

## ON CERTAIN EXPLICIT CONGRUENCES FOR MOCK THETA FUNCTIONS

MATTHIAS WALDHERR

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ABSTRACT. Recently, Garthwaite and Penniston have shown that the coefficients of Ramanujan’s mock theta function  $\omega$  satisfy infinitely many congruences of Ramanujan type. In this work we give the first explicit examples of congruences for Ramanujan’s mock theta function  $\omega$  and another mock theta function  $\mathcal{C}$ .

### 1. INTRODUCTION AND STATEMENT OF RESULTS

The famous “Ramanujan congruences” for the partition function

$$\begin{aligned}p(5n + 4) &\equiv 0 \pmod{5}, \\p(7n + 5) &\equiv 0 \pmod{7}, \\p(11n + 6) &\equiv 0 \pmod{11},\end{aligned}$$

found nearly 100 years ago, have since had an enormous impact on the theory of  $q$ -series arising in combinatorics. One was trying to understand why such congruences appear and whether there are analogous Ramanujan-type congruences for other interesting  $q$ -series. The first question is now well understood. The *partition rank* introduced by Dyson [Dys44] explains the first two congruences from a combinatorial point of view. The *crank* later defined by Andrews and Garvan [AG88] even explains all three congruences. For the second question, many examples for similar congruences have been found for other  $q$ -series, interestingly even for generating series of the rank and crank itself; see [BO10] and [Mah05].

Using computer calculations it is easy to find candidates for such congruences for a given  $q$ -series. However, there is no general method for proving them. In most cases the proofs rely on ingenious  $q$ -series manipulations and  $q$ -series identities. Only in the special case, when the  $q$ -series in question is a modular form, is there an approach that is capable of proving or disproving any given congruence. This method is based on the fact, known as Sturm’s theorem [Stu87], that it suffices to check the congruences for the coefficients of a modular form up to some explicitly computable bound in order to conclude that it holds for all coefficients.

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Many interesting examples of  $q$ -series which arise in combinatorics are not modular. Famous examples are given by Ramanujan's mock theta functions

$$f(\tau) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} = \sum_{n=0}^{\infty} a_f(n)q^n$$

and

$$\omega(\tau) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \cdots (1-q^{1+2n})^2} = \sum_{n=0}^{\infty} a_\omega(n)q^n,$$

where  $q = e^{2\pi i\tau}$  with  $\tau$  in the upper half-plane  $\mathbb{H}$ .

Over the years, there has been much effort by authors such as Watson [Wat36], Dragonette and Andrews [And66], [Dra52] to understand the mock theta functions. A new chapter in the study of mock theta functions, however, was opened only recently by Zwegers. In [Zwe01] and [Zwe02], he related Ramanujan's mock theta functions to harmonic weak Maass forms which are certain nonholomorphic generalizations of classical modular forms (for a precise definition we refer to the next chapter). This result turned out to be the starting point for further research in this area. Since questions about asymptotics, exact formulas and congruences are better understood in the context of Maass forms and modular forms, it became possible to prove long-standing conjectures on mock theta functions: In [Bri09], Bringmann proves asymptotic formulas for rank generating functions. In [BO06], Bringmann-Ono prove the Andrews-Dragonette conjecture concerning an exact formula for the coefficients of the mock theta function  $f$  and in [BO10], Bringmann-Ono prove the existence of infinitely many Ramanujan-type congruences for  $f$ . Garthwaite [Gar08] and Garthwaite-Penniston [GP08] have obtained similar results for  $\omega$ .

Despite the fact that there are infinitely many congruences, to the authors' knowledge not a single example has been exhibited yet. Using Borcherds' products, Bruinier-Ono [BO08] show congruences for  $\omega$ , which are, however, not of Ramanujan type.

In contrast to that, it is quite easy to find candidates for congruences simply by computing many coefficients and searching for congruence patterns. This was done by Jeremy Lovejoy for several  $q$ -series including  $\omega$  and the following function:

$$\mathcal{C}(\tau) := \sum_{n=0}^{\infty} (-1)^n \frac{(q; q^2)_n}{(-q; q)_n^2} = 2 \prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 - q^{2n}} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}n(n+1)}}{1 + q^n} = \sum_{n=0}^{\infty} a_{\mathcal{C}}(n)q^n,$$

which we will call the Cesaro function (since the series representation has to be interpreted in the Cesaro sense), and which is also related to Zwegers' work.

The objective of this paper is twofold. Firstly, we prove congruences for  $\omega$  and  $\mathcal{C}$ , thereby verifying the conjectures of Lovejoy and giving the first explicit examples of congruences for  $\omega$ . Our results can be summarized in the following theorem.

**Theorem 1.1.** *For all  $n \geq 0$ , the following congruences hold:*

$$\begin{aligned} a_\omega(40n + 27) &\equiv a_\omega(40n + 35) \equiv 0 \pmod{5}, \\ a_{\mathcal{C}}(3n + 1) &\equiv 0 \pmod{3}, \\ a_{\mathcal{C}}(7n + 2) &\equiv a_{\mathcal{C}}(7n + 3) \equiv a_{\mathcal{C}}(7n + 5) \equiv 0 \pmod{7}. \end{aligned}$$

Secondly, in the course of proving the theorem it will turn out that our approach is capable of proving or disproving a given congruence of Ramanujan type (i.e. the congruences that are supported on arithmetic progressions) for  $\omega$  and  $\mathcal{C}$  as long

as a certain condition is satisfied. To explain this condition we note that both  $\mathcal{C}$  and  $\omega$  will appear as holomorphic parts of harmonic weak Maass forms. The condition now requires that the congruences only involve coefficients of the Fourier expansion of the Maass form which belong to the holomorphic part. We reduce proving a congruence to a finite amount of computation in the same fashion as Sturm’s theorem does this for modular forms. In fact, the application of Sturm’s theorem is a crucial step in our argument.

*Remark.* After completing this work the author learned that Song Heng Chan in his recent preprint [Ch10] has given an alternative proof of the congruences for  $\mathcal{C}$ .

