PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 3, March 2011, Pages 1023–1032
S 0002-9939(2010)10638-X
Article electronically published on September 1, 2010

AXIAL SYMMETRY OF SOME STEADY STATE SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS

CHANGFENG GUI, ANDREA MALCHIODI, AND HAOYUAN XU

(Communicated by Matthew J. Gursky)

Abstract. In this paper, we show the axial symmetry of steady state solutions of nonlinear Schrodinger equations when the exponent of the nonlinearity is between the critical Sobolev exponent of n-dimensional space and $(n-1)$-dimensional space.

1. Introduction

In this paper, we shall consider anisotropic bounded entire solutions, which only decay to zero in certain directions, to the nonlinear stationary Schrödinger equation

$$\Delta u - F'(u) = 0, \quad u > 0, \quad \text{in } \mathbb{R}^n. \tag{1}$$

where F is a C^2 function and $F(0) = 0, F'(0) = 0, F''(0) > 0$.

A typical such nonlinear function is $F(u) = \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1}$ with $1 < p < p^* := \frac{n+1}{n-3}$ when $n > 3$ and $p > 1$ when $2 \le n \le 3$. Several new entire solutions of this type were obtained recently in [3], [9] and [4]. Some qualitative properties of these solutions are also studied in [6].

It is well-known that if $F(u) = \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1}$ with $1 < p < 2^* := \frac{n+2}{n-2}$ when $n > 2$ and $p > 1$ when $n = 2$, then (1) has a solution $U(z)$ which is radial in z and converges to 0 exponentially as z goes to infinity, where we use the variable $z = (z_1, z') = (z_1, z_2, z'') \in \mathbb{R}^n$ with $z' \in \mathbb{R}^{n-1}, z'' \in \mathbb{R}^{n-2}$.

The following proposition states the existence of a family of solutions periodic in z_1, as shown by Dancer in [3].

Proposition 1. *Assume $F(u) = \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1}$ with $1 < p < p^* := \frac{n+1}{n-3}$ when $n > 3$ and $p > 1$ when $2 \le n \le 3$. For sufficiently large L there exists a solution u_L to (1) such that $u_L(z)$ is even and periodic in z_1 with period L and radial in z'. Furthermore, $u_L(z)$ goes to zero exponentially as z' goes to infinity.*

Solution u_L may be called Dancer's solution. It provides some essential ingredients in several recent developments regarding entire solutions of the nonlinear Schrödinger equation (see, e.g., [4], [9]). Let us consider a bounded positive solution u of (1). We assume that $u(z)$ goes to zero uniformly in z_1 as $|z'| \to \infty$. It is interesting to ask: Are all such solutions of (1) periodic in z_1 and axially symmetric

Received by the editors March 28, 2010.
2010 *Mathematics Subject Classification.* Primary 35J15, 35J20, 35J60, 35J61.
Key words and phrases. Nonlinear Schrödinger equation, steady state solution, Hamiltonian identity, axial symmetry, the moving plane method.

about some line parallel to the z_1-direction, i.e., $u(z) = u(z_1, |z' - z_0'|)$ for some $z_0' \in \mathbb{R}^{n-1}$? In this paper we shall partially answer this question. Several interesting properties of entire solutions are studied in [5], [6]. In particular, Hamiltonian identities for very general elliptic nonlinear equations or systems of the form (1) are formulated and applied to various problems. Here we state the following special version of the Hamiltonian identity, whose proof will be given in Section 2 for the convenience of the reader. A similar version of the identity was also formulated in [1] earlier.

Proposition 2. *Assume that $u(z) \in C^2(\mathbb{R}^n)$ is a bounded solution of* (1) *and satisfies*

$$|u(z_1, z')| \le C|z'|^{-\frac{n-1+\epsilon}{2}}, \quad z \in \mathbb{R}^n \tag{2}$$

for some constants $\epsilon > 0, C > 0$. Then the following Hamiltonian identity holds:

$$H(z_1; u) := \int_{\mathbb{R}^{n-1}} \left(\frac{1}{2}\big(|\nabla_{z'} u|^2 - |u_{z_1}|^2\big) + F\big(u(z)\big) \right) dz' = C, \quad \forall z_1 \in \mathbb{R}. \tag{3}$$

Hence, we can define the following quantity, which may be called the Hamiltonian constant for the solution u:

$$H(u) := H(z_1; u). \tag{4}$$

Using the Hamiltonian constant and the moving plane method, we can prove the following.

