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p-CONVERGENT SEQUENCES AND BANACH SPACES IN WHICH p-COMPACT SETS ARE q-COMPACT

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Abstract. We introduce and investigate the notion of p-convergence in a Banach space. Among others, a Grothendieck-like result is obtained; namely, a subset of a Banach space is relatively p-compact if and only if it is contained in the closed convex hull of a p-null sequence. We give a description of the topological dual of the space of all p-null sequences which is used to characterize the Banach spaces enjoying the property that every relatively p-compact subset is relatively q-compact ($1 \leq q < p$). As an application, Banach spaces satisfying that every relatively p-compact set lies inside the range of a vector measure of bounded variation are characterized.

1. Introduction

By a well known characterization due to Grothendieck [7] (see, e.g., [9, p. 30]), a subset A of a Banach space X is relatively compact if and only if there exists (x_n) in $c_0(X)$ (the space of norm-null sequences in X) such that $A \subset \{\sum_n a_n x_n : \sum_n |a_n| \leq 1\}$. Since then, several authors have dealt with stronger forms of compactness, studying sets sitting inside the convex hulls of special types of null sequences. For instance, it was observed in [14] (see also [1]) that if one considers, instead of $c_0(X)$, the space of q-summable sequences $\ell_q(X)$, for some fixed $q \geq 1$, then this stronger form of compactness characterizes Reinov's approximation property of order p, $0 < p < 1$. This latter form of compactness was recently further strengthened by Sinha and Karn [15] as follows. Let $1 \leq p \leq \infty$ and let p' be the conjugate index of p (i.e., $1/p + 1/p' = 1$). The *p-convex hull* of a sequence $(x_n) \in \ell_p(X)$ is defined as $p\text{-}\operatorname{co}(x_n) = \{\sum_n a_n x_n : \sum_n |a_n|^{p'} \leq 1\}$ ($\sup |a_n| \leq 1$ if $p = 1$). A set $A \subset X$ is said to be *relatively p-compact* if there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $A \subset p\text{-}\operatorname{co}(x_n)$. (Note that similar notions with (x_n) being a weakly p-summable sequence were already considered in [2, p. 51].) Some results concerning this type of relatively compact set have been set in [3].

The aim of this article is to deepen the study of the geometry of Banach spaces related to p-compact sets. In this way, the notion of p-convergent sequence is introduced in Section 2 and a Grothendieck-like result is obtained; namely, a subset

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of a Banach space is relatively p-compact if and only if it is contained in the closed convex hull of a sequence p-convergent to zero (Theorem 2.5).

In [3, Theorem 3.14], Serrano and the authors have proved that every infinite dimensional Banach space has relatively compact sets failing to be q-compact for every $1 \leq q < \infty$. Section 3 is devoted to find out if this result is also true when we replace compact ($= \infty$-compact) with p-compact, $p > q$. In fact, we come to the same conclusion if $p > 2$ (Proposition 3.5): if $1 < q < p \leq 2$, it is shown that every relatively p-compact subset of X is relatively q-compact if and only if every q'-summing operator from c_0 to X^* is p'-summing (p' and q' are conjugate exponents of p and q, respectively). It is convenient to point out that the description of the topological dual of the space of the p-null sequences (Proposition 3.1) simplifies the proofs of this section.

Our notation is standard. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $i_X \colon X \to X^{**}$. We denote the closed unit ball of X by B_X. For Banach spaces X and Y, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. We also need the following operator ideals: $\mathcal{N}_p-$ p-nuclear operators, $\mathcal{QN}_p-$ quasi p-nuclear operators, $\mathcal{I}_p-$ p-integral operators and Π_p- p-summing operators. We refer to Pietsch's book [11] for operator ideals (see also [5] by Diestel, Jarchow and Tonge for common operator ideals $\mathcal{N}_p$ and Π_p, and [10] by Persson and Pietsch for $\mathcal{QN}_p$).

Let $1 \leq p \leq \infty$. The space of all weakly p-summable sequences (respectively, (strongly) p-summable sequences) in X is denoted by $\ell_p^w(X)$ (respectively, $\ell_p(X)$) endowed with its natural norm $\|(x_n)\|_p^w$ (respectively, $\|(x_n)\|_p$). We write $\ell_\infty(X)$ to describe the space of all bounded sequences (x_n) in X with the norm $\|(x_n)\|_\infty$.

Relying on the notion of p-compactness, the notion of p-compact operator is defined in an obvious way (see [15]): an operator $T \in \mathcal{L}(X, Y)$ is said to be *p-compact* if $T(B_X)$ is relatively p-compact in Y. The class of all p-compact operators between Banach spaces is denoted by $\mathcal{K}_p$. It is shown in [15] that $\mathcal{K}_p$ is an operator ideal. Even more, $\mathcal{K}_p$, equipped with the norm k_p defined by

$$k_p(T) = \inf\{\|(y_n)\|_p \colon\ (y_n) \in \ell_p(Y) \text{ and } T(B_X) \subset p\text{-}\operatorname{co}(y_n)\}$$

for all $T \in \mathcal{K}_p(X, Y)$, is a Banach operator ideal. In order to make the article self–contained, we list some properties related to p-compactness:

