Abstract. We introduce and investigate the notion of \( p \)-convergence in a Banach space. Among others, a Grothendieck-like result is obtained; namely, a subset of a Banach space is relatively \( p \)-compact if and only if it is contained in the closed convex hull of a \( p \)-null sequence. We give a description of the topological dual of the space of all \( p \)-null sequences which is used to characterize the Banach spaces enjoying the property that every relatively \( p \)-compact subset is relatively \( q \)-compact (\( 1 \leq q < p \)). As an application, Banach spaces satisfying that every relatively \( p \)-compact set lies inside the range of a vector measure of bounded variation are characterized.

1. Introduction

By a well known characterization due to Grothendieck [7] (see, e.g., [9, p. 30]), a subset \( A \) of a Banach space \( X \) is relatively compact if and only if there exists \((x_n)\) in \( c_0(X) \) (the space of norm-null sequences in \( X \)) such that \( A \subset \{ \sum_n a_n x_n : \sum_n |a_n| \leq 1 \} \). Since then, several authors have dealt with stronger forms of compactness, studying sets sitting inside the convex hulls of special types of null sequences. For instance, it was observed in [14] (see also [1]) that if one considers, instead of \( c_0(X) \), the space of \( q \)-summable sequences \( \ell_q(X) \), for some fixed \( q \geq 1 \), then this stronger form of compactness characterizes Reinov’s approximation property of order \( p \), \( 0 < p < 1 \). This latter form of compactness was recently further strengthened by Sinha and Karn [15] as follows. Let \( 1 \leq p \leq \infty \) and let \( p' \) be the conjugate index of \( p \) (i.e., \( 1/p + 1/p' = 1 \)). The \( p \)-convex hull of a sequence \((x_n) \in \ell_p(X) \) is defined as \( p \text{-co} \ (x_n) = \{ \sum_n a_n x_n : \sum_n |a_n|^{p'} \leq 1 \} \) (sup \( |a_n| \leq 1 \) if \( p = 1 \)). A set \( A \subset X \) is said to be relatively \( p \)-compact if there exists \((x_n) \in \ell_p(X) \) \((x_n) \in c_0(X) \) if \( p = \infty \) such that \( A \subset p \text{-co} (x_n) \). (Note that similar notions with \((x_n) \) being a weakly \( p \)-summable sequence were already considered in [2, p. 51].) Some results concerning this type of relatively compact set have been set in [3].

The aim of this article is to deepen the study of the geometry of Banach spaces related to \( p \)-compact sets. In this way, the notion of \( p \)-convergent sequence is introduced in Section 2 and a Grothendieck-like result is obtained; namely, a subset...
of a Banach space is relatively $p$-compact if and only if it is contained in the closed convex hull of a sequence $p$-convergent to zero (Theorem 2.5).

In [3, Theorem 3.14], Serrano and the authors have proved that every infinite dimensional Banach space has relatively compact sets failing to be $q$-compact for every $1 \leq q < \infty$. Section 3 is devoted to find out if this result is also true when we replace compact (= $\infty$-compact) with $p$-compact, $p > q$. In fact, we come to the same conclusion if $p > 2$ (Proposition 3.5): if $1 < q < p \leq 2$, it is shown that every relatively $p$-compact subset of $X$ is relatively $q$-compact if and only if every $q'$-summing operator from $c_0$ to $X^*$ is $p'$-summing ($p'$ and $q'$ are conjugate exponents of $p$ and $q$, respectively). It is convenient to point out that the description of the topological dual of the space of all bounded sequences (Proposition 3.1) simplifies the proofs of this section.

Our notation is standard. A Banach space $X$ will be regarded as a subspace of its bidual $X^{**}$ under the canonical embedding $i_X : X \to X^{**}$. We denote the closed unit ball of $X$ by $B_X$. For Banach spaces $X$ and $Y$, the space of all bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$. We also need the following operator ideals: $\mathcal{N}_p$—$p$-nuclear operators, $\mathcal{QN}_p$—quasi-$p$-nuclear operators, $\mathcal{I}_p$—$p$-integral operators and $\Pi_p$—$p$-summing operators. We refer to Pietsch’s book [11] for operator ideals (see also [5] by Diestel, Jarchow and Tonge for common operator ideals $\mathcal{K}_p$ and $\Pi_p$, and [10] by Persson and Pietsch for $\mathcal{QN}_p$).

Let $1 \leq p \leq \infty$. The space of all weakly $p$-summable sequences (respectively, (strongly) $p$-summable sequences) in $X$ is denoted by $\ell^w_p(X)$ (respectively, $\ell_p(X)$) endowed with its natural norm $\|(x_n)\|_p^w$ (respectively, $\|(x_n)\|_p$). We write $\ell_\infty(X)$ to describe the space of all bounded sequences $(x_n)$ in $X$ with the norm $\|(x_n)\|_\infty$.

