

LINEARIZATION AND LOCAL STABILITY OF RANDOM DYNAMICAL SYSTEMS

IGOR V. EVSTIGNEEV, SERGEY A. PIROGOV, AND KLAUS R. SCHENK-HOPPÉ

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ABSTRACT. The paper examines questions of local asymptotic stability of random dynamical systems. Results concerning stochastic dynamics in general metric spaces, as well as in Banach spaces, are obtained. The results pertaining to Banach spaces are based on the linearization of the systems under study. The general theory is motivated (and illustrated in this paper) by applications in mathematical finance.

1. STOCHASTIC DYNAMICS AND EQUILIBRIUM

Let (Ω, \mathcal{F}, P) be a probability space and $T : \Omega \rightarrow \Omega$ its endomorphism, i.e., an \mathcal{F} -measurable map preserving the probability P :

$$P(T^{-1}(\Gamma)) = P(\Gamma), \Gamma \in \mathcal{F}.$$

Let (X, \mathcal{X}) be a measurable space and $f(x, \omega)$ a jointly measurable mapping of $X \times \Omega$ into X . These data define a discrete-time (autonomous) random dynamical system with the state space X , the law of motion $f(x, \omega)$ and the time shift T . For each $t = 1, 2, \dots$, we put

$$(1.1) \quad f_t(x, \omega) := f(x, T^{t-1}\omega).$$

A sequence $x_0(\omega), x_1(\omega), \dots$ of random elements in X is called a *path* of the random dynamical system if

$$(1.2) \quad x_t(\omega) = f_t(x_{t-1}(\omega), \omega), \quad t = 1, 2, \dots$$

almost surely (a.s.). A random element $\bar{x}(\omega) \in X$ satisfying

$$(1.3) \quad \bar{x}(T\omega) = f(\bar{x}(\omega), \omega) \quad (\text{a.s.})$$

is termed a *stochastic equilibrium* (or a *random fixed point*, or a *stationary point*) of the system. The sequence $\bar{x}_t(\omega) := \bar{x}(T^t\omega)$, which satisfies $\bar{x}_t(\omega) = f_t(\bar{x}_{t-1}(\omega), \omega)$ (a.s.) by virtue of (1.3), represents the *equilibrium path* of the system generated by the random fixed point $\bar{x}(\omega)$. For each $a \in X$, we denote by $x_0^a(\omega), x_1^a(\omega), \dots$

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the path with the initial state a , i.e., the sequence of random elements generated recursively (for each ω and all $t \geq 1$) by equations (1.2) with $x_0(\omega) = a$.

2. BASIC ASSUMPTIONS

Let $\bar{x}(\omega)$ be a stochastic equilibrium and let $X(\omega) \subseteq X$ be a random set¹ such that $\bar{x}(\omega) \in X(\omega)$ and $f(x, \omega) \in X(T\omega)$ for each $x \in X(\omega)$ and $\omega \in \Omega$. Let ρ be a metric on X . For any number $\delta > 0$, define

$$\bar{X}(\omega, \delta) := \{x \in X(\omega) : \rho(x, \bar{x}(\omega)) \leq \delta\}.$$

We introduce two hypotheses, **(A)** and **(B)**, that will be needed for the statement of the main results.

(A) X is a complete separable metric space with the metric $\rho(x, y)$ and the Borel σ -algebra \mathcal{X} . There exist random variables $L(\omega) > 0$ and $\delta(\omega) > 0$ such that

$$(2.1) \quad E|\ln L| < +\infty, \quad E|\ln \delta| < +\infty,$$

$$(2.2) \quad E \ln L < 0,$$

and

$$(2.3) \quad \rho(f(x, \omega), f(\bar{x}(\omega), \omega)) \leq L(\omega)\rho(x, \bar{x}(\omega))$$

for all $x \in \bar{X}(\omega, \delta(\omega))$. Here, “ E ” stands for the expectation with respect to the probability measure P .