2. BASICS FACTS ABOUT HARMONIC WEAK MAASS FORMS

In this paper we use standard terminology of the theory of modular forms. For basic definitions the reader is referred to chapter 1 of [Ono04]. In addition to modular forms, harmonic Maass forms will play a crucial role. These functions were originally introduced by Bruinier-Funke in [BF04]. An excellent reference for harmonic Maass forms for the theta multiplier and its applications is [Ono09]. However, in this paper we also work with other multiplier systems. Therefore we give a definition of modular forms and harmonic weak Maass forms for arbitrary multiplier systems. A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is called a *weakly holomorphic modular form* of weight  $\frac{k}{2}$  with respect to a congruence subgroup  $\Gamma \leq \text{SL}_2(\mathbb{Z})$  and a multiplier system  $\nu$  if the following conditions hold:

- (1)  $f$  satisfies the modular transformation property:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \nu\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) (c\tau + d)^{\frac{k}{2}} f(\tau).$$

- (2)  $f$  is holomorphic on  $\mathbb{H}$ .
- (3)  $f$  has at most linear exponential growth at the cusps.

We call a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  a *harmonic weak Maass form* if the second condition is replaced by the weaker condition that  $f$  is annihilated by the weight  $\frac{k}{2}$  hyperbolic Laplacian  $\Delta_{\frac{k}{2}} := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{ik}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ . Note that this implies that  $f$  is a real-analytic function. We will write  $M_{\frac{k}{2}}^!(\Gamma, \nu)$  and  $H_{\frac{k}{2}}(\Gamma, \nu)$  for the space of weakly holomorphic modular forms and harmonic weak Maass forms, respectively. By  $M_{\frac{k}{2}}(\Gamma, \nu)$  we denote the space of holomorphic modular forms, i.e., weakly holomorphic modular forms which are holomorphic at all the cusps.

Let  $\chi$  be a Dirichlet character and define  $\nu_{\theta, \chi} : \Gamma_0(4) \rightarrow \{z \in \mathbb{C} \mid |z| = 1\}$  by

$$\nu_{\theta, \chi} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) := \chi(d) \left( \frac{c}{d} \right) \epsilon_d^{-1},$$

where  $\left( \frac{c}{d} \right)$  denotes the Jacobi symbol and

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

This function is called the  $\theta$ -multiplier, and a harmonic weak Maass form with respect to this multiplier will just be called a harmonic weak Maass form with character  $\chi$ . The  $\eta$ -multiplier  $\nu_{\eta} : \text{SL}_2(\mathbb{Z}) \rightarrow \{z \in \mathbb{C} \mid |z| = 1\}$  is defined by

$$\nu_{\eta} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) := \frac{1}{\sqrt{c\tau + d}} \frac{\eta\left(\frac{a\tau+b}{c\tau+d}\right)}{\eta(\tau)},$$

where  $\eta$  is the classical Dedekind  $\eta$ -function. Both the  $\theta$ - and the  $\eta$ -multiplier and any integral power of them are multiplier systems for all half-integral weights.

### 3. THE CESARO FUNCTION $\mathcal{C}$

In this section we prove the congruences for  $\mathcal{C}$  using results from Zwegers' thesis.

**3.1. Zwegers' results on  $\mu$ .** We first review some results of Zwegers [Zwe02], who defines the following function:

$$\mu(u, v; \tau) := \frac{e^{\pi i u}}{\theta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi i n v}}{1 - e^{2\pi i n \tau+2\pi i u}},$$

for  $\tau \in \mathbb{H}$  and  $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$  and where

$$\theta(v; \tau) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu(v + \frac{1}{2})}$$

is the classical Jacobi theta function. The function  $\mu$  is holomorphic but does not transform like a modular form. Zwegers shows that we can complete  $\mu$  by adding a nonholomorphic but real analytic correction term  $R$  in the following way:

$$\tilde{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau),$$

so that the resulting function  $\tilde{\mu}$  has nice transformation properties. We will not recall Zwegers' construction of the correction term  $R$ , since we do not use it. We need another description of  $R$  in terms of following unary theta series of weight  $\frac{3}{2}$ :

$$g_{a,b}(\tau) := \sum_{\nu \in a + \mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b},$$

where  $a, b \in \mathbb{R}$ .

**Theorem 3.1** ([Zwe02], Theorem 1.16). *For  $a \in ]\frac{1}{2}, \frac{1}{2}[$  and  $b \in \mathbb{R}$ , we have*

$$-e^{-\pi i a^2 \tau + 2\pi i a(b + \frac{1}{2})} R(a\tau + b; \tau) = \int_{-\bar{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2}, b+\frac{1}{2}}(t)}{\sqrt{-i(t+\tau)}} dt.$$

Zwegers then proves transformation formulas for  $\tilde{\mu}$ . From his results (Theorem 1.11 of [Zwe02]) we easily obtain the following proposition.

**Proposition 3.2.** *For any  $z \in \mathbb{C}$ , any  $\tau \in \mathbb{H}$  and any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , we have*

$$\tilde{\mu}\left(\frac{z}{c\tau+d}, \frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \nu_{\eta}^{-3} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (c\tau + d)^{\frac{1}{2}} \tilde{\mu}(z, z; \tau).$$

Furthermore, for any  $z \in \mathbb{C}$  and any  $\tau \in \mathbb{H}$ , we have

$$\tilde{\mu}(z + \tau, z + \tau; \tau) = \tilde{\mu}(z, z; \tau) \text{ and } \tilde{\mu}(z + 1, z + 1; \tau) = \tilde{\mu}(z, z; \tau).$$

**3.2. Modular transformation properties.** The holomorphic part of the function  $\tilde{\mu}(\frac{1}{2}, \frac{1}{2}; \tau)$  will turn out to be related to  $\mathcal{C}$ . In this section we will study the modular transformation properties of  $h_1(\tau) := \tilde{\mu}(\frac{1}{2}, \frac{1}{2}; \tau)$ . For that purpose we additionally need the functions  $h_2(\tau) := \tilde{\mu}(\frac{\tau}{2}, \frac{\tau}{2}; \tau)$  and  $h_3(\tau) := \tilde{\mu}(\frac{\tau+1}{2}, \frac{\tau+1}{2}; \tau)$ .

