PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 3, March 2011, Pages 1091–1093
S 0002-9939(2010)10553-1
Article electronically published on August 23, 2010

ON A QUESTION OF D. SHLYAKHTENKO

IONUT CHIFAN AND ADRIAN IOANA

(Communicated by Marius Junge)

Abstract. In this short paper we construct two countable, infinite conjugacy class (ICC) groups which admit free, ergodic, probability measure-preserving orbit equivalent actions but whose group von Neumann algebras are not (stably) isomorphic.

1. Introduction

Two countable, discrete groups Γ and Λ are *orbit equivalent* if they admit free, probability measure-preserving actions which generate isomorphic equivalence relations. They are W^**-equivalent* (or *von Neumann equivalent*) if their group von Neumann algebras are isomorphic. D. Shlyaktenko noticed that, for all known examples of orbit equivalent groups, their group von Neumann algebras are isomorphic and speculated that this might be the case in general. Subsequently, a number of people have also asked the question of whether orbit equivalence of groups implies von Neumann equivalence ([4],[8]). Motivated by this question, we prove:

Theorem 1. *There exist two countable, discrete, infinite conjugacy class groups Γ and Λ which are orbit equivalent but not W^*-equivalent. Moreover, the group von Neumann algebras $L\Gamma$ and $L\Lambda$ are not stably isomorphic.*

The construction of the groups Γ and Λ is based on the observation that being an infinite conjugacy class group is not an orbit equivalence invariant. Indeed, from [1] we have that $\Gamma_0 = S_\infty$, the group of finite permutations of $\mathbb{N}$, is orbit equivalent to $\Lambda_0 = \mathbb{Z}$. This example already shows that there are orbit equivalent groups which are not von Neumann equivalent. Further, notice that the map $\Gamma_0 \to \Gamma = (\Gamma_0 \times \mathbb{F}_2) \star \mathbb{Z}$ turns every pair (Γ_0, Λ_0) of orbit equivalent groups into a pair (Γ, Λ) of orbit equivalent, infinite conjugacy class groups. Finally, by applying the Kurosh type results for free product von Neumann algebras from [5] we derive that the group von Neumann algebras of Γ and Λ are not stably isomorphic.

2. Proof of theorem

Before proving the theorem, we recall the notion of stable isomorphism of II_1 factors. For a II_1 factor M and $0 < t \in \mathbb{R}$, the amplification M^t is defined as the isomorphism class of $p(\mathbb{M}_n(\mathbb{C}) \otimes M)p$, where $n > t$ is an integer and $p \in \mathbb{M}_n(\mathbb{C}) \otimes M$ is a projection of trace $\frac{t}{n}$. It is well known that this isomorphism class does not

Received by the editors September 3, 2009 and, in revised form, April 5, 2010.
2010 *Mathematics Subject Classification.* Primary 46L10; Secondary 37A20.
Key words and phrases. von Neumann algebra, W^*-equivalence, orbit equivalence.

depend on the choices of n and p. Then two II_1 factors are called *stably* isomorphic if one of them is isomorphic with an amplification of the other.

Proof. Let Γ_0 and Λ_0 be two infinite amenable groups and assume that Γ_0 is infinite conjugacy class (ICC) while Λ_0 is abelian. By [3], Γ_0 and Λ_0 are orbit equivalent. Further, by Section 2.2 in [2], the ICC groups $\Gamma = (\Gamma_0 \times \mathbb{F}_2) \star \mathbb{Z}$ and $\Lambda = (\Lambda_0 \times \mathbb{F}_2) \star \mathbb{Z}$ are orbit equivalent. We claim that the group von Neumann algebras $M = L\Gamma$ and $N = L\Lambda$ are not stably isomorphic. Let $M_0 = L\Gamma_0$ and $N_0 = L\Lambda_0$ and note that $M = (M_0 \overline{\otimes} L\mathbb{F}_2) \star L\mathbb{Z}$ and $N = (N_0 \overline{\otimes} L\mathbb{F}_2) \star L\mathbb{Z}$.

We suppose by contradiction that this is not the case. Therefore we can find an isomorphism $\theta : M^t \to N$, for some $t > 0$. Since M_0 is a factor we can view M_0^t as a subfactor of M^t. The commutant of M_0^t in M^t is then equal to $L\mathbb{F}_2$. Since the latter is a noninjective factor, by the first part of the proof of Theorem 3.3 in [5] we deduce that a corner of $\theta(M_0^t)$ can be embedded into $N_0 \overline{\otimes} L\mathbb{F}_2$ inside N in the sense of Popa (Theorem 2.1 in [7]).