Theorem 3. *Let u be a bounded positive solution of* (1) *such that $\lim_{|z'|\to\infty} u(z_1, z') = 0$ uniformly in z_1. If the Hamiltonian constant $H(u) \neq 0$, then $u(z) = u(z_1, |z' - z_0'|)$ for some $z_0' \in \mathbb{R}^{n-1}$.*

When $H(u) = 0$, we believe that the above result should still be true. Actually we suspect that when $H(u) = 0$, modulo translations, a positive bounded solution of (1) which decays in $|z'|$ should be the unique radial solution of (1) which decays in $|z|$. However, we cannot prove this yet. Under another balance condition, using the moving plane method, we can prove that

Theorem 4. *Let u be a bounded positive solution of* (1) *such that $\lim_{|z'|\to\infty} u(z_1, z') = 0$ uniformly in z_1. If in addition we assume that u is even in z_1, then $u(z) = u(z_1, |z' - z_0'|)$ for some $z_0' \in \mathbb{R}^{n-1}$.*

We like to mention that axial symmetry has also been shown for traveling wave solutions and saddle solutions of the Allen-Cahn equation in [7] and [8].

2. Proof of Theorem 3 and Theorem 4

We will first prove Proposition 2.

Proof of Proposition 2. First we note that, by the gradient estimate of elliptic equations, (2) implies that

$$|\nabla u(z)| \le C|z'|^{-\frac{n-1+\epsilon}{2}}, \quad z \in \mathbb{R}^n \tag{5}$$

for some constant $C > 0$. Hence the integral in (3) is finite.

Let us define

$$\rho_R(z_1) = \int_{\{|z'| \le R\}} \frac{1}{2}\big(|\nabla_{z'} u|^2 - |u_{z_1}|^2\big) + F\big((u(z)\big) dz'. \tag{6}$$

Then, using the equation and integrating by parts, we have

$$
\begin{aligned}
\rho_R'(z_1) &= \int_{\{|z'|\le R\}} (\nabla_{z'} u \cdot \nabla_{z'} u_{z_1} - u_{z_1} \cdot u_{z_1 z_1} + F'(u(x)) \cdot u_{z_1}) dx' \\
&= \int_{\{|z'|\le R\}} (\nabla_{z'} u \cdot \nabla_{z'} u_{z_1} + \Delta_{z'} u \cdot u_{z_1}) dz' \\
&= \int_{\partial\{|z'|\le R\}} \frac{\partial u}{\partial \nu_{z'}} \cdot u_{z_1} dS_{z'}.
\end{aligned} \tag{7}
$$

Hence, in view of (2) and (5) we obtain

$$
\begin{aligned}
|\rho_R(z_1) - \rho_R(0)| &= |\int_0^{z_1} \int_{\partial\{|z'|\le R\}} \frac{\partial u}{\partial \nu_{z'}}(z', s) \cdot u_{z_1}(z', s) dS_{z'} ds| \\
&\le C|z_1| R^{-1-\epsilon}.
\end{aligned} \tag{8}
$$

Letting R go to infinity, we derive (3).

□

Remark 5. From the above proof, we can see that if we define

$$D_L := \{z = (z_1, z') : \quad |z_1| \le L,\ z' \in \mathbb{R}^{n-1}\}$$

and let u be a nonnegative solution of (1) which goes to zero as $|z'| \to \infty$, then $H(u)$ is a constant for $-L \le z_1 \le L$.

Similarly, for $i = 2, ..., n$ we can define

$$H_i(z_1; u) := \int_{\mathbb{R}^{n-1}} u_{z_1} \cdot u_{z_i} dz'. \tag{9}$$

Proposition 6. *Assume that $u(z) \in C^2(\mathbb{R}^n)$ is a positive solution of* (1) *which, uniformly in z', also decays exponentially. Then, $H_i(z_1; u)$ is constant in z_1 and hence can be denoted by $H_i(u)$ and called the i-th Hamiltonian constant for $i = 2, ..., n$.*

Proof. We just need to show that $H_2(z_1; u)$ is constant, since the others are equivalent. For $i \ge 3$, integrating by parts first with respect to z_i we obtain

$$\int_{\mathbb{R}^{n-1}} u_{z_i z_i} \cdot u_{z_2} dz' = -\int_{\mathbb{R}^{n-1}} u_{z_i} \cdot u_{z_i z_2} dz' = 0.$$

Hence

$$
\begin{aligned}
\frac{d}{dz_1} H_2(z_1; u) &= \int_{\mathbb{R}^{n-1}} (u_{z_1 z_1} \cdot u_{z_1} + u_{z_1} \cdot u_{z_2 z_1}) dz' \\
&= \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} ((F'(u) - u_{z_2 z_2}) \cdot u_{z_2} - u_{z_1} \cdot u_{z_1 z_2}) dz_2 dz'' \\
&- \int_{\mathbb{R}^{n-1}} \sum_{i=3}^{n} u_{z_i z_i} \cdot u_{z_2} dz' = 0.
\end{aligned}
$$

Therefore $H_2(z_1; u)$ is constant in z_1. □

For a positive solution u of (1) which uniformly decays in z', it can be shown that u should decay exponentially in z'. The following lemma, which gives a quantitative decay rate in z', can be easily proven by using the maximum principle and the standard elliptic estimates.