It is assumed in **(A)** that the mapping $f(\cdot, \omega)$ is locally Lipschitzian at the equilibrium point $\bar{x}(\omega)$ (in a $\delta(\omega)$ -neighborhood of $\bar{x}(\omega)$) with the random Lipschitz constant $L(\omega)$. Conditions (2.1) guarantee that the random Lipschitz constant is not “too large” and the random $\delta(\omega)$ -neighborhood is not “too small”. (We could assume that $E \ln^+ L < +\infty$ and $E \ln^- \delta > -\infty$, but this would not lead to a gain in generality.) According to (2.2), $L(\omega)$ has negative expected logarithm, so that the mapping $f(\cdot, \omega)$ is *locally contracting on average*.

The next hypothesis pertains to the case of a linear space X . It provides conditions guaranteeing the validity of **(A)** formulated in terms of the linearization of the mapping $f(\cdot, \omega)$ at the equilibrium point $\bar{x}(\omega)$.

(B) X is a separable Banach space with the norm $\|\cdot\|$, the metric $\rho(x, y) := \|x - y\|$ and the Borel σ -algebra \mathcal{X} .

(B1) There exist random variables $L(\omega) > 0$ and $\delta(\omega) > 0$ for which conditions (2.1) hold and the mapping $f(x, \omega)$ satisfies (2.3) for all $x \in \bar{X}(\omega, \delta(\omega))$.

(B2) For each ω , there exists the derivative $f'(\bar{x}(\omega), \omega)$ of the mapping $f(\cdot, \omega)$ at the point $\bar{x}(\omega)$, and

$$E \ln \|f'(\bar{x}(\omega), \omega)\| < 0.$$

The derivative is understood in the strong (Fréchet) sense, as a continuous linear operator $F(\omega) := f'(\bar{x}(\omega), \omega)$ such that

$$(2.4) \quad f(\bar{x}(\omega) + h, \omega) = f(\bar{x}(\omega), \omega) + F(\omega)h + g(h, \omega)\|h\|, \quad h \in X,$$

where $g(h, \omega) \rightarrow 0$ as $\|h\| \rightarrow 0$. The norm $\|F\|$ of the operator F is defined as $\sup\{\|Fh\|/\|h\| : 0 \neq h \in X\}$. Note that hypotheses **(A)** and **(B1)**, in contrast with **(B2)**, impose assumptions on the behavior of the mapping $f(x, \omega)$ only in the intersection of a neighborhood of the point $\bar{x}(\omega)$ with the set $X(\omega)$ (which might be significantly smaller than the whole neighborhood).

¹We say that $X(\omega)$ is a random set if the graph $\{(u, \omega) \in X \times \Omega : u \in X(\omega)\}$ of the multivalued mapping $\omega \mapsto X(\omega)$ belongs to the σ -algebra $\mathcal{X} \otimes \mathcal{F}$.

Throughout the paper we will assume that the endomorphism T is ergodic: all invariant sets have either zero or full measure.

3. CENTRAL RESULT

Theorem 1. *Let hypothesis (A) or hypothesis (B) hold. Then there exist a random variable $\gamma(\omega) > 0$ and a constant $c < 0$ such that with probability one,*

$$(3.1) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq c$$

for all $a \in \bar{X}(\omega, \gamma(\omega))$.

The theorem shows that for all ω in a set Ω_1 of measure 1, the distance between the path $x_t^a(\omega)$ with the initial state a and the equilibrium path $\bar{x}_t(\omega)$ tends to zero at an exponential rate for all a in a sufficiently small neighborhood $\bar{X}(\omega, \gamma(\omega))$ of $\bar{x}(\omega)$. This convergence and its rate are uniform with respect to $\omega \in \Omega_1$ and $a \in \bar{X}(\omega, \gamma(\omega))$.