Relying on the notion of $p$-compactness, the notion of $p$-compact operator is defined in an obvious way (see [15]): an operator $T \in \mathcal{L}(X,Y)$ is said to be $p$-compact if $T(B_X)$ is relatively $p$-compact in $Y$. The class of all $p$-compact operators between Banach spaces is denoted by $\mathcal{K}_p$. It is shown in [15] that $\mathcal{K}_p$ is an operator ideal. Even more, $\mathcal{K}_p$, equipped with the norm $k_p$ defined by

$$k_p(T) = \inf \{\|(y_n)\|_p : (y_n) \in \ell_p(Y) \text{ and } T(B_X) \subset p\text{-co}(y_n)\}$$

for all $T \in \mathcal{K}_p(X,Y)$, is a Banach operator ideal. In order to make the article self-contained, we list some properties related to $p$-compactness:

- If $1 \leq q \leq p \leq \infty$, then every relatively $q$-compact set is $p$-compact.
- An operator $T : X \to Y$ belongs to $\mathcal{K}_p(X,Y)$ (respectively, $\mathcal{QN}_p(X,Y)$) if and only if $T^*$ belongs to $\mathcal{QN}_p(Y^*,X^*)$ (respectively, $\mathcal{K}_p(Y^*,X^*)$) [3, Corollary 3.4 and Proposition 3.8].
- A sequence $\hat{x} = (x_n)$ of $X$ is relatively $p$-compact if and only if $U_{\hat{x}} : e_n \in \ell_1 \mapsto x_n \in X$ is $p$-compact, where $(e_n)$ denotes the unit vector basis of $\ell_1$ [3, Proposition 3.5].

2. $p$-CONVERGENCE

**Definition 2.1.** Let $p \geq 1$. A sequence $(x_n)$ in a Banach space $X$ is said to be $p$-null if, for every $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $(z_k) \in \varepsilon B_{\ell_p(X)}$ such that $x_n \in p$-co $(z_k)$ for all $n \geq n_0$.

A sequence $(x_n)$ in $X$ is said to be $p$-convergent if there exists $x \in X$ such that $(x_n - x)$ is $p$-null.
Theorem 2.2. Since every $p$-null sequence $\hat{x} = (x_n)$ has relatively $p$-compact rank, it induces an operator $U_{\hat{x}} : \ell_p \to X$ defined by $U_{\hat{x}}(\alpha_n) = \sum_n \alpha_n x_n$ that is $p$-compact. It is easy to show that a bounded sequence $\hat{x} = (x_n)$ is $p$-null if and only if $U_{\hat{x}} = k_p^{-}\lim_n U_{\hat{x}(n)}$, where $\hat{x}(n) = (x_1, \ldots, x_n, 0, \ldots)$.

Remark 2.3. It is clear that $p$-summable sequences are $p$-null. For an example of a $p$-null sequence failing to be $p$-summable, consider $(n^{-1/p} e_n)$ (where $(e_n)$ denotes the unit vector basis in $\ell_2$) which is a null sequence in $\ell_2$. From the next proposition, that sequence is $p$-null in $c_0$. Nevertheless, it does not belong to $\ell_p(c_0)$.

In [3] Theorem 3.14, it is proved that an operator $T \in \mathcal{L}(X,Y)$ has $p$-summing adjoint if and only if $T$ maps relatively compact sets in $X$ to relatively $p$-compacts sets in $Y$. The notion of $p$-null sequence allows us to establish the sequential version of this result:

Proposition 2.4. Let $p \geq 1$. An operator $T \in \mathcal{L}(X,Y)$ has $p$-summing adjoint if and only if $T$ maps $p$-null sequences in $X$ to $p$-null sequences in $Y$.

Proof. Suppose that $T^{*}$ is $p$-summing and let $\hat{x} = (x_n)$ be a null sequence in $X$. Notice that $U_{T\hat{x}} - U_{T\hat{x}(n)} = T \circ (U_{\hat{x}} - U_{\hat{x}(n)})$ for each $n \in \mathbb{N}$. Now, the dual version of [3] Theorem 3.13] ensures that $T \circ (U_{\hat{x}} - U_{\hat{x}(n)})$ belongs to $K_p(\ell_1, Y)$ and

$$k_p(T \circ (U_{\hat{x}} - U_{\hat{x}(n)})) \leq \pi_p(T^{*}) \|U_{\hat{x}} - U_{\hat{x}(n)}\|.$$ 

Since $\|U_{\hat{x}} - U_{\hat{x}(n)}\| \rightarrow 0$, it follows that $U_{T\hat{x}} = k_p^{-}\lim_n U_{T\hat{x}(n)}$.

The converse is obvious in view of [3] Theorem 3.14].

We denote by $c_{0,p}(X)$ the vector space of all $p$-null sequences in $X$ endowed with the norm $k_p(\hat{x}) = k_p(U_{\hat{x}})$. This space can be identified with a closed subspace of $K_p(\ell_1, X)$ using a standard argument.

The following result is intended to make clear the analogy between the $p$-convergence and the convergence in a Banach space related to compactness.

Theorem 2.5. Let $p \geq 1$. A set in a Banach space $X$ is relatively $p$-compact if and only if it is contained in the closed convex hull of a $p$-null sequence.