Lemma 7 (Lemma 2.1 of [1])**.** *There exists $\epsilon_0 = \epsilon(n, F) > 0$ such that for all integers $k \in \mathbb{N}$ and for all multi-indices $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in \mathbb{N}^{n-1}$, with $k+|\alpha| \leq 3$, we have*

$$|\partial_{z_1}^k \partial_{z'}^\alpha u(z_1, z')| \leq C(k, \alpha, n, F, |u|_\infty) e^{-\epsilon_0 |z'|}, \quad \textit{for all } (z_1, z') \in \mathbb{R}^n.$$

Next we define for $i = 2, ..., n$,

$$E_i(u, z_1) = \int_{\mathbb{R}^{n-1}} z_i\Big(\frac{1}{2}\big(|\nabla_{z'} u|^2 - |u_{z_1}|^2\big) + F\big(u(z)\big)\Big) dz'. \tag{10}$$

Using integration by parts and the behavior of u as $|z'| \to \infty$, we can get

$$\frac{dE_i}{dz_1} = -\int_{\mathbb{R}^{n-1}} u_{z_1} \cdot u_{z_i} dz'. \tag{11}$$

For any solution u of (1), we choose any sequence $z_1^i \in \mathbb{R}$ and let $u_i(z) := u(z_1 - z_1^i, z')$. Then by the Arzelà-Ascoli theorem together with Lemma 7, there is a subsequence of u_i (still denoted by u_i) such that $v(z) = \lim_{i\to\infty} u_i(z)$ is still a solution of (1).

Then by the dominated convergence theorem and the exponential decay of u and its derivatives in the z'-direction, we can show that

$$H(v) = H(u) \quad \text{and} \quad H_i(v) = H_i(u). \tag{12}$$

We can show Lemmas 8 and 9 below.

Lemma 8. *$H_i(u) = 0$ for all $i = 2, ..., n$.*

For any λ, let $u_\lambda(z) = u(z_1, 2\lambda - z_2, z'')$ and $\Sigma_\lambda = \{z \in \mathbb{R}^n | z_2 < \lambda\}$.

Lemma 9. *Let u be a solution of* (1) *in Theorem* 3 *and u_λ be defined as above with $\lambda \geq \lambda_0$. Then $u - u_\lambda$ cannot achieve its negative infimum in Σ_λ with $z_2 \leq -C(\lambda_0)$ or $|z''| \geq C(\lambda_0)$ for some constant $C(\lambda_0)$ large enough.*

Proof. We first show that $u - u_\lambda$ cannot have an interior negative minimum at $z_0 = (z_1, z_2, z'')$ with $z_2 \leq -C(\lambda_0)$ or $|z''| \geq C(\lambda_0)$. Since u and u_λ are solutions of (1), we get

$$\Delta(u - u_\lambda) = F'(u) - F'(u_\lambda) = \frac{F'(u) - F'(u_\lambda)}{u - u_\lambda}(u - u_\lambda) \tag{13}$$

in Σ_λ.

If $u - u_\lambda$ achieves an interior negative minimum at z_0 with $z_2 \leq -C(\lambda_0)$ or $|z''| \geq C(\lambda_0)$, then we choose $C(\lambda_0)$ so large that

$$0 \leq u(z), u_\lambda(z) \leq \epsilon \quad \text{for all } z \in \Sigma_\lambda \text{ with } z_2 \leq -C(\lambda_0) \text{ or } |z''| \geq C(\lambda_0),$$

where $\epsilon > 0$ is small and satisfies $F''(s) > 0$ for all $0 \leq s \leq \epsilon$. This can be done as $u(z)$ has exponential decay in z' independent of z_1. Then the maximum principle is violated at z_0.

Next, suppose there is a sequence $z^i = (z_1^i, z_2^i, (z^i)'')$ with $z_2^i \leq -C(\lambda_0)$ or $|(z^i)''| \geq C(\lambda_0)$, such that $\liminf_{\Sigma_\lambda \cap \{z_2 \leq C(\lambda_0)\} \cap \{|z''| \geq C(\lambda_0)\}} (u - u_\lambda) = \lim_{i\to\infty} (u - u_\lambda)(z^i) < 0$. Let $v_i(z) = u(z - (z_1^i, 0, 0))$. Since $u(z), u_\lambda(z) \to 0$ as $|z'| \to \infty$, taking a subsequence if necessary, we can show that v_i converges to a positive solution v of (1), $z_2^i \to z_2$ and $(z^i)'' \to z''$ for some z_2, z''. Then $v - v_\lambda$ gets an interior negative minimum at $(0, z_2, z'')$. This leads to a contradiction, and the lemma is proved. $\square$

Lemma 10. *The moving plane procedure can start, i.e.,*

$$u - u_\lambda \geq 0, \quad \textit{in } \Sigma_\lambda$$

when λ is large enough.