The results of this paper are motivated by applications in evolutionary finance [12], a new and rapidly developing area in financial mathematics. The general mathematical framework for models in this area is the ergodic theory of random dynamical systems. For the analysis of evolutionary models of asset markets, one often needs to examine local stability of stochastic equilibria of some dynamical systems. Surprisingly, the literature does not contain results which would be quite suitable for applications in this field. The main tools offered for local analysis in the context of stochastic dynamics (see the classical monograph by L. Arnold [1]) are stochastic analogues of the Hartman-Grobman theorem [17] – [19] developed in [38] and [5] – [7], and closely related results on stable manifold theorems [4, 27, 29, 30, 34, 35]. For the most part, these are delicate results, the use of which requires the verification of complex conditions. They are much stronger than what is needed for our purposes and require much stronger assumptions. The need for suitable tools motivated us to develop the techniques described above. To use Theorem 1 under hypothesis (B) one has essentially to estimate only the expectation of one random variable: the logarithm of the norm of the derivative $f'(x, \omega)$ at the random fixed point $x = \bar{x}(\omega)$ (the local Lipschitz property (B1) usually causes no problems). The theorem represents a direct stochastic analogue of well-known deterministic results on local asymptotic stability. Its statement and proof are based on elementary notions and techniques.

By and large, stochastic equilibria, or random fixed points, exist under sufficiently strong assumptions.² In many cases such assumptions guarantee not only existence, but also global asymptotic stability established simultaneously as a consequence of a stochastic contraction principle [13, 36]. Global asymptotic stability (convergence to a limit from any initial state) implies that the limit forms an equilibrium. Of course in such cases, the local analysis is superfluous. This is the situation, for example, with stochastic equilibrium problems related to random Perron-Frobenius theorems [3, 10, 14, 22, 24]. Results in that field have been primarily inspired by the applications in the theory of Gibbs measures [33, 37], large deviations [23], models in mathematical biology [9], and others (see also [20, 21, 25]). In many models coming from economics and finance, the situation is different. The questions of existence are often separated from the questions of stability, and existence

²Counterexamples in [14, 31] substantiate this assertion.

proofs based on the above methods require too restrictive, unjustifiable assumptions. Typically, either the system has an obvious, even deterministic, fixed point, or the existence is proved by methods quite distinct from the above-mentioned arguments based on global stability and limiting procedures (see, e.g., studies on von Neumann-Gale dynamics [2, 15]). In the example we consider at the end of the paper, we deal with the former case: the system is random but the fixed point is deterministic.

The remainder of the paper is organized as follows. In section 4 we prove Theorem 1. At the end of the section, we provide a condition sufficient for the validity of hypothesis **(B1)**. Section 5 gives an extension of Theorem 1 applicable both to continuous and discrete time settings. Sections 6 and 7 analyze an example related to mathematical finance.

4. PROOF OF THEOREM 1

1st step. Assume that hypothesis **(A)** holds. Replacing Ω by its subset of full measure, we can (in view of (1.3)) assume without loss of generality that the equations $\bar{x}_{t+1}(\omega) = f_t(\bar{x}_t(\omega), \omega)$ hold for all t and all ω , rather than almost surely. Define $X_t(\omega) := X(T^t\omega)$, $\delta_t(\omega) := \delta(T^t\omega)$, $L_t(\omega) := L(T^{t-1}\omega)$ and

$$\bar{X}_t(\omega) := \{x \in X_t(\omega) : \rho(x, \bar{x}_t(\omega)) \leq \delta_t(\omega)\}.$$

It follows from (2.3) that

$$\sup_{x \in \bar{X}_{t-1}(\omega)} \rho(f_t(x, \omega), \bar{x}_t(\omega)) \leq L_t(\omega) \rho(x, \bar{x}_{t-1}(\omega)).$$

This implies the validity of the following assertion:

(*) If $x_{t-1}^a(\omega) \in \bar{X}_{t-1}(\omega)$, then

$$\rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq L_t(\omega) \rho(x_{t-1}^a(\omega), \bar{x}_{t-1}(\omega)).$$

We claim that there exists a random variable $\gamma(\omega) > 0$ for which the inequalities

$$(4.1) \quad \gamma(\omega) \leq \delta_0(\omega),$$

$$(4.2) \quad L_t(\omega) \dots L_1(\omega) \gamma(\omega) \leq \delta_t(\omega), \quad t = 1, 2, \dots$$

hold with probability one. Indeed, these inequalities are satisfied if and only if

$$(4.3) \quad \sigma(\omega) := \sup_{t=0,1,\dots} \frac{L_t(\omega) \dots L_0(\omega)}{\delta_t(\omega)} \leq \frac{1}{\gamma(\omega)},$$