**Lemma 3.3.** *We have*

$$\mathbf{h}(\tau + 1) = \begin{pmatrix} \nu_\eta^{-3}(T) & 0 & 0 \\ 0 & 0 & \nu_\eta^{-3}(T) \\ 0 & \nu_\eta^{-3}(T) & 0 \end{pmatrix} \mathbf{h}(\tau),$$

$$\frac{1}{\sqrt{\tau}} \mathbf{h}(-\frac{1}{\tau}) = \begin{pmatrix} 0 & \nu_\eta^{-3}(S) & 0 \\ \nu_\eta^{-3}(S) & 0 & 0 \\ 0 & 0 & \nu_\eta^{-3}(S) \end{pmatrix} \mathbf{h}(\tau),$$

where  $\mathbf{h} := (h_1, h_2, h_3)^T$ ,  $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

*Proof.* We only show the second statement. The first one follows analogously. By means of the first transformation formula in Proposition 3.2, applied to the matrix  $S$  and  $z = \frac{\tau}{2}$ , we find

$$h_1(-\frac{1}{\tau}) = \tilde{\mu}(\frac{1}{2}, \frac{1}{2}; -\frac{1}{\tau}) = \tilde{\mu}(\frac{z}{\tau}, \frac{z}{\tau}; -\frac{1}{\tau}) = \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}(z, z; \tau) = \nu_\eta^{-3}(S) \sqrt{\tau} h_2(\tau).$$

Similarly, using the same transformation formula for  $S$  and  $z = -\frac{1}{2}$  we obtain

$$h_2(-\frac{1}{\tau}) = \tilde{\mu}\left(\frac{-\frac{1}{2}}{\tau}, \frac{-\frac{1}{2}}{\tau}; -\frac{1}{\tau}\right) = \tilde{\mu}\left(\frac{z}{\tau}, \frac{z}{\tau}; -\frac{1}{\tau}\right) = \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}\left(-\frac{1}{2}, -\frac{1}{2}; \tau\right).$$

The second transformation formula in Proposition 3.2 then shows that

$$h_2(-\frac{1}{\tau}) = \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}\left(-\frac{1}{2}, -\frac{1}{2}; \tau\right) = \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}\left(\frac{1}{2}, \frac{1}{2}; \tau\right) = \nu_\eta^{-3}(S) \sqrt{\tau} h_1(\tau).$$

Finally, applying the first transformation formula in Proposition 3.2 to the matrix  $S$  and  $z = \frac{\tau-1}{2}$  yields

$$h_3(-\frac{1}{\tau}) = \tilde{\mu}\left(\frac{-\frac{1}{\tau}+1}{2}, \frac{-\frac{1}{\tau}+1}{2}; -\frac{1}{\tau}\right) = \tilde{\mu}\left(\frac{z}{\tau}, \frac{z}{\tau}; -\frac{1}{\tau}\right) = \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}(z, z; \tau)$$

$$= \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}\left(\frac{\tau-1}{2}, \frac{\tau-1}{2}; \tau\right) = \nu_\eta^{-3}(S) \sqrt{\tau} \tilde{\mu}\left(\frac{\tau+1}{2}, \frac{\tau+1}{2}; \tau\right) = \nu_\eta^{-3}(S) \sqrt{\tau} h_3(\tau).$$

□

The lemma shows that  $\mathbf{h}$  transforms as a vector-valued modular form. We will use this fact to derive the transformation properties for the first component  $h_1(\tau)$ .

**Proposition 3.4.** *The function  $h_1(\tau)$  is a harmonic weak Maass form of weight  $\frac{1}{2}$  with respect to the group  $\Gamma_0(2)$  with multiplier system  $\nu_\eta^{-3}$ .*

*Proof.* Using Lemma 3.3 one readily sees that under any of the transformations  $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , and  $ST^2S = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$ , the function  $h_1$  is mapped to a constant multiple of itself, where the factor is given by the multiplier system  $\nu_\eta^{-3}$ . It is easy to see that the group generated by  $-I, T, ST^2S$  is  $\Gamma_0(2)$ . The growth condition for  $h_1(\tau)$  can be deduced from its Fourier expansion, which we give in the next section. Furthermore it follows from Zwegers' results (see Proposition 4.2 in [Zwe02]) that  $h_1(\tau)$  is annihilated by  $\Delta_{\frac{1}{2}}$ . □

It is important to understand how  $h_1(\tau)$  transforms under all elements of  $SL_2(\mathbb{Z})$ .

**Proposition 3.5.** *Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . Then  $\frac{1}{\sqrt{c\tau+d}}h_1\left(\frac{a\tau+b}{c\tau+d}\right)$  is a nonzero multiple of*

- (1)  $h_1(\tau)$  if  $c$  is even and  $d$  is odd,
- (2)  $h_2(\tau)$  if  $c$  is odd and  $d$  is even,
- (3)  $h_3(\tau)$  if  $c$  and  $d$  are odd.

*Proof.* The first statement follows from Proposition 3.4. If  $c$  is odd and  $d$  is even, then

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = - \begin{pmatrix} a & b \\ c & d \end{pmatrix} S S = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} S = (-MS)S,$$

and we see that  $-MS \in \Gamma_0(2)$ . Now the result follows from Lemma 3.3. Analogously, if both  $c$  and  $d$  are odd, then we find  $M = (-MT^{-1}S)ST$  and  $-MT^{-1}S \in \Gamma_0(2)$ , and the result follows from Lemma 3.3.  $\square$

**3.3. Decomposition into holomorphic and nonholomorphic parts and Fourier expansions.** In this section we will see how  $\mathcal{C}$  is related to  $\mu(\frac{1}{2}, \frac{1}{2}; \tau)$ . It follows from Zwegers' results that

$$\tilde{\mu}\left(\frac{1}{2}, \frac{1}{2}; \tau\right) = \mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) - \frac{i}{2} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1}{2}, \frac{1}{2}}(t)}{\sqrt{-i(t+\tau)}} dt.$$

From this representation we can easily deduce the Fourier expansion of the non-holomorphic part of  $\tilde{\mu}(\frac{1}{2}, \frac{1}{2}; \tau)$ .

