{\text{alg}}$ and $\alpha(\mathcal{M}_0^{\text{alg}})$ contains more fixed points of α than $\mathcal{M}_0^{\text{alg}}$?

Remark 3.13. The condition $(*)$ can always be ensured by passing from the minimal stationary process \mathcal{M} to its saturation $\mathcal{M}_{\text{sat}} := (\mathcal{M}, \varphi, \alpha; \mathcal{M}_0 \vee \mathcal{N})$, where \mathcal{N} is as introduced in the proof of Theorem 3.10. Doing so, the $*$ -algebra $\mathcal{M}_0^{\text{alg}}$ needs to be replaced by $\text{alg}\{\mathcal{M}_0^{\text{alg}}, \mathcal{N}^{\text{alg}}\}$. This procedure has the same effect as Lehner's transfer from exchangeability systems to extended ones (see [Leh06, Remark and Definition 1.5]).

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