where $L_0(\omega) := 1$. It is sufficient to show that $P\{\sigma < \infty\} = 1$. Then relation (4.3), and hence inequalities (4.1) and (4.2), hold a.s. for the random variable γ defined as $\gamma(\omega) = 1/\sigma(\omega)$ if $\sigma(\omega) < \infty$ and $\gamma(\omega) = 1$ otherwise. To prove that $\sigma < \infty$ a.s., we use the ergodic theorem and obtain that

$$\frac{1}{t} \ln \frac{L_t \dots L_0}{\delta_t} = \frac{1}{t} \sum_{i=0}^t \ln L_i - \frac{1}{t} \ln \delta_t \rightarrow E \ln L < 0 \text{ (a.s.)},$$

since $E|\ln L| < \infty$ and $E|\ln \delta| < \infty$. Consequently, $L_t \dots L_0 \delta_t^{-1} \rightarrow 0$ a.s., which implies that $P\{\sigma < \infty\} = 1$.

Consider the random variable $\gamma(\omega) > 0$ constructed above and denote by Ω_1 the set of those ω for which inequalities (4.1) and (4.2) hold and, additionally,

$$(4.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=0}^t \ln L_i = E \ln L.$$

This relation holds a.s. by virtue of the ergodic theorem, and so $P(\Omega_1) = 1$. Take any $\omega \in \Omega_1$ and $a \in X(\omega)$ satisfying $\rho(a, \bar{x}(\omega)) \leq \gamma(\omega)$. Let us show by induction that for all $t \geq 0$,

$$(4.5) \quad x_t^a(\omega) \in \bar{X}_t(\omega),$$

$$(4.6) \quad \rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq L_t(\omega) \dots L_0(\omega) \gamma(\omega).$$

For $t = 0$, we have $a \in X_0(\omega) = X(\omega)$ by assumption and

$$(4.7) \quad \rho(a, \bar{x}(\omega)) \leq \gamma(\omega) \leq \delta_0(\omega)$$

by virtue of (4.1), so that $x_0^a(\omega) \in \bar{X}_0(\omega)$. Inequality (4.6) is true for $t = 0$ in view of (4.7) and because $\gamma(\omega) = L_0(\omega) \gamma(\omega)$. Suppose relations (4.5) and (4.6) are valid for some t . Then $x_{t+1}^a(\omega) \in X_{t+1}(\omega)$ because

$$\rho(x_{t+1}^a(\omega), \bar{x}_{t+1}(\omega)) \leq L_{t+1}(\omega) \rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq L_{t+1}(\omega) L_t(\omega) \dots L_0(\omega) \gamma(\omega),$$

where the first inequality follows from assertion (*) and (4.5), while the second is a consequence of (4.6). By using (4.2) (with $t + 1$ in place of t), we conclude that $\rho(x_{t+1}^a(\omega), \bar{x}_{t+1}(\omega)) \leq \delta_{t+1}(\omega)$, which proves that the analogues of relations (4.5) and (4.6) hold for $t + 1$. Inequality (4.6) combined with (4.4) implies (3.1) with $c := E \ln L$. \square

2nd step. To complete the proof of Theorem 1 we show that **(B)** implies **(A)**. In this connection, we make some comments regarding measurability. By applying (2.4) with $k^{-1}h$ in place of h , letting $k \rightarrow \infty$, and using the joint measurability of $f(x, \omega)$, we obtain that the mappings $F(\omega)h$ and $g(h, \omega)$, and hence the functions $\|F(\omega)h\|$ and $\|g(h, \omega)\|$, are jointly measurable in (h, ω) . The operator norm $F(\omega)$ depends measurably on ω because $\|F(\omega)\| := \sup_k \{\|F(\omega)h_k\|/\|h_k\|\}$, where $\{h_k\}$ is a countable dense subset in $X \setminus \{0\}$.