**Proposition 3.6.** *The function  $\tilde{\mu}(\frac{1}{2}, \frac{1}{2}; \tau)$  has the following decomposition into a holomorphic and a nonholomorphic part:*

$$\tilde{\mu}\left(\frac{1}{2}, \frac{1}{2}; \tau\right) = \mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) + \frac{i\sqrt{2}}{4\sqrt{\pi}} \sum_{n \in \mathbb{Z}} (-1)^n q^{-\frac{(2n+1)^2}{4}} \Gamma\left(\frac{1}{2}, \pi(2n+1)^2 y\right).$$

We next turn to the Fourier expansion of the holomorphic part. Indeed, we require the Fourier expansions of the holomorphic parts of  $h_1, h_2$ , and  $h_3$ . It is clear that these are  $\mu(\frac{1}{2}, \frac{1}{2}; \tau), \mu(\frac{\tau}{2}, \frac{\tau}{2}; \tau)$ , and  $\mu(\frac{\tau+1}{2}, \frac{\tau+1}{2}; \tau)$ , respectively.

**Proposition 3.7.** (1) *We have*

$$\mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) = -\frac{i}{\sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}n(n+1)}}{1+q^n} = \frac{1}{4i} q^{-\frac{1}{8}} \mathcal{C}(\tau).$$

Consequently the Fourier expansion of  $\mu(\frac{1}{2}, \frac{1}{2}; \tau)$  starts with

$$-\frac{i}{4} q^{-\frac{1}{8}} - \frac{3i}{4} q^{\frac{7}{8}} + \dots.$$

(2) *We have*

$$\mu\left(\frac{\tau}{2}, \frac{\tau}{2}; \tau\right) = -\frac{iq^{\frac{1}{4}}}{\sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n^2+2n+1)}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{1}{2}n(n+2)}}{1-q^{n+\frac{1}{2}}}.$$

Consequently the Fourier expansion of the holomorphic part of  $h_2$  starts as

$$\mu\left(\frac{\tau}{2}, \frac{\tau}{2}; \tau\right) = 2iq^{\frac{1}{4}} + 6iq^{\frac{3}{4}} + \dots.$$

(3) *We have*

$$\mu\left(\frac{\tau+1}{2}, \frac{\tau+1}{2}; \tau\right) = -\frac{iq^{\frac{1}{4}}}{\sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n^2+2n+1)}} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}n(n+2)}}{1+q^{n+\frac{1}{2}}}.$$

Consequently the Fourier expansion of the holomorphic part of  $h_3$  starts with

$$\mu\left(\frac{\tau+1}{2}, \frac{\tau+1}{2}; \tau\right) = -2iq^{\frac{1}{4}} + 6iq^{\frac{3}{4}} + \dots$$

*Proof.* We find that

$$\mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) = \frac{e^{\pi i \frac{1}{2}}}{\theta\left(\frac{1}{2}; \tau\right)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau + \pi i n}}{1 - e^{2\pi i n \tau + \pi i}} = \frac{i}{\theta\left(\frac{1}{2}; \tau\right)} \sum_{n \in \mathbb{Z}} (-1)^{2n} \frac{q^{\frac{1}{2}n(n+1)}}{1 + q^n}.$$

For the  $\theta$ -function we obtain the following series expansion:

$$\theta\left(\frac{1}{2}; \tau\right) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2 \tau + 2\pi i(n+\frac{1}{2})(\frac{1}{2}+\frac{1}{2})} = - \sum_{n \in \mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2 \tau} = - \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}.$$

Using the well-known product expansion for the  $\theta$ -function,

$$\theta(v; \tau) = -ie^{\frac{\pi i \tau}{4}} e^{-\pi i v} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \left(1 - e^{2\pi i v} e^{2\pi i(n-1)\tau}\right) (1 - e^{-2\pi i v} e^{2\pi i n \tau}),$$

we can also write

$$\theta\left(\frac{1}{2}; \tau\right) = -2q^{\frac{1}{8}} \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}}.$$

This proves the relation to  $\mathcal{C}$ . For the holomorphic parts of  $h_2$  and  $h_3$  we can derive the Fourier expansions in a similar manner.  $\square$

**3.4. Sieving out residue classes.** Since the holomorphic part of  $h_1(\tau)$  is (up to a constant)  $q^{-\frac{1}{8}}\mathcal{C}(\tau)$ , we find that the coefficients  $a_{\mathcal{C}}(3n + 1)$  correspond to the coefficients of  $h_1(\tau)$  with exponent  $24n + 7$  in the Fourier expansion given in terms of  $q^{\frac{1}{8}}$ . Similarly, we see that  $a_{\mathcal{C}}(7n + 2), a_{\mathcal{C}}(7n + 3)$ , and  $a_{\mathcal{C}}(7n + 5)$  correspond to the Fourier coefficients with exponents  $56n + 15, 56n + 23$  and  $56n + 39$ .

For a harmonic weak Maass form with Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n(y) q^{\frac{n}{w}}$$

with  $w \in \mathbb{N}$  and for  $r, m \in \mathbb{Z}$  we define the sieve operator by

$$U_{r,m}f(\tau) := \sum_{n \equiv r \pmod{m}} a_n(y) q^{\frac{n}{w}}.$$

For harmonic weak Maass forms with character, one can show in a standard manner that  $U_{r,m}f$  again is a harmonic weak Maass form. This result may be generalized to other multiplier systems. We need this result only for the multiplier system  $\nu_{\eta}^{-3}$ .