Proof. Let $C(\lambda_0)$ be as in Lemma 9 (we can just take $\lambda_0 = 0$). Letting $\lambda > C(\lambda_0)$, from Lemma 9, it follows that

$$u - u_\lambda \geq 0, \quad z \in \Sigma_\lambda : |z_2| \geq C(\lambda_0) \text{ or } |z''| \geq C(\lambda_0).$$

Let $\mathcal{O} = \{z \in \Sigma_\lambda : |z_2|, |z''| \leq C(\lambda)\}$. Then $u(z) \geq \epsilon$ for some $\epsilon > 0$ and all $z \in \mathcal{O}$. Then, by taking λ large enough, it is easy to see that

$$u - u_\lambda \geq 0, \quad z \in \mathcal{O}.$$

Combining the above together, we have $u - u_\lambda \geq 0$ in Σ_λ when λ is large enough. □

Define

$$\bar\lambda = \inf\{\lambda_0 | u - u_\lambda \geq 0, \forall z \in \Sigma_\lambda, \forall \lambda > \lambda_0\}.$$

Proof of Lemma 8. We will prove that $H_2(u) = 0$, since the other identities follow similarly. By the maximum principle, we get either $u \equiv u_{\bar\lambda}$ or $u > u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$. In the first case, $H_2(u) = 0$ follows easily by symmetry of u in the z_2 direction. Let us consider the case $u > u_{\bar\lambda}$. By the definition of $\bar\lambda$, we can choose a sequence $\lambda_i < \bar\lambda$ with $\lim_{i\to\infty} \lambda_i = \bar\lambda$, and $z_i = (z_1^i, z_2^i, (z^i)'') \in \Sigma_{\lambda_i}$, such that

$$u(z^i) < u_{\lambda_i}(z^i).$$

By Lemma 9, we can assume that z_2^i and $(z^i)''$ are uniformly bounded. Without loss of generality, we can assume $z_2^i \to z_2$ and $(z^i)'' \to z''$ as $i \to \infty$. If z_1^i has a bounded subsequence, we may assume $z_1^i \to z_1$. Then we get

$$u(z_1, z_2, z'') \leq u_{\bar\lambda}(z_1, z_2, z''),$$

which contradicts either Hopf's Lemma in the case $z_2 = \bar\lambda$ or the fact that $u > u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$ when $z_2 \neq \bar\lambda$. Hence z_1^i cannot have a bounded subsequence. We may assume that $z_1^i \to \infty$. Define

$$u_i(z) = u(z - (z_1^i, 0, 0)).$$

Then a subsequence of u_i will convergence to a positive solution of (1) with

$$v(z) \geq v_{\bar\lambda}(z), \quad \forall z \in \Sigma_{\bar\lambda} \text{ and } v(0, z_2, z'') \leq v_{\bar\lambda}(0, z_2, z'').$$

By the maximum principle or Hopf's Lemma, we can get $v(z) = v_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$. Hence we conclude that $H_2(v) = 0$ and $H_2(u) = 0$. □

Corollary 11. *$E_i(u) = E_i(v)$ is constant for $i = 2, ..., n$.*

Proof. The fact that $E_i(u)$ is a constant follows simply from Lemma 8, and $E_i(u) = E_i(v)$ can be derived from the dominated convergence theorem and the uniform exponential decay of u in z'. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. We will prove symmetry in the z_2 direction first. We start by decreasing λ from $+\infty$. From the proof of Lemma 8, we get the existence of $\bar\lambda$ such that either $u = u_{\bar\lambda}$ or $v = v_{\bar\lambda}$. We may assume that $\bar\lambda = 0$. It can be seen that $E_2(u) = 0$ by the fact that u or v is symmetric in the z_2-direction. If we start to increase λ from $-\infty$ we get similarly that either $u = u_{\underline\lambda}$ or there exists a positive solution $\bar{v}$ of (1) such that $\bar{v} = \bar{v}_{\underline{\lambda}}$ in $\Omega = \{z | z_2 > \underline{\lambda}\}$ for some $\underline{\lambda}$. In either case, we have (in the formula below, $\bar{v}$ can also be replaced by u)

$$\begin{aligned} E_2(u) &= E_2(\bar{v}) = \int_{\mathbb{R}^{n-1}} z_2\big(\tfrac{1}{2}\big(|\nabla_{z'}\bar{v}|^2 - |\bar{v}_{z_1}|^2\big) + F\big(\bar{v}(z)\big)\big)dz' \\ &= \int_{\mathbb{R}^{n-1}} \underline{\lambda}\big(\tfrac{1}{2}\big(|\nabla_{z'}\bar{v}|^2 - |\bar{v}_{z_1}|^2\big) + F\big(\bar{v}(z)\big)\big)dz' \\ &+ \int_{\mathbb{R}^{n-1}} (z_2 - \underline{\lambda})\big(\tfrac{1}{2}\big(|\nabla_{z'}\bar{v}|^2 - |\bar{v}_{z_1}|^2\big) + F\big(\bar{v}(z)\big)\big)dz' \\ &= \underline{\lambda} H(\bar{v}) = \underline{\lambda} H(u). \end{aligned} \tag{14}$$

In the above formula, we have used

$$\int_{\mathbb{R}^{n-1}} (z_2 - \underline{\lambda})\Big(\frac{1}{2}\big(|\nabla_{z'}\bar{v}|^2 - |\bar{v}_{z_1}|^2\big) + F\big(\bar{v}(z)\big)\Big)dz' = 0,$$

which follows from the fact that $\bar{v}$ is symmetric about the hyperplane $z_2 = \underline{\lambda}$.