Proof. If $A$ is relatively $p$-compact in $X$, there exists a sequence $\hat{z} = (z_n) \in \ell_p(X)$ so that $A \subset p\text{-co}(z_n)$. The sequence $(z_n)$ induces an operator $\phi_{\hat{z}} : c_{0,p} \rightarrow X$ satisfying $\phi_{\hat{z}}(B_{\ell_p'(0)}) = p\text{-co}(z_n)$. In order to obtain a factorization of $\phi_{\hat{z}}$, choose $(\alpha_n) \subseteq 0$ such that $(\alpha_n^{-1} z_n) \in \ell_p(X)$. Now, consider the compact operator $D : \ell_p'(0) \to \ell_p'$ and the $p$-compact operator $\phi : \ell_p' \to X$ defined by $D(\beta_n) = (\alpha_n \beta_n)$ and $\phi(\beta_n) = \sum_n \beta_n \alpha_n^{-1} x_n$. It is obvious that the following diagram is commutative:

$$\begin{array}{ccc}
\ell_p' & \xrightarrow{\phi_{\hat{z}}} & X \\
D \searrow & \Downarrow & \nearrow \phi \\
\ell_p' & \end{array}$$

As $D(B_{\ell_p'(0)})$ is relatively compact in $\ell_p'$, we can find a null sequence $(\gamma_n)$ in $\ell_p'$ with $D(B_{\ell_p'(0)}) \subset \text{co}(\gamma_n)$. Then

$$A \subset \phi_{\hat{z}}(B_{\ell_p'(0)}) = \phi(D(B_{\ell_p'(0)})) \subset \phi(\text{co}(\gamma_n)) \subset \text{co}(\phi(\gamma_n)).$$

Since $\phi$ is $p$-compact, $\phi^*$ is quasi $p$-nuclear [3] Corollary 3.4] and, in particular, $p$-summing. So, according to Proposition 2.3 the sequence $(\phi(\gamma_n))$ is $p$-null in $X$. 

For the converse, just notice that the closed convex hull of relatively $p$-compact sets are $p$-compact.

We finish this section with a characterization of $p$-null sequences in Banach spaces having the $k_p$-approximation property. A Banach space $X$ has the $k_p$-approximation property if, for every Banach space $Y$, the space $\mathcal{F}(Y, X)$ of all finite rank operators is $k_p$-dense in $K_p(Y, X)$ [4]. This property might be considered as a gradation of the classic approximation property. Since the classic approximation property implies the $k_p$-approximation property for all $p \geq 1$, the following result can be applied to a wide class of Banach spaces.

**Proposition 2.6.** If $X$ has the $k_p$-approximation property, then $\hat{x} = (x_n) \in c_0, p(X)$ if and only if $\hat{x} \in c_0(X)$ and has relatively $p$-compact rank.

**Proof.** Only the “if” part needs to be proved (the converse is straightforward from the definition of $p$-null sequences). So, given $\epsilon > 0$ and a sequence $(x_n) \in c_0(X)$ with relatively $p$-compact rank, there exists a finite rank operator $S: X \to X$ such that $k_p(U_{\epsilon} - S \circ U_{\epsilon}) < \epsilon/3$ [4 Theorem 2.1]. According to Proposition 2.4 the sequence $S\hat{x} = (Sx_n)$ is $p$-null and this allows us to find $n_0 \in \mathbb{N}$ for which $k_p(U_{\epsilon} - U_{\epsilon}(n)) < \epsilon/3$ whenever $n \geq n_0$. Finally, if $P_n: \ell_1 \to \ell_1$ denotes the projection onto the first $n$-th coordinates, we have, for every $n \geq n_0$,

$$k_p(U_{\epsilon} - U_{\epsilon}(n)) \leq k_p(U_{\epsilon} - S \circ U_{\epsilon}) + k_p(S \circ U_{\epsilon} - S \circ U_{\epsilon}(n)) + k_p(U_{\epsilon}(n) - U_{\epsilon}(n)) \leq k_p(U_{\epsilon} - S \circ U_{\epsilon}) + k_p(U_{\epsilon} - U_{\epsilon}(n)) \frac{\epsilon}{3}$$

In [4], it is shown that every Banach space has the $k_2$-approximation property. Nevertheless, we do not know whether the hypothesis of the $k_p$-approximation property can be omitted in the previous result for $p \neq 2$.

### 3. ON THE EQUALITY $\Pi_p(c_0, X^*) = \Pi_q(c_0, X^*)$

In [3], it is proved that, for every $q \geq 1$ and every infinite dimensional Banach space, there exist relatively compact subsets that are not relatively $q$-compact [3 Theorem 3.14]. The objective of this section is to find out if this result is also true when we replace compact ($= \infty$-compact) with $p$-compact, $p > q$. We begin with a description of the dual space $c_0, p(X)^*$ which will be very useful in several proofs. Recall that the trace functional (denoted by $\text{tr}$) is well defined and continuous on $\mathcal{N}_1(X, X^*)$ if and only if $X^*$ has the approximation property.

**Proposition 3.1.** Let $X$ be a Banach space and $p \geq 1$. The dual space $c_0, p(X)^*$ is isometrically isomorphic to $\Pi_{p'}(c_0, X^*)$.