**Proposition 3.8.** *Let  $f$  be a harmonic weak Maass form of weight  $\frac{1}{2}$  for  $\Gamma_0(2)$  with respect to  $\nu_{\eta}^{-3}$ . Suppose that  $f$  has a Fourier expansion in terms of  $q^{\frac{1}{8}}$ . Let  $r, m \in \mathbb{N}$ . Then  $U_{r,m}f$  is a harmonic weak Maass form of weight  $\frac{1}{2}$  with respect to the congruence subgroup*

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a, d \text{ coprime to } m, a \equiv d \pmod{m} \text{ and } 2m^2 \mid c \right\}.$$

We have  $\Gamma_1(2m^2) \leq \Gamma \leq \Gamma_0(2m^2)$  and

$$[\text{SL}_2(\mathbb{Z}) : \Gamma] = \frac{2m^4 \varphi(m)}{\varphi(2m^2)} \prod_{p \mid 2m^2} \left(1 - \frac{1}{p^2}\right),$$

where  $\varphi$  is Euler's  $\varphi$ -function.

We now apply this result to those Fourier coefficients of  $h_1$  in which we are interested.

**Proposition 3.9.** (1) *The function*

$$\mathcal{C}_{7(24)}(\tau) := \sum_{n \equiv 7 \pmod{24}} a_c \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}$$

is a weakly holomorphic modular form of weight  $\frac{1}{2}$  with respect to  $\nu_\eta^{-3}$  for some subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  which contains  $\Gamma_1(1152)$  and has index 9216 in  $\mathrm{SL}_2(\mathbb{Z})$ .

(2) *The function*

$$\mathcal{C}_{15,23,39(56)}(\tau) := \sum_{n \equiv 15,23,39 \pmod{56}} a_c \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}$$

is a weakly holomorphic modular form of weight  $\frac{1}{2}$  with respect to  $\nu_\eta^{-3}$  for some subgroup  $\Gamma'$  of  $\mathrm{SL}_2(\mathbb{Z})$  which contains  $\Gamma_1(6272)$  and has index 129024 in  $\mathrm{SL}_2(\mathbb{Z})$ .

*Proof.* We only show the first assertion; the second one is proven similarly. The first function is (up to a constant) the holomorphic part of the function obtained from  $h_1(\tau)$  by sieving out the coefficients  $24n+7$  with  $n \in \mathbb{Z}$ . By Proposition 3.8 and Proposition 3.4 this function is a harmonic weak Maass form for the congruence subgroup with the properties stated above. It remains to show that the function is holomorphic; i.e., its nonholomorphic part vanishes. This is due to the fact that by Proposition 3.6 the nonholomorphic part is supported at Fourier coefficients of the form  $q^{\frac{-2(2n+1)^2}{8}}$  with  $n \in \mathbb{Z}$  and the fact that  $24n+7$  can never be twice a negative square of an odd number. To see this, suppose that  $24n+7 \equiv -2(2X+1)^2 \pmod{8}$  with  $X \in \mathbb{Z}$ . This implies that  $7 \equiv -2 \pmod{8}$ , a contradiction.  $\square$

We now would like to apply Sturm’s theorem to the modular forms in Proposition 3.9. Sturm’s theorem is not applicable since it is only valid for modular forms which are holomorphic at the cusps. The next two sections will show that we may apply Sturm’s theorem if we first multiply with suitable cusp forms.

**3.5. Behaviour at the cusps.** The functions  $\mathcal{C}_{7(24)}$  and  $\mathcal{C}_{15,23,39(56)}$  arise from  $h_1(\tau)$  by applying the sieve operator. For an arbitrary harmonic weak Maass form  $f$  with Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n(y) q^{\frac{n}{w}},$$

the sieve operator  $U_{r,m}$  can be written as

$$U_{r,m}f(\tau) = \frac{1}{m} \sum_{s \pmod{m}} \zeta_m^{rs} f\left(\tau - \frac{ws}{m}\right),$$

where  $\zeta_m := e^{\frac{2\pi i}{m}}$ . Hence, in order to determine the behavior of the two functions above at the cusps, we investigate how the holomorphic part of  $h_1$  behaves under the translation  $\tau \mapsto \tau - \frac{ws}{m}$ . For this we require the following lemma, which can be proved by a straightforward calculation.

**Lemma 3.10.** *Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  and suppose  $0 \leq s < m$ . Set  $l := \mathrm{gcd}(cm, wsc + am)$ . Then define  $\tilde{a} := \frac{wsc+am}{l}$  and  $\tilde{c} := \frac{cm}{l}$ . Let  $\tilde{d}$  be such that  $\tilde{a}\tilde{d} - 1 \equiv 0 \pmod{\tilde{c}}$*



and  $\tilde{b} := \frac{\tilde{a}\tilde{d}-1}{\tilde{c}}$ . Finally define  $t := \frac{d\tilde{l}-\tilde{d}m}{\tilde{c}}$ . Then  $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  and we have for all  $\tau \in \mathbb{H}$ ,

$$\begin{pmatrix} 1 & \frac{s}{m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} l & t \\ 0 & \frac{m^2}{l} \end{pmatrix} \tau.$$

Now we determine the pole orders of the holomorphic parts of the shifts of  $h_1(\tau)$ .

**Lemma 3.11.** *Let  $\frac{a}{c}$  be a cusp and suppose that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  is a matrix which maps  $\infty$  to this cusp. Furthermore suppose that  $m$  is some integer and  $0 \leq s < m$ .*

- (1) *The Fourier expansion of the holomorphic part of  $h_1(\tau + \frac{ws}{m})$  at the cusp  $\frac{a}{c}$  is the Fourier expansion at  $\infty$  of  $h_1(\tau)$  under the transformation*

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} l & t \\ 0 & \frac{m^2}{l} \end{pmatrix}$$

*with notation as in Lemma 3.10.*

- (2) *The first Fourier coefficient of the holomorphic part of  $h_1(\tau)$  at the cusp  $\frac{a}{c}$  up to a nonzero constant is given by*
  - (a)  $q^{-\frac{l^2}{8m^2}}$  *if  $\tilde{c}$  is even and  $\tilde{d}$  is odd,*
  - (b)  $q^{\frac{l^2}{4m^2}}$  *if  $\tilde{c}$  is odd and  $\tilde{d}$  is even,*
  - (c)  $q^{\frac{l^2}{4m^2}}$  *if  $\tilde{c}$  is odd and  $\tilde{d}$  is odd.*