Since $E_2(u) = 0$ and $H(u) \neq 0$, we get $\underline{\lambda} = 0$, which gives

$$u(z_1, z_2, z'') = u(z_1, -z_2, z'').$$

Using the moving plane method as above in any direction ν which is perpendicular to z_1, we can conclude that u is axially symmetric about a line parallel to the z_1-direction and passing through $z_0' \in \mathbb{R}^{n-1}$ in the z' hyperplane. The proof of Theorem 3 is complete. □

Given a solution u of (1) we can define a set of bounded positive functions from the solution u of equation (1) by

$$\begin{aligned} \delta_{\pm\infty} = \big\{v(z_1, z') \mid \quad &\text{there exist a sequence} \quad x_i \to \pm\infty, \\ &\text{such that } v(z_1, z') = \lim_{x_i \to \pm\infty} u(z_1 + x_i, z')\big\}. \end{aligned}$$

Remark 12. From the proof of Theorem 3, we know that the functions in $\delta_{+\infty}$ (or in $\delta_{-\infty}$) are axially symmetric about a line parallel to the z_1-axis. Moreover, when $H(u) \neq 0$, and this line is the same for both $\delta_{\pm\infty}$, it will be interesting to study the sets $\delta_{\pm\infty}$. For example, what is the relation of u with v in $\delta_{\pm\infty}$? More information seems to be needed to answer those questions. For example, in addition to the condition of Theorem 3, if we assume that $u(z_1, z')$ is monotone in z_1 for $z_1 > 0$, then $v(z') = \lim_{z_1 \to \infty} u(z_1, z')$ is a nonnegative solution of (1) in $\mathbb{R}^{n-1}$. Only two cases are possible. If $v(z') = 0$ (in this case, monotonicity is indeed not needed), we can show that the moving plane can start from the positive z_1 direction. If the moving plane stops somewhere, we can get $u(z_1, z') = u(|z|)$. Otherwise, if the moving plane never stops, due to the fact of constant energy $H(u)$, we can get $u \equiv 0$. If on the other hand $v(z')$ is the unique positive solution of (1) in $\mathbb{R}^{n-1}$, the argument on p. 965 of [2] shows that $u(z_1, z') = u(z')$ (note, in this case, monotonicity in z_1 is important). In [1], J. Busca and P. Felmer proved that when $\delta_{\pm\infty} = \{v(z'), \text{the unique positive radial decaying solution of (1) in } \mathbb{R}^{n-1}\}$, some closeness of v to u is needed to ensure that $u(z) = v(z')$.

Now we apply the moving plane method in the z_2-direction with z_1 fixed. Since the moving plane procedure stops at a finite number, we define the following two functions $\bar{\lambda}(z_1)$ and $\underline{\lambda}(z_1)$ as follows:

$$\bar{\lambda}(z_1) = \inf\{\lambda_0 | u(z_1, z') - u_\lambda(z_1, z') \geq 0, \forall z_2 \leq \lambda, \forall \lambda > \lambda_0\}$$

and

$$\underline{\lambda}(z_1) = \sup\{\lambda_0 | u(z_1, z') - u_\lambda(z_1, z') \geq 0, \forall z_2 \geq \lambda, \forall \lambda < \lambda_0\}.$$

If u is a nonzero positive solution of (1), it is easy to see that $\underline{\lambda}(z_1)$ and $\bar{\lambda}(z_1)$ are well defined bounded functions with $\underline{\lambda}(z_1) \leq \bar{\lambda}(z_1)$ for all z_1. We will show some properties for those two functions.

Lemma 13. *$\underline{\lambda}$ (or $\bar{\lambda}$ respectively) cannot have an interior local minimum (or interior local maximum respectively) if it is not a constant function.*