- If $1 \leq q \leq p \leq \infty$, then every relatively q-compact set is p-compact.
- An operator $T \colon X \longrightarrow Y$ belongs to $\mathcal{K}_p(X, Y)$ (respectively, $\mathcal{QN}_p(X, Y)$) if and only if T^* belongs to $\mathcal{QN}_p(Y^*, X^*)$ (respectively, $\mathcal{K}_p(Y^*, X^*)$) [3, Corollary 3.4 and Proposition 3.8].
- A sequence $\hat{x} = (x_n)$ of X is relatively p-compact if and only if $U_{\hat{x}} \colon e_n \in \ell_1 \longmapsto x_n \in X$ is p-compact, where (e_n) denotes the unit vector basis of ℓ_1 [3, Proposition 3.5].

2. p-convergence

Definition 2.1. Let $p \geq 1$. A sequence (x_n) in a Banach space X is said to be *p-null* if, for every $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $(z_k) \in \varepsilon B_{\ell_p(X)}$ such that $x_n \in p\text{-}\operatorname{co}(z_k)$ for all $n \geq n_0$.

A sequence (x_n) in X is said to be *p-convergent* if there exists $x \in X$ such that $(x_n - x)$ is p-null.

Remark 2.2. Since every p-null sequence $\hat{x} = (x_n)$ has relatively p-compact rank, it induces an operator $U_{\hat{x}} \colon \ell_1 \longrightarrow X$ defined by $U_{\hat{x}}(\alpha_n) = \sum_n \alpha_n x_n$ that is p-compact. It is easy to show that a bounded sequence $\hat{x} = (x_n)$ is p-null if and only if $U_{\hat{x}} = k_p\text{-}\lim_n U_{\hat{x}(n)}$, where $\hat{x}(n) = (x_1, \ldots, x_n, 0, \ldots)$.

Remark 2.3. It is clear that p-summable sequences are p-null. For an example of a p-null sequence failing to be p-summable, consider $(n^{-1/p} e_n)$ (where (e_n) denotes the unit vector basis in ℓ_2) which is a null sequence in ℓ_2. From the next proposition, that sequence is p-null in c_0. Nevertheless, it does not belong to $\ell_p(c_0)$.

In [3, Theorem 3.14], it is proved that an operator $T \in \mathcal{L}(X, Y)$ has p-summing adjoint if and only if T maps relatively compact sets in X to relatively p-compacts sets in Y. The notion of p-null sequence allows us to establish the sequential version of this result:

Proposition 2.4. *Let $p \geq 1$. An operator $T \in \mathcal{L}(X, Y)$ has p-summing adjoint if and only if T maps null sequences in X to p-null sequences in Y.*

Proof. Suppose that T^* is p-summing and let $\hat{x} = (x_n)$ be a null sequence in X. Notice that $U_{T\hat{x}} - U_{T\hat{x}(n)} = T \circ (U_{\hat{x}} - U_{\hat{x}(n)})$ for each $n \in \mathbb{N}$. Now, the dual version of [3, Theorem 3.13] ensures that $T \circ (U_{\hat{x}} - U_{\hat{x}(n)})$ belongs to $\mathcal{K}_p(\ell_1, Y)$ and

$$k_p\big(T \circ (U_{\hat{x}} - U_{\hat{x}(n)})\big) \leq \pi_p(T^*) \| U_{\hat{x}} - U_{\hat{x}(n)} \|.$$

Since $\| U_{\hat{x}} - U_{\hat{x}(n)} \| \xrightarrow{n} 0$, it follows that $U_{T\hat{x}} = k_p\text{-}\lim_n U_{T\hat{x}(n)}$.

The converse is obvious in view of [3, Theorem 3.14]. □

We denote by $c_{0,p}(X)$ the vector space of all p-null sequences in X endowed with the norm $k_p(\hat{x}) = k_p(U_{\hat{x}})$. This space can be identified with a closed subspace of $\mathcal{K}_p(\ell_1, X)$ using a standard argument.

The following result is intended to make clear the analogy between the p-convergence and the convergence in a Banach space related to compactness.

Theorem 2.5. *Let $p \geq 1$. A set in a Banach space X is relatively p-compact if and only if it is contained in the closed convex hull of a p-null sequence.*

Proof. If A is relatively p-compact in X, there exists a sequence $\hat{z} = (z_n) \in \ell_p(X)$ so that $A \subset p\text{-}\operatorname{co}(z_n)$. The sequence (z_n) induces an operator $\phi_{\hat{z}} \colon e_n \in \ell_{p'} \longmapsto z_n \in X$ satisfying $\phi_{\hat{z}}(B_{\ell_{p'}}) = p\text{-}\operatorname{co}(z_n)$. In order to obtain a factorization of $\phi_{\hat{z}}$, choose $(\alpha_n) \searrow 0$ such that $(\alpha_n^{-1} z_n) \in \ell_p(X)$. Now, consider the compact operator $D \colon \ell_{p'} \longrightarrow \ell_{p'}$ and the p-compact operator $\phi \colon \ell_{p'} \longrightarrow X$ defined by $D(\beta_n) = (\alpha_n \beta_n)$ and $\phi(\beta_n) = \sum_n \beta_n \alpha_n^{-1} x_n$. It is obvious that the following diagram is commutative:

$$\begin{array}{ccc} \ell_{p'} & \xrightarrow{\phi_{\hat{z}}} & X \\ D \searrow & & \nearrow \phi \\ & \ell_{p'} & \end{array} .$$