**Proof.** Given $S \in \Pi_{p'}(c_0, X^*)$, we consider the linear form $f_S$ on $c_0, p(X)$ defined by $f_S(\hat{x}) = \text{tr}(U_{\hat{x}} \circ S)$. Since $U_{\hat{x}}$ is $p$-compact, the adjoint $U_{\hat{x}}^*: X^* \to \ell_\infty$ is quasi $p$-nuclear [3 Corollary 3.4]; in particular, $U_{\hat{x}}^*$ is $p$-summing and the following inequalities hold:

$$\nu_p(U_{\hat{x}}^*) \leq \nu_p^0(U_{\hat{x}}^*) \leq k_p(U_{\hat{x}}).$$

Then, the composition $U_{\hat{x}}^* \circ S \in \Pi_1(c_0, \ell_\infty) = \mathcal{N}_1(c_0, \ell_\infty)$ and $\nu_1(U_{\hat{x}}^* \circ S) = \nu_1(U_{\hat{x}}^* \circ S) \leq \pi_p(U_{\hat{x}}^* \circ S) \pi_{p'}(S)$ [17 p. 55]. Since $\ell_1$ has the approximation property, the map $f_S$ is well defined and

$$|\text{tr}(U_{\hat{x}}^* \circ S)| \leq \nu_1(U_{\hat{x}}^* \circ S) \leq k_p(U_{\hat{x}}) \pi_{p'}(S).$$
Therefore, \( f_S \in c_{0,p}(X)^* \) and \( \|f_S\| \leq \pi_p(S) \).

On the other hand, if \( f \in c_{0,p}(X)^* \), we are able to define the operator \( S_f: \alpha \in c_0 \rightarrow S_f \alpha \in X^* \) so that \( \langle S_f \alpha, x \rangle = f(\alpha \otimes x) \) for all \( x \in X \), where \( \alpha \otimes x := (\alpha_n x) \) if \( \alpha = (\alpha_n) \). To see that \( S_f \in \Pi_{p'}(c_0, X^*) \), consider \( (\alpha_k) \in l_{p'}^w(c_0) \) with \( \|\langle \alpha_k \rangle\|_{p'} \leq 1 \) and let us prove

\[
\left( \sum_n \|S_f \alpha_k\|_{p'}^p \right)^{1/p'} \leq \|f\|.
\]

Fix \( N \in \mathbb{N} \), \( x_1, \ldots, x_N \in B_X \) and \( (\beta_k)_{k=1}^N \in B_{\ell_p^N} \). We have

\[
\left| \sum_{k=1}^N \langle S_f \alpha_k, x_k \rangle \beta_k \right| = \left| f \left( \sum_{k=1}^N \alpha_k \otimes \beta_k x_k \right) \right| \leq \|f\| \left( \sum_{k=1}^N \|\alpha_k \| \|\beta_k x_k\| \right).
\]

Set \( \hat{y} = \sum_{k=1}^N \alpha_k \otimes \beta_k x_k \). Notice that \( (\alpha_k^* \beta_k)_{k=1}^N \in B_{\ell_{p'}^N} \) for each \( k \in \mathbb{N} \), so we can ensure that \( U_{\hat{y}} e_n^* = \sum_{k=1}^N \alpha_k^* \beta_k x_k \) belongs to the closed and absolutely convex set \( p\text{-}co (\beta_k x_k)_{k=1}^N \) for each \( n \in \mathbb{N} \) (here, \((e_n^*)\) denotes the unit vector basis of \( \ell_1 \)). From this, it is clear that

\[
k_p(\hat{y}) = k_p(U_{\hat{y}}) \leq \| (\beta_k x_k)_{k=1}^N \|_p \leq 1.
\]

Summing up,

\[
\left| \sum_{k=1}^N \langle S_f \alpha_k, x_k \rangle \beta_k \right| \leq \|f\|
\]

for all \( N \in \mathbb{N} \), \( x_1, \ldots, x_N \in B_X \) and \( (\beta_k)_{k=1}^N \in B_{\ell_p^N} \), from which (3.1) is obtained, using a standard argument. So \( S_f \) is \( p'\text{-}summing \) and, moreover, \( \pi_{p'}(S) \leq \|f\| \).

Now, the functionals \( f_{S_j} \) and \( f \) coincide on finitely supported sequences in \( c_{0,p}(X) \). Indeed, no matter how we choose \( N \in \mathbb{N} \) and \( \hat{x}_0 = (x_1, \ldots, x_N, 0,0,\ldots) \),

\[
f(\hat{x}_0) = \sum_{k=1}^N f(e_k \otimes x_k) = \sum_{k=1}^N \langle S_f e_k, U_{\hat{x}_0} e_k^* \rangle
\]

\[
= \sum_{k=1}^N \langle (U_{\hat{x}_0}^* \circ S_f) e_k, e_k^* \rangle = \text{tr} \left( U_{\hat{x}_0}^* \circ S_f \right) = f_S(\hat{x}_0).
\]

Since \( \hat{x} = k_p - \lim_n \hat{x}(n) \) for every \( \hat{x} \in c_{0,p}(X) \) (Remark 2.2), it is clear that the functionals \( f_{S_j} \) and \( f \) coincide on \( c_{0,p}(X) \). \( \square \)

The following result turns out to be the key to answer the question posed at the beginning of this section. As usual, if \( X \) and \( Y \) are Banach spaces and \( x^* \in X^* \) and \( y \in Y \) are fixed, \( x^* \otimes y \) denotes the one dimensional operator from \( X \) to \( Y \) defined by \( (x^* \otimes y)x = (x^*, x)y \) for all \( x \in X \).