Proof. We just prove the lemma for $\underline{\lambda}$. Without loss of generality, we can assume that $\underline{\lambda}(0)$ is a local minimum, say in $-L \leq z_1 \leq L$ for some L positive and $\underline{\lambda}(0) < \min\{\underline{\lambda}(-L), \underline{\lambda}(L)\}$. This is true as long as $\underline{\lambda}$ is not a constant function. We start the moving plane procedure from the negative z_2-direction in D_L (see Remark 5 for the notation) and conclude that the moving plane will stop at $\underline{\lambda}(0)$; i.e, that for all $\lambda \leq \underline{\lambda}(0)$,

$$u \geq u_\lambda, \quad \forall z \in \{z_2 \geq \lambda\} \cap D_L.$$

By the maximum principle, we have either $u \equiv u_{\underline{\lambda}(0)}$ or $u > u_{\underline{\lambda}(0)}$ in $\{z_2 > \underline{\lambda}(0)\} \cap D_L$. In the first case, since u is nonzero, this contradicts the fact that $\underline{\lambda}(L) > \underline{\lambda}(0)$. In the second case, we can easily see that the moving plane can go a little further in D_L in the negative z_2-direction, which contradicts the definition of $\underline{\lambda}(0)$. This contradiction can be easily derived: if we have a decreasing sequence of $\lambda_i \to \underline{\lambda}(0)$ with $u(z_i) < u_{\lambda_i}(z_i)$ for some $z_i \in \{z_2 > \lambda_i\} \cap D_L$, then z_i cannot be on the boundary of D_L by the proof of Lemma 9. Hence we know that $|z_i'|$ is uniformly bounded. Taking a subsequence, we can get a contradiction with Hopf's Lemma or the maximum principle. □

Now by Lemma 13, we can define $\bar{\lambda} = \lim_{z_1 \to \infty} \bar{\lambda}(z_1)$. Similarly we can define $\underline{\lambda} = \lim_{z_1 \to \infty} \underline{\lambda}(z_1)$. Letting $e = (1, ..., 0) \in \mathbb{S}^{n-1}$, we have, for any $\epsilon > 0$ small, and z_1 large enough (it may depend on ϵ),

$$u(z_1, z') \geq u(z_1, (2\bar{\lambda} + 2\epsilon)e - z'), \quad \text{for} \quad z' \cdot e \leq \bar{\lambda} + \epsilon. \tag{15}$$

Repeating the moving plane procedure for any $e \in \mathbb{S}^{n-1}$, we define a function k on $\mathbb{S}^{n-1}$ by

$$k(e) = \bar{\lambda}. \tag{16}$$

It is easy to see that the above function k is well defined for any $e \in \mathbb{S}^{n-1}$ with the inequality (15) being satisfied when z_1 is large enough.

Now consider any rotation σ of $\mathbb{R}^n$ which keeps the positive z_1-direction: we have

$$u(z_1, z') \geq u(z_1, (2k(\sigma e) + 2\epsilon)\sigma e - z'), \quad \text{for} \quad z' \cdot \sigma e \leq k(\sigma e) + \epsilon \tag{17}$$

whenever z_1 is large enough (it may depend on ϵ).

Lemma 14. *It follows that*

$$\lim_{z_1 \to \infty} \bar{\lambda}(z_1) = \lim_{z_1 \to \infty} \underline{\lambda}(z_1); \lim_{z_1 \to -\infty} \bar{\lambda}(z_1) = \lim_{z_1 \to -\infty} \underline{\lambda}(z_1).$$

Proof. Take any rotation σ which keeps the positive z_1-direction and is close to the identity map. If we denote

$$\Omega_e = \{z' \in \mathbb{R}^{n-1} | z' \cdot e \leq k(e)\},$$

then we have $\Omega_e \cap \Omega_{\sigma e} \neq \emptyset$. Now take a sequence of $z_1^i \to \infty$, such that $v(z_1, z') = \lim_{z_1^i \to \infty} u(z_1 + z_1^i, z')$ satisfies (by the proof of Lemma 8)

$$v(z_1, z') = v(z_1, 2k(e)e - z'), \quad \text{for} \quad z' \in \Omega_e.$$

By equation (17), we have

$$v(z_1, z') \geq v(z_1, (2k(\sigma e) + \epsilon)\sigma e - z'), \quad \text{for} \quad z' \in \Omega_{\sigma e}.$$

Since ϵ can be taken to be arbitrary, letting $\epsilon \to 0$, we get

$$v(z_1, z') \geq v(z_1, 2k(\sigma e)\sigma e - z'), \quad \text{for} \quad z' \in \Omega_{\sigma e}.$$

Now since $\Omega_e \cap \Omega_{\sigma e} \neq \emptyset$, by the maximum principle, we get

$$v(z_1, z') = v(z_1, 2k(\sigma e)\sigma e - z'), \quad \text{for} \quad z' \in \Omega_{\sigma e}. \tag{18}$$

Since such kinds of rotations σe cover all of $\mathbb{S}^{n-1}$, we conclude that for any $e \in \mathbb{S}^{n-1}$, equation (18) is satisfied. Then it follows that $\lim_{z_1 \to \infty} \bar{\lambda}(z_1) = \lim_{z_1 \to \infty} \underline{\lambda}(z_1)$.

Similarly, we can show that $\lim_{z_1 \to -\infty} \bar{\lambda}(z_1) = \lim_{z_1 \to -\infty} \underline{\lambda}(z_1)$. The lemma is proven. □

Remark 15. From Lemma 14, the limiting function $v(z_1, z')$ is axially symmetric about some line parallel to the z_1-axis.