As $D(B_{\ell_{p'}})$ is relatively compact in $\ell_{p'}$, we can find a null sequence (γ_n) in $\ell_{p'}$ with $D(B_{\ell_{p'}}) \subset \overline{\operatorname{co}}(\gamma_n)$. Then

$$A \subset \phi_{\hat{z}}(B_{\ell_{p'}}) = \phi\big(D(B_{\ell_{p'}})\big) \subset \phi\big(\overline{\operatorname{co}}(\gamma_n)\big) \subset \overline{\operatorname{co}}\big(\phi(\gamma_n)\big).$$

Since ϕ is p-compact, ϕ^* is quasi p-nuclear [3, Corollary 3.4] and, in particular, p-summing. So, according to Proposition 2.4, the sequence $\big(\phi(\gamma_n)\big)$ is p-null in X.

For the converse, just notice that the closed convex hull of relatively p-compact sets are p-compact. $\square$

We finish this section with a characterization of p-null sequences in Banach spaces having the k_p-approximation property. A Banach space X has the k_p-approximation property if, for every Banach space Y, the space $\mathcal{F}(Y,X)$ of all finite rank operators is k_p-dense in $\mathcal{K}_p(Y,X)$ [4]. This property might be considered as a gradation of the classic approximation property. Since the classic approximation property implies the k_p-approximation property for all $p \geq 1$, the following result can be applied to a wide class of Banach spaces.

Proposition 2.6. *If X has the k_p-approximation property, then $\hat{x} = (x_n) \in c_{0,p}(X)$ if and only if $\hat{x} \in c_0(X)$ and has relatively p-compact rank.*

Proof. Only the "if" part needs to be proved (the converse is straightforward from the definition of p-null sequences). So, given $\varepsilon > 0$ and a sequence $(x_n) \in c_0(X)$ with relatively p-compact rank, there exists a finite rank operator $S\colon X \longrightarrow X$ such that $k_p(U_{\hat{x}} - S \circ U_{\hat{x}}) < \varepsilon/3$ [4, Theorem 2.1]. According to Proposition 2.4, the sequence $S\hat{x} = (Sx_n)$ is p-null and this allows us to find $n_0 \in \mathbb{N}$ for which $k_p(U_{S\hat{x}} - U_{S\hat{x}(n)}) < \varepsilon/3$ whenever $n \geq n_0$. Finally, if $P_n\colon \ell_1 \longrightarrow \ell_1$ denotes the projection onto the first n-th coordinates, we have, for every $n \geq n_0$,

$$\begin{aligned} k_p(U_{\hat{x}} - U_{\hat{x}(n)}) &\leq k_p(U_{\hat{x}} - S \circ U_{\hat{x}}) + k_p(S \circ U_{\hat{x}} - S \circ U_{\hat{x}(n)}) + k_p(S \circ U_{\hat{x}(n)} - U_{\hat{x}(n)}) \\ &\leq k_p(U_{\hat{x}} - S \circ U_{\hat{x}}) + k_p(U_{S\hat{x}} - U_{S\hat{x}(n)}) + k_p(S \circ U_{\hat{x}} - U_{\hat{x}})\,\|P_n\| \\ &< \varepsilon. \end{aligned}$$ $\square$

In [4], it is shown that every Banach space has the k_2-approximation property. Nevertheless, we do not know whether the hypothesis of the k_p-approximation property can be omitted in the previous result for $p \neq 2$.

3. On the equality $\Pi_p(c_0, X^*) = \Pi_q(c_0, X^*)$

In [3], it is proved that, for every $q \geq 1$ and every infinite dimensional Banach space, there exist relatively compact subsets that are not relatively q-compact [3, Theorem 3.14]. The objective of this section is to find out if this result is also true when we replace compact ($=\infty$-compact) with p-compact, $p > q$. We begin with a description of the dual space $c_{0,p}(X)^*$ which will be very useful in several proofs. Recall that the trace functional (denoted by tr) is well defined and continuous on $\mathcal{N}_1(X, X^{**})$ if and only if X^* has the approximation property.