**Theorem 3.2.** Let \( X \) and \( Y \) be Banach spaces and let \( 1 \leq q < p < \infty \). The following statements are equivalent for an operator \( T \in \mathcal{L}(X,Y) \):

a) The operator \( T \) maps relatively \( p\text{-}compact \) subsets of \( X \) to relatively \( q\text{-}compact \) subsets of \( Y \).

b) The operator \( T^* \circ u \in \Pi_{p'}(c_0, X^*) \) whenever \( u \in \Pi_{q'}(c_0, Y^*) \) \((u \in \mathcal{L}(c_0, Y^*), \) in case \( q = 1 \)).
Proof: a) ⇒ b) It is easy to prove that the operator \( V: u \in K_p(\ell_1, X) \mapsto T \circ u \in K_q(\ell_1, Y) \) has closed graph. Then there exists a positive constant \( C \) such that
\[
 k_q(T \circ u) \leq C k_p(u) \quad \text{for all } u \in K_p(\ell_1, X).
\]
In view of this, for every \( p \)-null sequence \( \hat{x} \) we have
\[
k_q(U_T x - U_{T_{x(n)}}) = k_q(T \circ (U_x - U_{x(n)})) \leq C k_p(U_x - U_{x(n)}).
\]
So, from Remark 2.2, \( T \) maps \( p \)-null sequences to \( q \)-null sequences and, therefore, we can consider the operator \( \hat{T}: (x_n) \in c_0,p(X) \mapsto (Tx_n) \in c_0,q(Y) \). It only remains to prove that the adjoint \( \hat{T}^*: \Pi'_p(c_0,Y^*) \mapsto \Pi'_q(c_0, Y^*) \) is defined by \( \hat{T}^*(u) = T^* \circ u \) for all \( u \in \Pi'_q(c_0,Y^*) \). To see this, consider \( x \in X, \beta = (\beta_n) \in c_0 \) and \( u \in \Pi'_q(c_0,Y^*) \) and denote the sequence \( (\beta_n,x) \) by \( \beta \otimes x \). On one hand, we have
\[
 \langle \hat{T}^*(u), \beta \otimes x \rangle = \text{tr} \left( U_{\beta \otimes x} \circ \hat{T}^*(u) \right) = \text{tr} \left( (x \otimes \beta) \circ \hat{T}^*(u) \right) = \text{tr} \left( \hat{T}^*(u^*) x \otimes \beta \right) = \langle \hat{T}^*(u^*) x, \beta \rangle = \langle x, \hat{T}^*(u) \beta \rangle.
\]
On the other hand,
\[
 \langle \hat{T}(\beta \otimes x), u \rangle = \text{tr} \left( U_{\beta \otimes x} \circ u \right) = \text{tr} \left( (x \otimes \beta) \circ u \right) = \text{tr} \left( u^*(Tx) \otimes \beta \right) = \langle u^*(Tx), \beta \rangle = \langle x, (T^* \circ u) \beta \rangle.
\]
As \( \langle \hat{T}^*(u), \beta \otimes x \rangle = \langle \hat{T}(\beta \otimes x), u \rangle \), it follows that \( \langle x, \hat{T}^*(u) \beta \rangle = \langle x, (T^* \circ u) \beta \rangle \) for all \( x \in X \) and \( \beta \in c_0 \), which yields \( \hat{T}^*(u) = T^* \circ u \).

b) ⇒ a) It is a standard argument to prove that the operator
\[
 u \in \Pi'_q(c_0,Y^*) \mapsto T^* \circ u \in \Pi'_q(c_0, X^*)
\]
has closed graph and, therefore, that it is continuous. If \( \Phi \) denotes the restriction of this operator to \( \mathcal{N}_q(c_0,Y^*) \), then \( \Phi^*: c_0,p(X)^{**} \mapsto \Pi_q(Y^*, \ell_\infty) \) is defined by \( \Phi^*(\hat{x}) = U_\hat{x} \circ T^* \) for all \( \hat{x} \in c_0,p(X) \) (just using a similar argument as in a) ⇒ b) above). Since \( U_\hat{x} \circ T^* = U_{T_\hat{x}} \), the restriction of \( \Phi^* \) to \( c_0,p(X) \) is the operator
\[
(3.2) \quad \hat{x} \in c_0,p(X) \mapsto U_{T_\hat{x}} \in \Pi_q(Y^*, \ell_\infty).
\]
Now, we will show that \( \Phi^* \) maps \( c_0,p(X) \) to \( \mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty) \). According to [16, Theorem 2.6], the \( q \)-summing norm and the quasi \( q \)-nuclear norm of a finite rank operator coincide. So, given \( \hat{x} \in c_0,p(X) \), we have for \( m > n \):
\[
 \nu_q^Q \left( \Phi^*(\hat{x}(m)) - \Phi^*(\hat{x}(n)) \right) = \pi_q \left( \Phi^*(\hat{x}(m)) - \Phi^*(\hat{x}(n)) \right) \leq \| \Phi^* \| k_p \left( \hat{x}(m) - \hat{x}(n) \right).
\]
This proves that \( \left( \Phi^*(\hat{x}(n)) \right) \) is a Cauchy sequence in \( \mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty) \) from which it is easy to conclude that \( \Phi^*(\hat{x}) \in \mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty) \). Hence, we can replace \( \Pi_q(Y^*, \ell_\infty) \) with \( \mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty) \) in [16]). In view of [3, Proposition 3.5], this means that \( T \) maps \( p \)-null sequences to sequences with relatively \( q \)-compact rank. The proof concludes by just invoking Theorem 2.5. \( \square \)