Proof of Theorem 4. By Lemmas 14 and 13, $\bar{\lambda}(z_1)$ and $\underline{\lambda}(z_1)$ are monotone functions of z_1. Since u is even in z_1, this results in that $\bar{\lambda}(z_1)$, $\underline{\lambda}(z_1)$ are constants. We can use Lemma 14 again to show that $\bar{\lambda}(z_1) = \underline{\lambda}(z_1) = \textit{constant}$. This proves the theorem. □

Due to (3) (see also [5] and [9]), we have

$$\int_{D_L} \left(\frac{1}{2}\big(|\nabla_{z'} u|^2 - |u_{z_1}|^2\big) + F\big(u(z)\big) \right) dz = 2L \cdot H(u). \tag{19}$$

On the other hand, for $i = 2, ..., n$, the Pohazaev identity applied to D_L leads to

$$\int_{D_L} \left(\frac{1}{2}\big(\sum_{j \neq i} u_{z_j}^2 - u_{z_i}^2\big) + F\big(u(z)\big) \right) dz = -\int_{\mathbb{R}^{n-1}} u_{z_1} u_{z_i} z_i dz' \Big|_{-L}^{L}. \tag{20}$$

Here we have used the fact that $u(z)$ goes to zero exponentially in $|z'|$ as z' tends to infinity.

Summing (20) for $i = 2, ..., n$, we get

$$\begin{aligned} &\textstyle\int_{D_L} \big(\frac{1}{2}\big((n-3)|\nabla_{z'} u|^2 + (n-1)u_{z_1}^2\big) + (n-1)F\big(u(z)\big)\big)\, dz \\ &\qquad = -\textstyle\int_{\mathbb{R}^{n-1}} u_{z_1}(\nabla_{z'} u \cdot z') dz' \Big|_{-L}^{L}. \end{aligned} \tag{21}$$

Furthermore, multiplying (1) by u and integrating on D_L yields

$$\int_{D_L} \big(|\nabla_{z'} u|^2 + |u_{z_1}|^2 + uF'\big(u(z)\big)\big)\, dz = \int_{\mathbb{R}^{n-1}} u_{z_1} u dz' \Big|_{-L}^{L}. \tag{22}$$

Due to Lemma 7, the right hand sides of (20), (21) and (22) are uniformly bounded independently of L. Combining (21) and (22), we obtain

$$\begin{aligned} \textstyle\int_{D_L} |\nabla_{z'} u|^2 &= \tfrac{n-1}{2} \textstyle\int_{D_L} \big[2F\big(u(z)\big) - uF'\big(u(z)\big)\big] dz \\ &\quad + \big(\textstyle\int_{\mathbb{R}^{n-1}} \tfrac{n-1}{2} u_{z_1} u + u_{z_1}(\nabla_{z'} u \cdot z') dz'\big) \Big|_{-L}^{L} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \int_{D_L}|u_{z_1}|^2 &= \int_{D_L}\big[\tfrac{n-3}{2}uF'\big(u(z)\big)-(n-1)F\big(u(z)\big)\big]dz \\ &\quad -\big(\int_{\mathbb{R}^{n-1}}\tfrac{n-3}{2}u_{z_1}u+u_{z_1}(\nabla_{z'}u\cdot z')dz'\big)\big|_{-L}^{L}. \end{aligned} \tag{24}$$

Using (19), we conclude

Theorem 16. *The Hamiltonian constant for the solution u in Theorem 3 satisfies*

$$\begin{aligned} H(u) =& \frac{1}{2L}\int_{D_L}\big[nF\big(u(z)\big)-\frac{n-2}{2}uF'\big(u(z)\big)\big]dz \\ &+\frac{1}{2L}\left(\int_{\mathbb{R}^{n-1}}\frac{n-2}{2}u_{z_1}u+u_{z_1}(\nabla_{z'}u\cdot z')dz'\right)\Big|_{-L}^{L} \\ =&\frac{1}{2L}\int_{D_L}\big[\frac{1}{n-1}|\nabla_{z'}u|^2-|u_{z_1}|^2\big]dz \\ &-\frac{1}{2L(n-1)}\left(\int_{\mathbb{R}^{n-1}}u_{z_1}(\nabla_{z'}u\cdot z')dz'\right)\Big|_{-L}^{L} \\ =&\frac{1}{2L}\int_{D_L}[u_{z_i}^2-u_{z_1}^2]dz-\frac{1}{2L}\left(\int_{\mathbb{R}^{n-1}}z_iu_{z_1}u_{z_i}dz'\right)\Big|_{-L}^{L} \end{aligned} \tag{25}$$

for any $L>0$. Moreover, $H(u)\geq 0$ if $nF\big(u(z)\big)-\frac{n-2}{2}uF'\big(u(z)\big)>0, \quad \forall u>0$.