By using (2.4) and (2.3), we obtain that for each $x \in \bar{X}(\omega, \delta(\omega))$,

$$\begin{aligned} \|g(x - \bar{x}(\omega), \omega)\| \|x - \bar{x}(\omega)\| &\leq \|f(x, \omega) - f(\bar{x}(\omega), \omega)\| + \|F(\omega)(x - \bar{x}(\omega))\| \\ &\leq L(\omega) \|x - \bar{x}(\omega)\| + \|F(\omega)\| \|x - \bar{x}(\omega)\|, \end{aligned}$$

which implies

$$(4.8) \quad \|g(x - \bar{x}(\omega), \omega)\| \leq L(\omega) + \|F(\omega)\|.$$

Further, if $x \in \bar{X}(\omega, \delta(\omega))$, then

$$(4.9) \quad \|f(x, \omega) - f(\bar{x}(\omega), \omega)\| \leq [\|F(\omega)\| + \|g(x - \bar{x}(\omega), \omega)\|] \|x - \bar{x}(\omega)\|.$$

Define

$$(4.10) \quad g_k(\omega) := \sup\{\|g(x - \bar{x}(\omega), \omega)\| : x \in \bar{X}(\omega, k^{-1}\delta(\omega))\}.$$

The function $g_k(\omega)$ is measurable with respect to the completion \mathcal{F}^P of the σ -algebra \mathcal{F} by P -null sets because $\|g(x - \bar{x}(\omega), \omega)\|$ is jointly measurable in (x, ω) and $X(\omega)$ is a random set. This follows from the fact that the projection of a set in $\mathcal{X} \otimes \mathcal{F}$ on Ω is \mathcal{F}^P -measurable (see, e.g., [8], Theorem III.33).

Define $D_k(\omega) := \|F(\omega)\| + g_k(\omega)$. By virtue of (2.4), $g_k(\omega) \rightarrow 0$ for each ω . Furthermore, in view of (4.8) we have

$$\ln D_k(\omega) \leq \ln[2\|F(\omega)\| + L(\omega)] \leq \ln 4 + \max[\ln \|F(\omega)\|, \ln L(\omega)] =: \Xi(\omega),$$

where $E|\Xi(\omega)| < \infty$. By using Fatou's lemma, we get

$$\limsup_{k \rightarrow \infty} E \ln D_k(\omega) \leq E \lim_{k \rightarrow \infty} \ln D_k(\omega) = E \ln \|F(\omega)\| < 0.$$

Thus there exists k such that $E \ln D_k(\omega) < 0$ and (by virtue of (4.9) and (4.10))

$$\|f(x, \omega) - f(\bar{x}(\omega), \omega)\| \leq D_k(\omega)\|x - \bar{x}(\omega)\|$$

for each $x \in \bar{X}(\omega, \delta(\omega)/k)$. This completes the proof of Theorem 1. □

Remark 1. The following assumption is sufficient for condition **(B1)** to hold.

(B3) There exist random variables $L(\omega) > 0$ and $\delta(\omega) > 0$ satisfying (2.1) such that for all $x \in \bar{X}(\omega, \delta(\omega))$, the set $X(\omega)$ contains the segment $[\bar{x}(\omega), x]$ connecting $\bar{x}(\omega)$ and x , the mapping $f(x, \omega)$ is differentiable at the point x , and the norm of the derivative $\|f'(x, \omega)\|$ is bounded from above by $L(\omega)$.

To deduce **(B1)** from **(B3)** it suffices to observe that the inequality $\|f'(x, \omega)\| \leq L(\omega)$ implies (2.3) by virtue of the generalized mean value theorem:

$$\|f(x, \omega) - f(\bar{x}(\omega), \omega)\| \leq \|x - \bar{x}(\omega)\| \sup_{y \in [\bar{x}(\omega), x]} \|f'(y, \omega)\|$$

(which holds even if $f'(x, \omega)$ is the weak rather than the strong derivative); see [26], Section X.1.3.