Proposition 3.1. *Let X be a Banach space and $p \geq 1$. The dual space $c_{0,p}(X)^*$ is isometrically isomorphic to $\Pi_{p'}(c_0, X^*)$.*

Proof. Given $S \in \Pi_{p'}(c_0, X^*)$, we consider the linear form f_S on $c_{0,p}(X)$ defined by $f_S(\hat{x}) = \mathrm{tr}\,(U_{\hat{x}}^* \circ S)$. Since $U_{\hat{x}}$ is p-compact, the adjoint $U_{\hat{x}}^*\colon X^* \longrightarrow \ell_\infty$ is quasi p-nuclear [3, Corollary 3.4]; in particular, $U_{\hat{x}}^*$ is p-summing and the following inequalities hold:

$$\pi_p(U_{\hat{x}}^*) \leq \nu_p^Q(U_{\hat{x}}^*) \leq k_p(U_{\hat{x}}).$$

Then, the composition $U_{\hat{x}}^* \circ S \in \Pi_1(c_0, \ell_\infty) = \mathcal{N}_1(c_0, \ell_\infty)$ and $\nu_1(U_{\hat{x}}^* \circ S) \leq \pi_p(U_{\hat{x}}^*)\pi_{p'}(S)$ [17, p. 55]. Since ℓ_1 has the approximation property, the map f_S is well defined and

$$|\mathrm{tr}\,(U_{\hat{x}}^* \circ S)| \leq \nu_1(U_{\hat{x}}^* \circ S) \leq k_p(U_{\hat{x}})\pi_{p'}(S).$$

Therefore, $f_S \in c_{0,p}(X)^*$ and $\|f_S\| \le \pi_{p'}(S)$.

On the other hand, if $f \in c_{0,p}(X)^*$, we are able to define the operator $S_f \colon \alpha \in c_0 \longmapsto S_f\alpha \in X^*$ so that $\langle S_f\alpha, x\rangle = f\big(\alpha \otimes x\big)$ for all $x \in X$, where $\alpha \otimes x := (\alpha_n x)$ if $\alpha = (\alpha_n)$. To see that $S_f \in \Pi_{p'}(c_0, X^*)$, consider $(\alpha_k) \in \ell^w_{p'}(c_0)$ with $\|(\alpha_k))\|^w_{p'} \le 1$ and let us prove

$$\left(\sum_k \|S_f\alpha_k\|^{p'}\right)^{1/p'} \le \|f\|. \tag{3.1}$$

Fix $N \in \mathbb{N}$, $x_1, \ldots, x_N \in B_X$ and $(\beta_k)_{k=1}^N \in B_{\ell_p^N}$. We have

$$\left|\sum_{k=1}^N \langle S_f\alpha_k, x_k\rangle\beta_k\right| = \left|f\left(\sum_{k=1}^N \alpha_k \otimes \beta_k x_k\right)\right| \le \|f\|\ k_p\left(\sum_{k=1}^N \alpha_k \otimes \beta_k x_k\right).$$

Set $\hat{y} = \sum_{k=1}^N \alpha_k \otimes \beta_k x_k$. Notice that $(\alpha_k^n)_n \in B_{\ell_{p'}}$ for each $k \in \mathbb{N}$, so we can ensure that $U_{\hat{y}}e_n^* = \sum_{k=1}^N \alpha_k^n \beta_k x_k$ belongs to the closed and absolutely convex set $p\text{-}\operatorname{co}(\beta_k x_k)_{k=1}^N$ for each $n \in \mathbb{N}$ (here, (e_n^*) denotes the unit vector basis of ℓ_1). From this, it is clear that

$$k_p(\hat{y}) = k_p(U_{\hat{y}}) \le \|(\beta_k x_k)_{k=1}^N\|_p \le 1.$$

Summing up,

$$\left|\sum_{k=1}^N \langle S_f\alpha_k, x_k\rangle\beta_k\right| \le \|f\|$$

for all $N \in \mathbb{N}$, $x_1, \ldots, x_N \in B_X$ and $(\beta_k)_{k=1}^N \in B_{\ell_p^N}$, from which (3.1) is obtained, using a standard argument. So S_f is p'-summing and, moreover, $\pi_{p'}(S) \le \|f\|$.

Now, the functionals f_{S_f} and f coincide on finitely supported sequences in $c_{0,p}(X)$. Indeed, no matter how we choose $N \in \mathbb{N}$ and $\hat{x}_0 = (x_1, \ldots, x_N, 0, 0, \ldots)$,

$$\begin{aligned} f(\hat{x}_0) &= \sum_{k=1}^N f(e_k \otimes x_k) = \sum_{k=1}^N \langle S_f e_k, U_{\hat{x}_0} e_k^*\rangle \\ &= \sum_{k=1}^N \langle (U_{\hat{x}_0}^* \circ S_f)e_k, e_k^*\rangle = \operatorname{tr}(U_{\hat{x}_0}^* \circ S_f) = f_S(\hat{x}_0). \end{aligned}$$

Since $\hat{x} = k_p - \lim_n \hat{x}(n)$ for every $\hat{x} \in c_{0,p}(X)$ (Remark 2.2), it is clear that the functionals f_{S_f} and f coincide on $c_{0,p}(X)$. $\square$

The following result turns out to be the key to answer the question posed at the beginning of this section. As usual, if X and Y are Banach spaces and $x^* \in X^*$ and $y \in Y$ are fixed, $x^* \otimes y$ denotes the one dimensional operator from X to Y defined by $(x^* \otimes y)x = \langle x^*, x\rangle y$ for all $x \in X$.