Let \( \mathcal{P}_{p,q} \) \( (1 \leq q < p) \) be the class consisting of all Banach spaces with the property that every relatively \( p \)-compact subset is relatively \( q \)-compact.
Corollary 3.3. Let $1 \leq q < p < \infty$. The following statements are equivalent for a Banach space $X$:

a) $X \in \mathcal{P}_{p,q}$.

b) $\Pi_q'(c_0, X^*) = \Pi_p'(c_0, X^*)$ if $q = 1$.

c) $\Pi_2(c_0, Y) = \mathcal{L}(c_0, Y)$ for all $Y$ satisfying (3.3).

d) $\Pi_1(Y, X) = \Pi_1(Y, X)$ for every $L_\infty$-space $Y$.

e) $\Pi_2(Y, X) = \Pi_2(Y, X)$ for some infinite dimensional $L_\infty$-space $Y$.

In [6], the class of Banach spaces $Y$ satisfying (3.3) is deeply studied. Having in mind that $\mathcal{L}(c_0, Y) = L_\infty(c_0, Y)$, Corollary 3.3 reveals that a dual Banach space $Y = X^*$ enjoys (3.3) if and only if relatively 2-compact subsets of $X$ are relatively 1-compact. Related to the ideas of [6], several conditions equivalent to Corollary 3.3(b) are established in the following result, the proof of which is omitted since it is quite similar to the proof in [6] Proposition 2.1.

Proposition 3.4. Let $X$ be a Banach space and let $1 \leq r < s$. The following statements are equivalent:

a) $\Pi_q(Y, X) = \Pi_s(Y, X)$ for every $L_\infty$-space $Y$.

b) $\Pi_q(Y, X) = \Pi_s(Y, X)$ for some infinite dimensional $L_\infty$-space $Y$.

c) There exists a positive constant $C$ such that $\pi_r(u) \leq C \pi_s(u)$ for all $n \in \mathbb{N}$ and $u \in L(\ell_n^0, X)$.

Proposition 3.5. Let $1 \leq q < p$ and let $X$ be a Banach space.

1) If $X \in \mathcal{P}_{p,q}$ and $E$ is a closed subspace of $X$, then $X/E \in \mathcal{P}_{p,q}$.

2) $X^{**} \in \mathcal{P}_{p,q}$ if and only if $X \in \mathcal{P}_{p,q}$.

3) If $2 \leq q < p$, then $\mathcal{P}_{p,q}$ does not contain any infinite dimensional Banach space.

Proof. To prove 1), notice that $u \in \Pi_q(c_0, (X/E)^*) = \Pi_q(c_0, E^*)$ if and only if $i_{E^*} \circ u \in \Pi_q(c_0, X^*)$, where $i_{E^*}$ denotes the inclusion map from $E^*$ into $X^*$. By hypothesis and using the injectivity of the ideal of $p'$-summing operators, it follows that $u \in \Pi_p(c_0, (X/E)^*)$.

If $X^{**} \in \mathcal{P}_{p,q}$, then a similar argument allows us to obtain that $X \in \mathcal{P}_{p,q}$. The converse follows easily from Proposition 3.4(c) using the local reflexivity principle.

Let us argue by contradiction to show 3). Suppose there exists an infinite dimensional Banach space $X \in \mathcal{P}_{p,q}$. By virtue of Corollary 3.3 there exists a positive constant $C$ such that (3.4)

$$\pi_{p'}(u) \leq C \pi_q(u)$$

d for all $u \in \Pi_q(c_0, X^*)$. Now, for every $\gamma = (\gamma_n) \in \ell_{q'}$, consider the operator $D_\gamma : (x_n^*) \in \ell_{q'}(X^*) \mapsto (\gamma_n x_n^*) \in \ell_{p'}(X^*)$.

To see that $D_\gamma$ is well defined, take $(x_n^*) \in \ell_{q'}(X^*)$ and define the operators $A : c_0 \rightarrow \ell_{q'}$ and $B : \ell_{q'} \rightarrow X^*$ by $A(\alpha_n) = (\gamma_n \alpha_n)$ and $B(\beta_n) = \sum_n \beta_n x_n^*$. It is easy to prove that $A$ is $q'$-summing with $\pi_q'(A) \leq \|\gamma\|_{q'}$ and $\|B\| \leq 1$. Hence, $B \circ A$ is $q'$-summing and, by hypothesis, $p'$-summing. Moreover, in view of (3.4), we have