5. EXTENSION TO DISCRETE AND CONTINUOUS TIME COCYCLES

The next result provides an extension of Theorem 1 to random dynamical systems defined in terms of cocycles in discrete and continuous time. Let \mathbb{T} be either the set of non-negative integers or the set of non-negative real numbers, and let $T^t, t \in \mathbb{T}$, be a semigroup of ergodic endomorphisms of the probability space (Ω, \mathcal{F}, P) . For each $t \in \mathbb{T}$, let $C_t(x, \omega)$ be a jointly measurable mapping of $X \times \Omega$ into X . Assume that the family of mappings $C_t(x, \omega), t \in \mathbb{T}$, forms a *cocycle*, i.e.,

$$C_{t+s}(\cdot, \omega) = C_s(\cdot, T^t\omega) \circ C_t(\cdot, \omega), \quad C_0(x, \omega) = x,$$

for all t, s, x and ω . The cocycle defines the law of motion in the system, whose *paths* are random functions $x_t(\omega), t \in \mathbb{T}$, such that with probability one, $x_t(\omega) = C_t(x_0(\omega), \omega)$ for all $t \in \mathbb{T}$. Random dynamical systems of this kind can be generated by stochastic or random differential equations in continuous time and by products of random mappings in discrete time (see [1]). We will assume that a random set $X(\omega)$ is given such that $C_t(x, \omega) \in X(T^t\omega)$ for all $x \in X(\omega)$.

Let $\bar{x}_t(\omega), t \in \mathbb{T}$, be an *equilibrium path*, i.e., a path satisfying a.s. $\bar{x}_t(\omega) = \bar{x}_0(T^t\omega)$ for all $t \in \mathbb{T}$. For each $a \in X$, define $x_t^a(\omega) := C_t(a, \omega)$ (the random path with the initial state a).

Theorem 2. *Let the following assumptions hold:*

(C1) *There exists $M \in \mathbb{T}$ such that the mapping $C_M(\cdot, \omega)$ satisfies condition **(A)** or **(B)** with $\bar{x}(\omega) = \bar{x}_0(\omega)$.*

(C2) *There are random variables $H(\omega) > 0, \kappa(\omega) > 0$ and a constant $b > 0$ such that $E|\ln H(\omega)| < +\infty, E|\ln \kappa(\omega)| < +\infty$ and with probability one,*

$$(5.1) \quad \rho(C_t(x, \omega), C_t(\bar{x}(\omega), \omega)) \leq H(\omega)\rho(x, \bar{x}(\omega))^b$$

for all $x \in \bar{X}(\omega, \kappa(\omega))$ and $t \in \mathbb{T}$ satisfying $0 \leq t \leq M$.

Then there exist a random variable $\gamma(\omega) > 0$ and a constant $d < 0$ such that almost surely

$$(5.2) \quad \limsup_{\mathbb{T} \ni t \rightarrow \infty} \frac{1}{t} \ln \rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq d$$

for all $a \in \bar{X}(\omega, \gamma(\omega))$.

Since $d < 0$, (5.2) implies that $\rho(x_t^a(\omega), \bar{x}_t(\omega)) \rightarrow 0$ a.s. at an exponential rate. This convergence, as well as its rate, are uniform with respect to ω in a set Ω_1 of full measure and all a in the neighborhood $\bar{X}(\omega, \gamma(\omega))$ of $\bar{x}(\omega) = \bar{x}_0(\omega)$. Property (5.1) represents a Hölder condition on the cocycle $C_t(x, \omega)$.

Proof of Theorem 2. Since **(B)** implies **(A)**, as we demonstrated in section 4, it is sufficient to prove the theorem under assumption **(A)**. Define $t(n) := nM$, $n = 0, 1, \dots$. By applying Theorem 1 to the mapping $f(x, \omega) := C_M(x, \omega)$, we obtain that there exist a random variable $\gamma(\omega) > 0$ and a constant $c < 0$ such that with probability one,

$$(5.3) \quad \limsup_{n \rightarrow \infty} \frac{1}{t(n)} \ln \rho(x_{t(n)}^a(\omega), \bar{x}_{t(n)}(\omega)) \leq c$$

for all $a \in \bar{X}(\omega, \gamma(\omega))$. From (5.3) we obtain that for each $\varepsilon \in (0, 1)$ with probability one,

$$(5.4) \quad \rho_n(\omega) := \rho(x_{t(n)}^a(\omega), \bar{x}_{t(n)}(\omega)) \leq \exp[t(n)c(1 - \varepsilon)]$$