Theorem 3.2. *Let X and Y be Banach spaces and let $1 \le q < p < \infty$. The following statements are equivalent for an operator $T \in \mathcal{L}(X, Y)$:*

a) *The operator T maps relatively p-compact subsets of X to relatively q-compact subsets of Y.*
b) *The operator $T^* \circ u \in \Pi_{p'}(c_0, X^*)$ whenever $u \in \Pi_{q'}(c_0, Y^*)$ ($u \in \mathcal{L}(c_0, Y^*)$, in case $q = 1$).*

Proof. *a*) $\Rightarrow$ *b*) It is easy to prove that the operator $V\colon u \in \mathcal{K}_p(\ell_1, X) \longmapsto T \circ u \in \mathcal{K}_q(\ell_1, Y)$ has closed graph. Then there exists a positive constant C such that

$$k_q(T \circ u) \leq C\, k_p(u) \quad \text{for all } u \in \mathcal{K}_p(\ell_1, X).$$

In view of this, for every p-null sequence $\hat{x}$ we have

$$k_q(U_{T\hat{x}} - U_{T\hat{x}(n)}) = k_q(T \circ (U_{\hat{x}} - U_{\hat{x}(n)})) \leq C\, k_p(U_{\hat{x}} - U_{\hat{x}(n)}).$$

So, from Remark 2.2, T maps p-null sequences to q-null sequences and, therefore, we can consider the operator $\widehat{T}\colon (x_n) \in c_{0,p}(X) \to (Tx_n) \in c_{0,q}(Y)$. It only remains to prove that the adjoint $\widehat{T}^*\colon \Pi_{q'}(c_0, Y^*) \longrightarrow \Pi_{p'}(c_0, X^*)$ is defined by $\widehat{T}^*(u) = T^* \circ u$ for all $u \in \Pi_{q'}(c_0, Y^*)$. To see this, consider $x \in X$, $\beta = (\beta_n) \in c_0$ and $u \in \Pi_{q'}(c_0, Y^*)$ and denote the sequence $(\beta_n x)$ by $\beta \otimes x$. On one hand, we have

$$\begin{aligned}\langle \widehat{T}^*(u), \beta \otimes x\rangle &= \operatorname{tr}\Big(U^*_{\beta\otimes x} \circ\ \widehat{T}^*(u)\Big) = \operatorname{tr}\Big((x \otimes \beta)\ \circ\ \widehat{T}^*(u)\Big)\\ &= \operatorname{tr}\Big(\big(\widehat{T}^*(u)^*x\big)\ \otimes\ \beta\Big) = \langle \widehat{T}^*(u)^*x, \beta\rangle = \langle x, \widehat{T}^*(u)\beta\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle u, \widehat{T}(\beta \otimes x)\rangle &= \operatorname{tr}\Big(U^*_{\beta\otimes Tx} \circ\ u\Big) = \operatorname{tr}\Big((Tx \otimes \beta) \circ u\Big)\\ &= \operatorname{tr}\Big(u^*(Tx)\ \otimes\ \beta\Big) = \langle u^*(Tx), \beta\rangle = \langle x, (T^* \circ u)\beta\rangle.\end{aligned}$$

As $\langle \widehat{T}^*(u), \beta \otimes x\rangle = \langle u, \widehat{T}(\beta \otimes x)\rangle$, it follows that $\langle x, \widehat{T}^*(u)\beta\rangle = \langle x, (T^* \circ u)\beta\rangle$ for all $x \in X$ and $\beta \in c_0$, which yields $\widehat{T}^*(u) = T^* \circ u$.

b) $\Rightarrow$ *a*) It is a standard argument to prove that the operator

$$u \in \Pi_{q'}(c_0, Y^*) \longmapsto T^* \circ u \in \Pi_{p'}(c_0, X^*)$$

has closed graph and, therefore, that it is continuous. If Φ denotes the restriction of this operator to $\mathcal{N}_{q'}(c_0, Y^*)$, then $\Phi^*\colon c_{0,p}(X)^{**} \longrightarrow \Pi_q(Y^*, \ell_\infty)$ is defined by $\Phi^*(\hat{x}) = U^*_{\hat{x}} \circ T^*$ for all $\hat{x} \in c_{0,p}(X)$ (just using a similar argument as in *a*) $\Rightarrow$ *b*) above). Since $U^*_{\hat{x}} \circ T^* = U^*_{T\hat{x}}$, the restriction of Φ^* to $c_{0,p}(X)$ is the operator

$$\hat{x} \in c_{0,p}(X) \longmapsto U^*_{T\hat{x}} \in \Pi_q(Y^*, \ell_\infty). \tag{3.2}$$

Now, we will show that Φ^* maps $c_{0,p}(X)$ to $\mathcal{QN}_q(Y^*, \ell_\infty)$. According to [16, Theorem 2.6], the q-summing norm and the quasi q-nuclear norm of a finite rank operator coincide. So, given $\hat{x} \in c_{0,p}(X)$, we have for $m > n$:

$$\nu_q^Q\Big(\Phi^*\big(\hat{x}(m)\big) - \Phi^*\big(\hat{x}(n)\big)\Big) = \pi_q\Big(\Phi^*\big(\hat{x}(m)\big) - \Phi^*\big(\hat{x}(n)\big)\Big) \leq \|\Phi^*\|\, k_p\Big(\hat{x}(m) - \hat{x}(n)\Big).$$