*Proof.* The statement (1) is clear. The three statements in (2) are proved by applying Proposition 3.5 as follows: If  $\tilde{c}$  is even and  $\tilde{d}$  is odd, then by Proposition 3.5 we know that under  $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ , the function  $h_1(\tau)$  is mapped to a multiple of  $h_1(\tau)$ . The holomorphic part of  $h_1(\tau)$  is up to a constant  $q^{-\frac{1}{8}}\mathcal{C}(\tau)$ , which has a Fourier expansion starting with  $q^{-\frac{1}{8}}$ . Hence, applying the matrix  $\begin{pmatrix} l & t \\ 0 & \frac{m^2}{l} \end{pmatrix}$  shows that the Fourier expansion starts with  $q^{-\frac{l^2}{8m^2}}$  in this case.

In the other two cases,  $h_1(\tau)$  is mapped to a multiple of  $h_2(\tau)$  or  $h_3(\tau)$  under  $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$  by Proposition 3.5. The Fourier expansion of the holomorphic part of both these functions starts with  $q^{\frac{1}{4}}$ , and applying the matrix  $\begin{pmatrix} l & t \\ 0 & \frac{m^2}{l} \end{pmatrix}$  gives the result. □

**3.6. Application of Sturm’s theorem.** In this section we reduce proving the congruences for  $\mathcal{C}$  to a finite computation by applying Sturm’s theorem. Before doing this we have to get rid of the poles by multiplying with a suitable cusp form.

**Proposition 3.12.** *We have*

- (1) *The function  $\eta^{12}(24\tau)\Delta(\tau)$  is a cusp form of weight 18 for  $\Gamma_0(1152)$ . Furthermore the function*

$$\tilde{\mathcal{C}}_{7(24)}(\tau) := \eta^{12}(24\tau)\Delta(\tau) \sum_{n \equiv 7 \pmod{24}} a_{\mathcal{C}} \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}$$

*is holomorphic at every cusp of  $\Gamma_1(1152)$ .*

(2) The function  $\eta^{48}(56\tau)\Delta^2(\tau)$  is a cusp form of weight 48 for  $\Gamma_0(6272)$ . Furthermore the function

$$\tilde{\mathcal{C}}_{15,23,39(56)}(\tau) := \eta^{48}(56\tau)\Delta^2(\tau) \sum_{n \equiv 15,23,39 \pmod{24}} a_C \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}$$

is holomorphic at every cusp of  $\Gamma_1(6272)$ .

*Proof.* The statements for the  $\eta$ -product follow from Theorem 1.65 of [Ono04]. In the first case we take  $r_{24} = 12$ ,  $r_1 = 12$ , and  $r_d = 0$  for all other divisors  $d$  of  $N = 1152$  as in the statement of the theorem. In the second case we choose  $r_{56} = 48$ ,  $r_1 = 24$ , and again  $r_d = 0$  for all other divisors of  $N = 6272$ . The assertion about the holomorphicity at the cusps is checked by a computer as follows: For any cusp of  $\Gamma_1(1152)$  and  $\Gamma_1(6272)$  respectively we find a representative in the form  $\frac{a}{c}$ . Then we use Lemma 3.11 to find upper bounds for the pole orders of  $\sum_{n \equiv 7 \pmod{24}} a_C \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}$  and  $\sum_{n \equiv 15,23,39 \pmod{24}} a_C \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}$  at the cusp. Using Theorem 1.65 of [Ono04] we may compare the pole orders to the order of vanishing of the  $\eta$ -product.  $\square$

The next result now follows easily from Proposition 3.12 and Sturm’s theorem.

**Proposition 3.13.** (1) The congruence  $a_C(3n+1) \equiv 0 \pmod{3}$  holds for all  $n \in \mathbb{N}$  if it holds for all  $n \in \mathbb{N}$  with  $24n + 7 \leq 7104$ .

(2) The congruence  $a_C(7n+2) \equiv a_C(7n+3) \equiv a_C(7n+5) \equiv 0 \pmod{7}$  holds for all  $n \in \mathbb{N}$  if it holds for all  $n \in \mathbb{N}$  with  $56n + 15, 56n + 23, 56n + 39 \leq 260734$ .

*Proof.* We consider the functions  $\tilde{\mathcal{C}}_{7(24)}(\tau)$  and  $\tilde{\mathcal{C}}_{15,23,39(56)}(\tau)$ . We first note that these functions have integral Fourier coefficients, since they are defined by a product of functions all of which have integral Fourier coefficients. Using Proposition 3.12 and Proposition 3.9 the first product is a modular form of weight  $18 + \frac{1}{2}$  for some subgroup  $\Gamma$  of  $\text{SL}_2(\mathbb{Z})$  satisfying  $\Gamma_1(1152) \leq \Gamma$  and  $[\text{SL}_2(\mathbb{Z}) : \Gamma] = 9216$ . Furthermore Proposition 3.12 implies that this form is holomorphic at the cusps of  $\Gamma_1(1152)$  and hence also at all cusps of  $\Gamma$ . Completely analogously we find, using Proposition 3.12 and Proposition 3.9, that the second product is a modular form of weight  $48 + \frac{1}{2}$  for a group  $\Gamma'$  satisfying  $[\text{SL}_2(\mathbb{Z}) : \Gamma'] = 129024$  and is holomorphic at the cusps. To both forms we apply Sturm’s theorem (see [Stu87], Theorem 1), which states that all coefficients of a modular form of weight  $\frac{k}{2}$  on  $\Gamma$  are divisible by a prime  $p$  iff this holds for all the coefficients up to the explicit bound

$$\frac{k}{24} [\text{SL}_2(\mathbb{Z}) : \Gamma].$$

In our case, this implies that all coefficients of the first form are divisible by 3 if this is true for the first 7104 ones. For the second form we find that the first 260736 coefficients have to be checked. The number  $a_C(3n+1)$  appears as the coefficient  $q^{\frac{24n+7}{8}}$  in the expansion of

$$\mathcal{C}_{7(24)}(\tau) = \sum_{n \equiv 7 \pmod{24}} a_C \left( \frac{n+1}{8} \right) q^{\frac{n}{8}}.$$