$$\left(\sum_n \|\gamma_n x_n^*\|_{p'}^{r'}\right)^{1/r'} = \left(\sum_n \|(B \circ A)e_n\|_{p'}^{r'}\right)^{1/r'} \leq \pi_{p'}(B \circ A) \leq C \pi_{q'}(B \circ A) \leq C \|\gamma\|_{q'} \|x_n^*\|_{q'}^w.$$
This shows that \( D_n \) is well defined and continuous for all \( \gamma \in \ell_{q'} \). In particular, the inequality
\[
(3.5) \quad \left( \sum_{n=1}^{N} \| \gamma_n x_n^* \|^{p'} \right)^{1/p'} \leq C \| \gamma \|_{q'} \| (x_n^*)_{n=1}^{N} \|_{q'}
\]
holds for all \( N \in \mathbb{N} \) and \( x_1^*, \ldots, x_N^* \in X^* \). According to Dvoretzky–Rogers’ Theorem [11, p. 38], given \( \varepsilon > 0 \), for every \( N \in \mathbb{N} \) there exist unitary vectors \( x_1^*, \ldots, x_N^* \in X^* \) such that \( \| (x_n^*)_{n=1}^{N} \|_2 \leq (1 + \varepsilon) \). As \( q \geq 2 \), applying (3.5) to these vectors, it follows that
\[
\left( \sum_{n=1}^{N} |\gamma_n|^{p'} \right)^{1/p'} \leq C \| \gamma \|_{q'} (1 + \varepsilon)
\]
for all \( N \in \mathbb{N} \). This is a contradiction if we take \( \gamma \in \ell_{q'} \setminus \ell_{p'} \). \( \square \)

If \( 1 \leq q < p \leq 2 \), the following proposition shows that there are (infinite dimensional) spaces in which relatively \( p \)-compact sets are relatively \( q \)-compact.

**Proposition 3.6.** The following statements are equivalent for a Banach space \( X \):

a) \( X^* \) has finite cotype.

b) There exist \( p, q \in \mathbb{R} \), \( 1 \leq q < p \leq 2 \), such that \( X \in \mathcal{P}_{p,q} \).

In addition, if \( X^* \) has cotype \( s > 2 \) (respectively, \( s = 2 \)), then \( X \in \mathcal{P}_{p,q} \) for every \( 1 \leq q < p < s' \) (respectively, \( 1 \leq q < p \leq 2 \)).

**Proof.** a) \( \Rightarrow \) b) Suppose \( X^* \) has finite cotype \( s > 2 \) (respectively, \( s = 2 \)). According to [5, Theorem 11.14], we have \( \Pi_r(c_0, X^*) = \mathcal{L}(c_0, X^*) \) for all \( r > s \) (respectively, \( \Pi_2(c_0, X^*) = \mathcal{L}(c_0, X^*) \)). So, by Corollary 5.3 \( X \) belongs to \( \mathcal{P}_{r',1} \) (respectively, \( X \) belongs to \( \mathcal{P}_{2,1} \)).

b) \( \Rightarrow \) a) By contradiction, if \( X^* \) does not have finite cotype, then \( X^* \) contains \( \ell_{\infty} \) uniformly [5, Theorem 14.1]. Thus, there is a constant \( \lambda > 0 \) such that, for all \( n \in \mathbb{N} \), there exists an isomorphism \( J_n \) from \( \ell_{\infty} \) onto a finite dimensional subspace \( E_n \) of \( X^* \) satisfying
\[
(3.6) \quad \| J_n \| \| J_n^{-1} \| \leq \lambda \quad \text{for all } n \in \mathbb{N}.
\]
Suppose that \( p \) and \( q > 1 \) are real numbers satisfying b) (if \( q = 1 \), the proof is quite similar). A call to Corollary 3.3 tells us that there exists a constant \( C > 0 \) such that
\[
(3.7) \quad \pi_{p'}(u) \leq C \pi_{q'}(u) \quad \text{for all } u \in \mathcal{L}(\ell_{\infty}, X^*) \text{ and } n \in \mathbb{N}.
\]
If \( I_n \) denotes the identity map on \( \ell_{\infty} \), (3.6) and (3.7) yield
\[
(3.8) \quad \pi_{p'}(I_n) \leq \| J_n \| \| \pi_{p'}(J_n) \| \leq C \| J_n^{-1} \| \| \pi_{q'}(J_n) \| \leq C \lambda \pi_{q'}(I_n).
\]
On one hand, we have the estimation
\[
(3.9) \quad n^{1/p'} = \left( \sum_{k=1}^{n} \| I_n(e_k) \|^{p'} \right)^{1/p'} \leq \pi_{p'}(I_n),
\]
where \( (e_n) \) is the canonical vector basis in \( \ell_{\infty} \). On the other hand, as \( I_n \) admits the \( q' \)-nuclear representation \( I_n = \sum_{k=1}^{n} e_k \otimes e_k \), it follows that
\[
(3.10) \quad \nu_{q'}(I_n) \leq \| (e_k) \|_{q'} \| (e_k) \|_{q}^{w} \leq n^{1/q'}.
\]
Since \( \pi_q'(I^q_n) = \nu_q'(I^q_n) \), \( \| x \| \leq C \lambda n^{1/q'} \) for all \( n \in \mathbb{N} \), which is a contradiction since \( q' > p' \).