This proves that $\Big(\Phi^*\big(\hat{x}(n)\big)\Big)$ is a Cauchy sequence in $\mathcal{QN}_q(Y^*, \ell_\infty)$ from which it is easy to conclude that $\Phi^*(\hat{x}) \in \mathcal{QN}_q(Y^*, \ell_\infty)$. Hence, we can replace $\Pi_q(Y^*, \ell_\infty)$ with $\mathcal{QN}_q(Y^*, \ell_\infty)$ in (3.2). In view of [3, Proposition 3.5], this means that T maps p-null sequences to sequences with relatively q-compact rank. The proof concludes by just invoking Theorem 2.5. $\square$

Let $\mathcal{P}_{p,q}$ $(1 \leq q < p)$ be the class consisting of all Banach spaces with the property that every relatively p-compact subset is relatively q-compact.

Corollary 3.3. *Let $1 \leq q < p < \infty$. The following statements are equivalent for a Banach space X:*

a) $X \in \mathcal{P}_{p,q}$.
b) $\Pi_{q'}(c_0, X^*) = \Pi_{p'}(c_0, X^*)$ ($\mathcal{L}(c_0, X^*) = \Pi_{p'}(c_0, X^*)$ *if* $q = 1$).

In [6], the class of Banach spaces Y satisfying

$$\Pi_2(c_0, Y) = \mathcal{L}(c_0, Y) \tag{3.3}$$

is deeply studied. Having in mind that $\mathcal{L}(c_0, Y) = \Pi_\infty(c_0, Y)$, Corollary 3.3 reveals that a dual Banach space $Y = X^*$ enjoys (3.3) if and only if relatively 2-compact subsets of X are relatively 1-compact. Related to the ideas of [6], several conditions equivalent to Corollary 3.3(b) are established in the following result, the proof of which is omitted since it is quite similar to the proof in [6, Proposition 2.1].

Proposition 3.4. *Let X be a Banach space and let $1 \leq r < s$. The following statements are equivalent:*

a) $\Pi_r(Y, X) = \Pi_s(Y, X)$ *for every $\mathcal{L}_\infty$-space Y.*
b) $\Pi_r(Y, X) = \Pi_s(Y, X)$ *for some infinite dimensional $\mathcal{L}_\infty$-space Y.*
c) *There exists a positive constant C such that $\pi_r(u) \leq C\,\pi_s(u)$ for all $n \in \mathbb{N}$ and $u \in \mathcal{L}(\ell_\infty^n, X)$.*

Proposition 3.5. *Let $1 \leq q < p$ and let X be a Banach space.*

1) *If $X \in \mathcal{P}_{p,q}$ and E is a closed subspace of X, then $X/E \in \mathcal{P}_{p,q}$.*
2) *$X^{**} \in \mathcal{P}_{p,q}$ if and only if $X \in \mathcal{P}_{p,q}$.*
3) *If $2 \leq q < p$, then $\mathcal{P}_{p,q}$ does not contain any infinite dimensional Banach space.*

Proof. To prove 1), notice that $u \in \Pi_{q'}(c_0, (X/E)^*) = \Pi_{q'}(c_0, E^\perp)$ if and only if $i_{E^\perp} \circ u \in \Pi_{q'}(c_0, X^*)$, where $i_{E^\perp}$ denotes the inclusion map from $E^\perp$ into X^*. By hypothesis and using the injectivity of the ideal of p'-summing operators, it follows that $u \in \Pi_{p'}(c_0, (X/E)^*)$.

If $X^{**} \in \mathcal{P}_{p,q}$, then a similar argument allows us to obtain that $X \in \mathcal{P}_{p,q}$. The converse follows easily from Proposition 3.4(c) using the local reflexivity principle.

Let us argue by contradiction to show 3). Suppose there exists an infinite dimensional Banach space $X \in \mathcal{P}_{p,q}$. By virtue of Corollary 3.3, there exists a positive constant C such that

$$\pi_{p'}(u) \leq C\,\pi_{q'}(u) \tag{3.4}$$

for all $u \in \Pi_{q'}(c_0, X^*)$. Now, for every $\gamma = (\gamma_n) \in \ell_{q'}$, consider the operator

$$D_\gamma \colon (x_n^*) \in \ell_q^w(X^*) \longmapsto (\gamma_n x_n^*) \in \ell_{p'}(X^*).$$

To see that D_γ is well defined, take $(x_n^*) \in \ell_q^w(X^*)$ and define the operators $A\colon c_0 \longrightarrow \ell_{q'}$ and $B\colon \ell_{q'} \longrightarrow X^*$ by $A(\alpha_n) = (\gamma_n \alpha_n)$ and $B(\beta_n) = \sum_n \beta_n x_n^*$. It is easy to prove that A is q'-summing with $\pi_{q'}(A) \leq \|\gamma\|_{q'}$ and $\|B\| \leq 1$. Hence, $B \circ A$ is q'-summing and, by hypothesis, p'-summing. Moreover, in view of (3.4), we have

$$\begin{aligned}\left(\sum_n \|\gamma_n x_n^*\|^{p'}\right)^{1/p'} &= \left(\sum_n \|(B \circ A)e_n\|^{p'}\right)^{1/p'} \\ &\leq \pi_{p'}(B \circ A) \leq C\,\pi_{q'}(B \circ A) \leq C\,\|\gamma\|_{q'}\,\|(x_n^*)\|_q^w.\end{aligned}$$