Suppose the assertion in the statement of the theorem is satisfied; i.e., suppose the first 7104 coefficients of  $\mathcal{C}_{7(24)}$  are divisible by 3. Then, also the first 7104

coefficients of

$$\eta^{12}(24\tau)\Delta(\tau) \sum_{n \equiv 7 \pmod{24}} a_C \left(\frac{n+1}{8}\right) q^{\frac{n}{8}}$$

are divisible by 3. But then the consequence of Sturm’s theorem is that all coefficients of the product are divisible by 3. Since the leading coefficient of  $\eta^{12}(24\tau)\Delta(\tau)$  is 1, we can argue with induction and conclude that all coefficients of  $\mathcal{C}_{7(24)}$  are divisible by 3. This establishes the first claim. The second claim follows analogously.  $\square$

4. THE MOCK THETA FUNCTION  $\omega$

In this section we complete the proof of Theorem 1.1 by proving the congruences for  $\omega$ . We will carry out a similar program as for  $\mathcal{C}$ . We recall how  $\omega$  relates to Zwegers’ results and show that it is the holomorphic part of a harmonic weak Maass form. Then we sieve out for this Maass form those coefficients which are related to the presumed congruences and obtain a weakly holomorphic modular form. We will study its behavior at the cusps and apply Sturm’s theorem. Since this program so closely parallels our treatment of  $\mathcal{C}$ , we will omit most of the proofs.

4.1. **Work of Zwegers and Garthwaite-Penniston on  $\omega$ .** Following Zwegers [Zwe01] we define

$$\begin{aligned} F_1(\tau) &:= q^{-\frac{1}{24}} f(q), \\ F_2(\tau) &:= 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}), \\ F_3(\tau) &:= 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}), \end{aligned}$$

and the vector-valued function  $\mathbf{F}(\tau) := (F_1(\tau), F_2(\tau), F_3(\tau))^T$  for  $\tau \in \mathbb{H}$ . For  $z \in \mathbb{C}$  define

$$\begin{aligned} G_1(z) &:= -\sum_{n \in \mathbb{Z}} (n + \frac{1}{6}) e^{3\pi i (n + \frac{1}{6})^2 z}, \\ G_2(z) &:= \sum_{n \in \mathbb{Z}} (-1)^n (n + \frac{1}{3}) e^{3\pi i (n + \frac{1}{3})^2 z}, \\ G_3(z) &:= \sum_{n \in \mathbb{Z}} (n + \frac{1}{3}) e^{3\pi i (n + \frac{1}{3})^2 z}, \end{aligned}$$

and finally for  $\tau \in \mathbb{H}$ :

$$\mathbf{G}(\tau) := 2i\sqrt{3} \int_{-\bar{\tau}}^{i\infty} \frac{(G_1(z), G_2(z), G_3(z))^T}{\sqrt{-i(\tau+z)}} dz.$$

We can now state the main theorem of [Zwe01].

**Theorem 4.1** ([Zwe01], Theorem 3.6). *The function  $\mathbf{H}(\tau) := \mathbf{F}(\tau) - \mathbf{G}(\tau)$  is a vector-valued real analytic modular form of weight  $\frac{1}{2}$  satisfying*

$$\mathbf{H}(\tau + 1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} \mathbf{H}(\tau)$$

and

$$\frac{1}{\sqrt{-i\tau}} \mathbf{H}\left(-\frac{1}{\tau}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{H}(\tau).$$

Furthermore  $\mathbf{H}$  is annihilated by  $\Delta_{\frac{1}{2}}$ .

The fact that  $H_2(\tau)$  is not mapped to a multiple of itself under translation causes some problems. We circumvent this by studying instead  $H_2(6\tau)$ . For this function the transformation properties have already been studied completely.

**Theorem 4.2** ([GP08], Corollary 4.2). *The function  $H_2(6\tau)$  is a harmonic weak Maass form of weight  $\frac{1}{2}$  on  $\Gamma_0(144)$  with character  $\chi_{12}$ .*

Next we find the Fourier expansion of  $H_2(6\tau)$ .

**Lemma 4.3.** *The Fourier expansion of  $H_2(6\tau)$  has the form*

$$H_2(6\tau) = 2q^2\omega(q^3) + \frac{1}{\sqrt{\pi}} \sum_{n \equiv 1 \pmod{3}}^{\infty} 2(-1)^{\frac{n-1}{3}} q^{-n^2} \Gamma\left(\frac{1}{2}, 4\pi n^2 y\right).$$

Still we need to know how  $H_2(\tau)$  behaves under all transformations of  $\text{SL}_2(\mathbb{Z})$ .

**Proposition 4.4.** *Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ . Then  $\frac{1}{\sqrt{c\tau+d}} H_2\left(\frac{a\tau+b}{c\tau+d}\right)$  is a constant multiple of*

- (1)  $H_1(\tau)$ , if  $b$  is odd and  $a$  is even,
- (2)  $H_2(\tau)$ , if  $b$  is even and  $a$  is odd,
- (3)  $H_3(\tau)$ , if  $b$  and  $a$  are odd.