**Remark 3.7.** The additional information in the above result cannot be improved for spaces whose dual has cotype \( s > 2 \). Indeed, if \( s' \leq p \leq 2 \), let us show that \( X = \ell_{s'} \not\in \mathcal{P}_{p,q} \) for all \( q < p \). First, put \( p = s' \) and suppose, by contradiction, that there exists \( q < s' \) satisfying \( \ell_{s'} \in \mathcal{P}_{s',q} \). In view of Proposition 3.6, \( \ell_{s'} \) belongs to \( \mathcal{P}_{s',1} \); that is, \( \mathcal{L}(\ell_{s'2}, \ell_{s'2}) = \Pi_{s'}(\ell_{s'2}, \ell_{s'2}) \), which contradicts \( \mathcal{L}(\ell_{s'2}, \ell_{s'2}) \). Now, for the case \( p > s' \), it suffices to see that \( X = \ell_{s'} \not\in \mathcal{P}_{p,q} \) when \( q \) satisfies \( s' \leq q < p \). Under this assumption, consider a sequence \( (\lambda_n) \in \ell_{p} \setminus \ell_{q} \). The rank of the sequence \( (\lambda_n) \) is obviously relatively \( p \)-compact in \( X \). Arguing again by contradiction, if the sequence is relatively \( q \)-compact, then the operator \( U : \ell_{n} \rightarrow \lambda_n e_n \in \mathcal{P}_{p,q} \). Therefore, \( U^* \) is quasi \( q \)-nuclear \( \mathcal{L}(\ell_{s'2}, \ell_{s'2}) \) and, in particular, \( q \)-summing. As \( (e_n^*) \in \ell_{p}^\infty(X^*) \subset \ell_{q}^\infty(X^*) \), it follows that

\[
\sum_n |\lambda_n|^q = \sum_n \| U^* e_n^* \| \_q^p < +\infty,
\]

which is a contradiction (here, \( (e_n^*) \) designs the unit vector basis of \( X^* = \ell_{s'} \)). Finally, notice that this latter fact can be extended to \( \mathcal{L}_{s'} \)-spaces, since every infinite dimensional \( \mathcal{L}_{s'} \)-space has a complemented subspace isomorphic to \( \ell_{s'} \).

**Remark 3.8.** In general, the classes \( \mathcal{P}_{p,q} \) are not closed under closed subspaces. For an example, consider \( X = \ell_{\infty}^* \); \( X^* \) has cotype 2 since it is an \( \mathcal{L}_{1} \)-space. Therefore, \( \ell_{\infty} \) belongs to \( \mathcal{P}_{2,1} \). Nevertheless, \( \ell_{1} \) is isometrically isomorphic to a closed subspace of \( \ell_{\infty} \) but \( \ell_{1} \not\in \mathcal{P}_{2,1} \) (Proposition 3.6).

We finish with an application of the preceding results to the theory of vector measures. It is well known that only finite dimensional Banach spaces \( X \) have the property that every relatively compact subset of \( X \) lies inside the range of a vector measure of bounded variation \( \mathcal{L}(\ell_{\infty}, \ell_{\infty}) \). We are going to prove that if we replace compact (\( \mathcal{L}(\ell_{\infty}, \ell_{\infty}) \)) with \( p \)-compact \( (p \leq 2) \), then there exist infinite dimensional Banach spaces with such a property. It suffices to deal with \( p \)-null sequences instead of relative \( p \)-compact sets (Theorem 2.5), so the following lemma gains importance for getting our objective:

**Lemma (122 Lemma 2)).** Let \( \hat{x} \) be a bounded sequence in a Banach space \( X \).

1) The rank of \( \hat{x} \) lies inside the range of a \( X^{**} \)-valued measure of bounded variation if and only if the operator \( U_{\hat{x}} : \ell_{1} \rightarrow X \) is 1-integral.

2) If \( U_{\hat{x}} \) is 1-nuclear, then the rank of \( \hat{x} \) lies inside the range of an \( X \)-valued measure of bounded variation.

**Proposition 3.9.** Let \( X \) be a Banach space and let \( 1 \leq p < +\infty \). Every relatively \( p \)-compact subset of \( X \) lies inside the range of a vector measure of bounded variation if and only if \( X \) belongs to \( \mathcal{P}_{p,1} \).

**Proof.** If every relatively \( p \)-compact subset of \( X \) lies inside the range of a vector measure of bounded variation, the previous lemma guarantees that the operator \( \Phi : \hat{x} \in c_{0,p}(X) \mapsto U_{\hat{x}} \in \mathcal{L}(\ell_{1}, X) \) is well defined. A standard argument shows
that $\Phi$ has closed graph and, therefore, is continuous. Actually, $\Phi$ maps $c_{0,p}(X)$ into $N_1(\ell_1,X)$, since $\hat{x} = k_p - \lim_n \hat{x}(n)$ whenever $\hat{x} \in c_{0,p}(X)$ and using that $N_1(\ell_1,X)$ is isometrically isomorphic to a closed subspace of $Z(\ell_1,X)$ [11 p. 132]. Obviously, this implies that every relatively $p$-compact subset of $X$ is relatively $1$-compact.

For the converse, notice that the operator $U_{\hat{x}}$ belongs to $K_1(\ell_1,X)$ whenever $\hat{x}$ is a relatively $p$-compact sequence in $X$. Since $K_1(\ell_1,X)$ can be isometrically identified with $N_1(\ell_1,X)$, the proof is concluded via [12, Lemma 2]. □

In view of this result and Proposition 3.6, for a fixed $p \in [1, 2)$ (respectively, $p = 2$), every Banach space $X$ such that $X^*$ has cotype $s > p'$ (respectively, $s = 2$) satisfies that its relatively $p$-compact sets lies in the range of an $X$-valued measure of bounded variation.

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References


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