This shows that D_γ is well defined and continuous for all $\gamma \in \ell_{q'}$. In particular, the inequality

$$\left(\sum_{n=1}^{N} \|\gamma_n x_n^*\|^{p'}\right)^{1/p'} \leq C\,\|\gamma\|_{q'}\,\|(x_n^*)_{n=1}^N\|_q^w \tag{3.5}$$

holds for all $N \in \mathbb{N}$ and $x_1^*, \ldots, x_N^* \in X^*$. According to Dvoretzky–Rogers' Theorem [11, p. 38], given $\varepsilon > 0$, for every $N \in \mathbb{N}$ there exist unitary vectors $x_1^*, \ldots, x_N^* \in X^*$ such that $\|(x_n^*)_{n=1}^N\|_2^w \leq (1+\varepsilon)$. As $q \geq 2$, applying (3.5) to these vectors, it follows that

$$\left(\sum_{n=1}^{N} |\gamma_n|^{p'}\right)^{1/p'} \leq C\,\|\gamma\|_{q'}\,(1+\varepsilon)$$

for all $N \in \mathbb{N}$. This is a contradiction if we take $\gamma \in \ell_{q'} \backslash \ell_{p'}$. □

If $1 \leq q < p \leq 2$, the following proposition shows that there are (infinite dimensional) spaces in which relatively p-compact sets are relatively q-compact.

Proposition 3.6. *The following statements are equivalent for a Banach space X:*

a) *X^* has finite cotype.*

b) *There exist $p, q \in \mathbb{R}$, $1 \leq q < p \leq 2$, such that $X \in \mathcal{P}_{p,q}$.*

In addition, if X^* has cotype $s > 2$ (respectively, $s = 2$), then $X \in \mathcal{P}_{p,q}$ for every $1 \leq q < p < s'$ (respectively, $1 \leq q < p \leq 2$).

Proof. *a)* ⇒ *b)* Suppose X^* has finite cotype $s > 2$ (respectively, $s = 2$). According to [5, Theorem 11.14], we have $\Pi_r(c_0, X^*) = \mathcal{L}(c_0, X^*)$ for all $r > s$ (respectively, $\Pi_2(c_0, X^*) = \mathcal{L}(c_0, X^*)$). So, by Corollary 3.3, X belongs to $\mathcal{P}_{r',1}$ (respectively, X belongs to $\mathcal{P}_{2,1}$).

b) ⇒ *a)* By contradiction, if X^* does not have finite cotype, then X^* contains ℓ_∞^n uniformly [5, Theorem 14.1]. Thus, there is a constant $\lambda > 0$ such that, for all $n \in \mathbb{N}$, there exists an isomorphism J_n from ℓ_∞^n onto a finite dimensional subspace E_n of X^* satisfying

$$\|J_n\|\|J_n^{-1}\| \leq \lambda \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Suppose that p and $q > 1$ are real numbers satisfying *b)* (if $q = 1$, the proof is quite similar). A call to Corollary 3.3 tells us that there exists a constant $C > 0$ such that

$$\pi_{p'}(u) \leq C\,\pi_{q'}(u) \quad \text{for all } u \in \mathcal{L}(\ell_\infty^n, X^*) \text{ and } n \in \mathbb{N}. \tag{3.7}$$

If I_∞^n denotes the identity map on ℓ_∞^n, (3.6) and (3.7) yield

$$\pi_{p'}(I_\infty^n) \leq \|J_n^{-1}\|\pi_{p'}(J_n) \leq C\,\|J_n^{-1}\|\pi_{q'}(J_n) \leq C\,\lambda\,\pi_{q'}(I_\infty^n). \tag{3.8}$$

On one hand, we have the estimation

$$n^{1/p'} = \left(\sum_{k=1}^{n} \|I_\infty^n(e_k)\|^{p'}\right)^{1/p'} \leq \pi_{p'}(I_\infty^n), \tag{3.9}$$

where (e_n) is the canonical vector basis in ℓ_∞^n. On the other hand, as I_∞^n admits the q'-nuclear representation $I_\infty^n = \sum_{k=1}^n e_k^* \otimes e_k$, it follows that

$$\nu_{q'}(I_\infty^n) \leq \|(e_k^*)\|_{q'}\|(e_k)\|_q^w \leq n^{1/q'}. \tag{3.10}$$