This amounts to the existence of two nonzero projections $q_1 \in M_0^t$, $p \in N_0 \overline{\otimes} L\mathbb{F}_2$, a nonzero partial isometry $v \in N$ and a unital injective homomorphism $\phi \colon \theta(q_1 M_0^t q_1) \to p(N_0 \overline{\otimes} L\mathbb{F}_2)p$ such that $xv = v\phi(x)$ for every $x \in \theta(q_1 M_0^t q_1)$. Moreover, v satisfies $vv^* \in \theta(q_1 M_0^t q_1)' \cap \theta(q_1) N \theta(q_1)$ and $v^*v \in \phi(\theta(q_1 M_0^t q_1))' \cap pNp$ and therefore [6] together with the factoriality of M_0 implies that $vv^* \in \theta((\mathbb{C}q_1) \overline{\otimes} L\mathbb{F}_2)$ and $v^*v \in p(N_0 \overline{\otimes} L\mathbb{F}_2)p$. Hence $vv^* = \theta(q_1 \otimes q_2)$ for some nonzero projection $q_2 \in L\mathbb{F}_2$, and let $u \in N$ be a unitary such that $\theta(q_1 \otimes q_2)u = v$. By combining these relations with $xv = v\phi(x)$ we obtain that

$$u^*\theta((q_1 \otimes q_2)(M_0^t \overline{\otimes} (\mathbb{C}1))(q_1 \otimes q_2))u = v^*\theta(M_0^t)v = (v^*v)\phi(\theta(q_1 M_0^t q_1)) \subset N_0 \overline{\otimes} L\mathbb{F}_2.$$

By [6] this further implies that $u^*\theta((q_1 \otimes q_2)(M_0 \overline{\otimes} L\mathbb{F}_2)^t (q_1 \otimes q_2))u \subset N_0 \overline{\otimes} L\mathbb{F}_2$. Replacing $\mathrm{Ad}(u^*) \circ \theta$ with θ and then denoting by $q = q_1 \otimes q_2 \in M_0 \overline{\otimes} L\mathbb{F}_2$ and $r = \theta(q) \in N \overline{\otimes} L\mathbb{F}_2$ we can therefore assume that $\theta(q(M_0 \overline{\otimes} L\mathbb{F}_2)^t q) \subset r(N_0 \overline{\otimes} L\mathbb{F}_2)r$.

Since the center, $\mathcal{Z}$, of $r(N_0 \overline{\otimes} L\mathbb{F}_2)r$ is diffuse and θ is an isomorphism, we derive that $\theta^{-1}(\mathcal{Z})$ is also a diffuse subalgebra which is contained in the relative commutant of $q(M_0 \overline{\otimes} L\mathbb{F}_2)^t q$ in $qM^t q$. On the other hand, by [6], the relative commutant of $q(M_0 \overline{\otimes} L\mathbb{F}_2)^t q$ in $qM^t q$ is equal to the center of $q(M_0 \overline{\otimes} L\mathbb{F}_2)^t q$, which consists only in the scalars because M_0 is a factor. Therefore we have reached a contradiction. $\square$

We would like to end this note by mentioning the following open question: Are there any examples of W^*-equivalent groups which are not orbit equivalent?

Acknowledgments

The authors are grateful to Professors Sorin Popa and Yehuda Shalom for useful discussions and encouragement.

References

[1] H.A. Dye, *On groups of measure preserving transformations.* II, Amer. J. Math. **85** (1963), 551–576. MR0158048 (28:1275)

[2] D. Gaboriau, *Examples of groups that are measure equivalent to the free group*, Ergodic Theory and Dynam. Systems **25(6)** (2005), 1809–1827. MR2183295 (2006i:22024)

[3] D. Ornstein and B. Weiss, *Ergodic theory of the amenable groups. I. The Rokhlin Lemma*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 161–164. MR551753 (80j:28031)

[4] N. Ozawa, *Amenable actions and applications*, International Congress of Mathematicians, vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1563-1580. MR2275659 (2008k:46185)

[5] N. Ozawa, *A Kurosh-type theorem for type* II_1 *factors*, Int. Math. Res. Not. (2006), Art. ID 97560, 21 pp. MR2211141 (2006m:46078)

[6] S. Popa, *Orthogonal pairs of* $*$*-subalgebras in finite von Neumann algebras*, J. Operator Theory **9(2)** (1983), 253–268. MR703810 (84h:46077)

[7] S. Popa, *Strong rigidity for* II_1 *factors arising from malleable actions of w-rigid groups.* I, Invent. Math. **165(2)** (2006), 369–408. MR2231961 (2007f:46058)

[8] S. Popa, *Revisiting some problems in* W^**-rigidity*, 2009. Available at `http://www.math.ucla.edu/~popa/tamu0809rev.pdf`

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240 – and – Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania

E-mail address: `ionut.chifan@vanderbilt.edu`

Department of Mathematics, University of California Los Angeles, Los Angeles, California 90095 – and – Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania

E-mail address: `aioana@caltech.edu`