**4.2. Congruences for  $\omega$ .** The coefficients of  $\omega$  for which we expect the congruences as stated in Theorem 1.1 are exactly the Fourier coefficients of  $H_2(6\tau)$  at those powers of  $q$  which have the forms  $q^{120n+83}$  and  $q^{120n+107}$ . In order to sieve out these coefficients, we do not apply the sieving operator  $U_{r,m}$  directly, but we sieve with twists of quadratic characters, because this yields modular forms on bigger groups, which is favorable for computational reasons. Consider the characters  $\chi_3(n) := \left(\frac{n}{3}\right)$  and  $\chi_5(n) := \left(\frac{n}{5}\right)$  and the quadratic characters  $\pmod{8}$  given by

	0	1	2	3	4	5	6	7
$\chi_8^{(0)}$	0	1	0	1	0	1	0	1
$\chi_8^{(1)}$	0	1	0	1	0	-1	0	-1
$\chi_8^{(2)}$	0	1	0	-1	0	1	0	-1
$\chi_8^{(3)}$	0	1	0	-1	0	-1	0	1

Then it is easy to verify that

$$\begin{aligned} \frac{1}{2} (\chi_3(n) - 1) \chi_3(n) &= \begin{cases} 1 & \text{if } n \equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 0, 1 \pmod{3}, \end{cases} \\ \frac{1}{2} (1 + \chi_5(n)) \chi_5(n) &= \begin{cases} 1 & \text{if } n \equiv 2, 3 \pmod{5}, \\ 0 & \text{if } n \equiv 0, 1, 4 \pmod{5}, \end{cases} \\ \frac{1}{4} (\chi_8^{(0)}(n) + \chi_8^{(1)}(n) - \chi_8^{(2)}(n) - \chi_8^{(3)}(n)) &= \begin{cases} 1 & \text{if } n \equiv 3 \pmod{8}, \\ 0 & \text{if } n \not\equiv 3 \pmod{8}. \end{cases} \end{aligned}$$

The product of the above functions is exactly the characteristic function of the set  $\{120n + 83, 120n + 107 | n \in \mathbb{Z}\}$ . It is easy to prove that the twisted forms of

$H_2(6\tau)$  are harmonic weak Maass forms for a certain subgroup of  $SL_2(\mathbb{Z})$  which may be explicitly computed. More precisely we obtain the following result.

**Proposition 4.5.** *The function*

$$\sum_{n \equiv 27,35 \pmod{40}} a_\omega(n)q^{3n+2}$$

is a weakly holomorphic modular form of weight  $\frac{1}{2}$  on  $\Gamma_0(86400)$  with character  $\chi_{12}$ .

In order to determine the behavior of this function at the cusps of  $\Gamma_0[86400]$  we prove the following analogue of Lemma 3.10.

**Lemma 4.6.** *Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$  and suppose  $0 \leq s < m$  and  $6|m$ . Set  $l := \gcd(\frac{cm}{6}, sc + am)$ . Then define  $\tilde{a} := \frac{sc+am}{l}$  and  $\tilde{c} := \frac{cm}{6l}$ . Let  $\tilde{d}$  be such that  $\tilde{a}\tilde{d} - 1 \equiv 0 \pmod{\tilde{c}}$  and  $\tilde{b} := \frac{\tilde{a}\tilde{d}-1}{\tilde{c}}$ . Finally define  $t := \frac{dl-dm}{c}$ . Then for all  $\tau \in \mathbb{H}$  and  $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in SL_2(\mathbb{Z})$  we have*

$$\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} l & t \\ 0 & \frac{m^2}{6l} \end{pmatrix} \tau.$$

Using this lemma and the transformation properties of  $H_2(\tau)$  we can deduce the following lemma with a proof completely analogous to the proof of Lemma 3.11.

**Lemma 4.7.** *Let  $\frac{a}{c}$  be a cusp and suppose that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$  is a matrix which maps  $\infty$  to this cusp. Let  $m$  be some integer divisible by 6 and  $0 \leq s < m$ .*

- (1) *The Fourier expansion of the holomorphic part of  $H_2(6(\tau + \frac{s}{m}))$  at this cusp is the Fourier expansion at  $\infty$  of  $H_2(\tau)$  under the following transformation:*

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} l & t \\ 0 & \frac{m^2}{6l} \end{pmatrix}$$

with notation as in Lemma 4.6.

- (2) *The Fourier expansion of the holomorphic part of  $H_2(6\tau)$  at this cusp starts (up to a constant) with*
  - (a)  $q^{-\frac{l^2}{4m^2}}$  if  $\tilde{a}$  is even and  $\tilde{b}$  is odd,
  - (b)  $q^{\frac{l^2}{18m^2}}$  if  $\tilde{a}$  is odd and  $\tilde{b}$  is even,
  - (c)  $q^{\frac{l^2}{18m^2}}$  if  $\tilde{a}$  is odd and  $\tilde{b}$  is odd.

Our next task is to construct a suitable cusp form which we multiply with

$$\sum_{n \equiv 27,35 \pmod{40}} a_\omega(n)q^{3n+2}$$

in order to get a modular form that is holomorphic at the cusps to which we may apply Sturm's Theorem. Our result is as follows.

**Proposition 4.8.** *The function  $\eta^{240}(120\tau)\Delta^2(\tau)$  is a cusp form of weight 144 for  $\Gamma_0(86400)$ . Furthermore the function*

$$\eta^{240}(120\tau)\Delta^2(\tau) \sum_{n \equiv 27,35 \pmod{40}} a_\omega(n)q^{3n+2}$$

is holomorphic at every cusp of  $\Gamma_0(86400)$ .

The proof is analogous to the proof of Proposition 3.12. As an easy consequence of this proposition and Sturm’s theorem we get the following result.

**Proposition 4.9.** *The congruences*

$$a_\omega(40n + 27) \equiv a_\omega(40n + 35) \equiv 0 \pmod{5}$$

hold for all  $n$  if they hold for all  $n$  for which  $40n + 27$  or  $40n + 35$  is less than or equal to 832 320.

Using a computer program, which can be found on the homepage of the author <http://www.mi.uni-koeln.de/~mwaldher/>, we computed enough coefficients of both  $\omega$  and  $\mathcal{C}$  in order to deduce Theorem 1.1 from Proposition 3.13 and Proposition 4.9.

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MATHEMATICAL INSTITUTE, UNIVERSITY OF COLOGNE, WEYERTAL 86-90, 50931 COLOGNE,  
GERMANY

*E-mail address:* mwaldher@math.uni-koeln.de