Since $\pi_{q'}(I_\infty^n) = \nu_{q'}(I_\infty^n)$ [17, p. 49], (3.8), (3.9) and (3.10) lead us to conclude

$$n^{1/p'} \le \pi_{p'}(I_\infty^n) \le C\,\lambda\,\pi_{q'}(I_\infty^n) \le C\,\lambda\,n^{1/q'}$$

for all $n \in \mathbb{N}$, which is a contradiction since $q' > p'$. □

Remark 3.7. The additional information in the above result cannot be improved for spaces whose dual has cotype $s > 2$. Indeed, if $s' \le p \le 2$, let us show that $X = \ell_{s'} \notin \mathcal{P}_{p,q}$ for all $q < p$. First, put $p = s'$ and suppose, by contradiction, that there exists $q < s'$ satisfying $\ell_{s'} \in \mathcal{P}_{s',q}$. In view of Proposition 3.6, $\ell_{s'}$ belongs to $\mathcal{P}_{s',1}$; that is, $\mathcal{L}(\ell_\infty, \ell_s) = \Pi_s(\ell_\infty, \ell_s)$, which contradicts [8, Theorem 7]. Now, for the case $p > s'$, it suffices to see that $X = \ell_{s'} \notin \mathcal{P}_{p,q}$ when q satisfies $s' \le q < p$. Under this assumption, consider a sequence $(\lambda_n) \in \ell_p \backslash \ell_q$. The rank of the sequence $(\lambda_n e_n)$ is obviously relatively p-compact in X. Arguing again by contradiction, if the sequence is relatively q-compact, then the operator $U\colon e_n \in \ell_1 \longmapsto \lambda_n e_n \in X$ is q-compact. Therefore, U^* is quasi q-nuclear [3, Corollary 3.4] and, in particular, q-summing. As $(e_n^*) \in \ell_{s'}^w(X^*) \subset \ell_q^w(X^*)$, it follows that

$$\sum_n |\lambda_n|^q = \sum_n \|U^* e_n^*\|_\infty^q < +\infty,$$

which is a contradiction (here, (e_n^*) designs the unit vector basis of $X^* = \ell_s$). Finally, notice that this latter fact can be extended to $\mathcal{L}_{s'}$-spaces, since every infinite dimensional $\mathcal{L}_r$-space has a complemented subspace isomorphic to ℓ_r.

Remark 3.8. In general, the classes $\mathcal{P}_{p,q}$ are not closed under closed subspaces. For an example, consider $X = \ell_\infty$; X^* has cotype 2 since it is an $\mathcal{L}_1$-space. Therefore, ℓ_∞ belongs to $\mathcal{P}_{2,1}$. Nevertheless, ℓ_1 is isometrically isomorphic to a closed subspace of ℓ_∞ but $\ell_1 \notin \mathcal{P}_{2,1}$ (Proposition 3.6).

We finish with an application of the preceding results to the theory of vector measures. It is well known that only finite dimensional Banach spaces X have the property that every relatively compact subset of X lies inside the range of a vector measure of bounded variation [13, Theorem 3.6]. We are going to prove that if we replace compact ($= \infty$-compact) with p-compact ($p \le 2$), then there exist infinite dimensional Banach spaces with such a property. It suffices to deal with p-null sequences instead of relative p-compact sets (Theorem 2.5), so the following lemma gains importance for getting our objective:

Lemma ([12, Lemma 2])**.** *Let $\hat{x}$ be a bounded sequence in a Banach space X.*

1) *The rank of $\hat{x}$ lies inside the range of a X^{**}-valued measure of bounded variation if and only if the operator $U_{\hat{x}}\colon \ell_1 \longrightarrow X$ is 1-integral.*
2) *If $U_{\hat{x}}$ is 1-nuclear, then the rank of $\hat{x}$ lies inside the range of an X-valued measure of bounded variation.*

Proposition 3.9. *Let X be a Banach space and let $1 \le p < +\infty$. Every relatively p-compact subset of X lies inside the range of a vector measure of bounded variation if and only if X belongs to $\mathcal{P}_{p,1}$.*

Proof. If every relatively p-compact subset of X lies inside the range of a vector measure of bounded variation, the previous lemma guarantees that the operator $\Phi\colon \hat{x} \in c_{0,p}(X) \longmapsto U_{\hat{x}} \in \mathcal{I}_1(\ell_1, X)$ is well defined. A standard argument shows that Φ has closed graph and, therefore, is continuous. Actually, Φ maps $c_{0,p}(X)$ into $\mathcal{N}_1(\ell_1, X)$, since $\hat{x} = k_p - \lim_n \hat{x}(n)$ whenever $\hat{x} \in c_{0,p}(X)$ and using that $\mathcal{N}_1(\ell_1, X)$ is isometrically isomorphic to a closed subspace of $\mathcal{I}_1(\ell_1, X)$ [11, p. 132]. Obviously, this implies that every relatively p-compact subset of X is relatively 1-compact.

For the converse, notice that the operator $U_{\hat{x}}$ belongs to $\mathcal{K}_1(\ell_1, X)$ whenever $\hat{x}$ is a relatively p-compact sequence in X. Since $\mathcal{K}_1(\ell_1, X)$ can be isometrically identified with $\mathcal{N}_1(\ell_1, X)$, the proof is concluded via [12, Lemma 2]. □

In view of this result and Proposition 3.6, for a fixed $p \in [1, 2)$ (respectively, $p = 2$), every Banach space X such that X^* has cotype $s > p'$ (respectively, $s = 2$) satisfies that its relatively p-compact sets lies in the range of an X-valued measure of bounded variation.

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