In particular, if $F(u)=\frac{1}{2}u^2-\frac{1}{p+1}u^{p+1}$ with $\frac{n+2}{n-2}\leq p<\frac{n+1}{n-3}$ when $n>3$, and $p\geq 5$ when $n=3$, we have

$$H(u)>0. \tag{26}$$

Remark 17. The second statement of the theorem follows directly from the first statement by letting L go to infinity. Similar results for Dancer's solution were first obtained in [6].

Remark 18. If $F(u)=\frac{1}{2}u^2-\frac{1}{p+1}u^{p+1}$ with $\frac{n+2}{n-2}\leq p<\frac{n+1}{n-3}$, $n>3$ or $p\geq 5$ when $n=3$, then $H(u)=0$ implies $||u||_{L^2(\mathbb{R}^n)}=0$ and hence $u\equiv 0$.

Remark 19. For the positive solution obtained in [9], it is known that $H(L):=H(u_L)>0$ and $H(L)$ is monotonically decreasing when L is sufficiently large (see Lemma 7.1 in [9]). It is natural to ask whether $H(L)$ is always positive. The positivity of $H(L)$ is an important feature which may be interpreted as concentration energy in comparison to the interface energy in phase separation (see [5]). We will be addressing this more later.

Remark 20. When $F(u)=\frac{1}{2}u^2-\frac{1}{p+1}u^{p+1}$ with $p\geq\frac{n+1}{n-3}$, there is no positive solution to equation (1) which decays in the z'-direction. Indeed, we can see that the trivial solution $u\equiv 0$ is the only bounded solution (not necessarily positive) of

$$\Delta u-u+|u|^{p-1}u=0, \quad p\geq\frac{n+1}{n-3}, \quad x\in\mathbb{R}^n$$

which decays to zero in the z'-direction. To see this, we use (24) and $p\geq\frac{n+1}{n-3}$ to get

$$\int_{D_L}|u_{z_1}|^2dz=\int_{D_L}\left(-u^2-(\frac{n-3}{2}-\frac{n-1}{p+1})|u|^{p+1}\right)dz+O(1),$$

where $O(1)$ is uniformly bounded independently of L. Letting $L \to \infty$, we get $u \in W^{1,2}(\mathbb{R}^n)$. The above shows that $u(z_1, z') \to 0$ as $z_1 \to \pm\infty$ uniformly in z'. Going back to (24) and letting L tend to infinity, we get

$$\int_{\mathbb{R}^n} \left(|u_{z_1}|^2 + u^2 + (\frac{n-3}{2} - \frac{n-1}{p+1})u^{p+1} \right) dz = 0.$$

Hence $u \equiv 0$.

Acknowledgements

This research is partially supported by U.S. National Science Foundation grant DMS-0500871. The first author would like to thank Professor Patricio Felmer for bringing to his attention a very stimulating article [1].

References

[1] J. Busca and P. Felmer, Qualitative properties of some bounded positive solutions to scalar field equations. *Calc. Var. Partial Differential Equations* **13** (2001) pp. 191-211. MR1861097 (2002g:35063)

[2] E. N. Dancer, On the uniqueness of the positive solution of a singularly perturbed problem, *Rocky Mountain Journal of Mathematics* **25** (1995) pp. 957-975. MR1357103 (96j:35021)

[3] E. N. Dancer, New solutions of equations on $\mathbb{R}^n$, *Annali Della Scuola Normale Superiore di Pisa* **30** (2001) pp. 535-563. MR1896077 (2003g:35057)

[4] M. Del Pino, M. Kowalzyck, F. Pacard and J. Wei, The Toda system and multiple-end solutions of autonomous planar elliptic problems, preprint, 2007.

[5] Changfeng Gui, Hamiltonian identities for elliptic partial differential equations, *Journal of Functional Analysis* **254** (2008), no. 4, pp. 904–933. MR2381198 (2009b:35112)

[6] Changfeng Gui, Hamiltonian constants for several new entire solutions, *Front. Math. China* **3** (2008) pp. 195-204. MR2395216 (2009e:35075)

[7] C. Gui, Symmetry of traveling wave solutions to the Allen-Cahn equation in $\mathbb{R}^2$, preprint.

[8] C. Gui, Symmetry of certain saddle solutions to the Allen-Cahn equation, preprint.

[9] Andrea Malchiodi, Some new entire solutions of semilinear elliptic equations on $\mathbb{R}^n$, preprint, 2007.

Department of Mathematics, U-9, University of Connecticut, Storrs, Connecticut 06269

E-mail address: gui@math.uconn.edu

Sector of Mathematical Analysis, SISSA, Via Beirut 2-4, 34014 Trieste, Italy

E-mail address: malchiod@sissa.it

Department of Mathematics, U-9, University of Connecticut, Storrs, Connecticut 06269

E-mail address: haoyuan@math.uconn.edu