for all sufficiently large n . Furthermore, almost surely

$$(5.5) \quad \kappa_n(\omega) := \kappa(T^{t(n)}\omega) \geq \exp[t(n)c(1 - \varepsilon)]$$

for all n large enough. Indeed, the last inequality holds if $Mc(1 - \varepsilon) \leq n^{-1} \ln \kappa_n(\omega)$, which is true for all n large enough because $\lim_{n \rightarrow \infty} n^{-1} \ln \kappa_n(\omega) \rightarrow 0$ a.s. (this follows from the assumption $E|\ln \kappa(\omega)| < +\infty$). From (5.4) and (5.5) we obtain that the inequalities

$$(5.6) \quad \rho_n(\omega) \leq \kappa_n(\omega)$$

a.s. hold for all n large enough. By using **(C2)** and (5.6), we obtain that for each $\varepsilon > 0$ with probability one,

$$\begin{aligned} \eta_n(\omega) &:= \sup_{t(n) \leq t \leq t(n+1)} \frac{1}{t} \ln \rho(x_t^a(\omega), \bar{x}_t(\omega)) \\ &\leq \sup_{t(n) \leq t \leq t(n+1)} \left[\frac{\ln^+ H(T^{t(n)}\omega)}{t} + \frac{b}{t} \ln \rho(x_{t(n)}^a(\omega), \bar{x}_{t(n)}(\omega)) \right] \\ &\leq \frac{\ln^+ H(T^{t(n)}\omega)}{t(n)} + \sup_{t(n) \leq t \leq t(n+1)} \frac{bt(n)c(1 - \varepsilon)}{t} = \frac{\ln^+ H(T^{t(n)}\omega)}{t(n)} + \frac{bnc(1 - \varepsilon)}{n + 1} \end{aligned}$$

for all n greater than some $n(\omega)$. The first summand in the last expression converges to zero a.s. by virtue of the assumption $E|\ln H| < +\infty$. The second summand tends to $bc(1 - \varepsilon)$. Consequently, with probability one, there exists $k(\omega)$ such that $\eta_n(\omega) \leq bc(1 - \varepsilon)^2$ for all $n \geq k(\omega)$. Denote by $N(t)$ the natural number such that $MN(t) \leq t < M[N(t) + 1]$. Then $t^{-1} \ln \rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq \eta_{N(t)}(\omega)$. Thus if $t \geq Mk(\omega)$, then $N(t) \geq k(\omega)$, and so

$$t^{-1} \ln \rho(x_t^a(\omega), \bar{x}_t(\omega)) \leq \eta_{N(t)}(\omega) \leq bc(1 - \varepsilon)^2.$$

Since ε is any number in $(0, 1)$, we obtain that (5.2) holds with $d := bc (< 0)$. The proof is complete. \square

Remark 2. Applying Theorem 2 in the discrete-time case, where $\mathbb{T} = \{0, 1, \dots\}$, to the cocycle defined by

$$C_t(\cdot, \omega) := f_t(\cdot, \omega) \circ \dots \circ f_1(\cdot, \omega), \quad t \geq 1,$$

we obtain a version of Theorem 1 in which condition **(A)** or **(B)** is imposed not on the given mapping $f(\cdot, \omega)$, but on the product $C_M(\cdot, \omega) = f_M(\cdot, \omega) \circ \dots \circ f_1(\cdot, \omega)$ of the mappings $f_t(\cdot, \omega)$. In this case, hypothesis **(C2)** is fulfilled under the following assumption.

(C3) There exist random variables $L(\omega) > 0$ and $\delta(\omega) > 0$ for which conditions (2.1) and (2.3) hold.

Indeed, assume that **(C3)** is satisfied and put

$$\kappa(\omega) := \min_{0 \leq t \leq M} \frac{\delta_t(\omega)}{L_0(\omega) \dots L_t(\omega)},$$

where L_t and δ_t are defined in section 4. Arguing by induction and using (2.3), we obtain that

$$\rho(C_t(x, \omega), C_t(\bar{x}(\omega), \omega)) \leq L_0(\omega) \dots L_t(\omega) \rho(x, \bar{x}(\omega)) \leq \delta_t(\omega)$$

($t = 1, 2, \dots, M$), as long as $\rho(x, \bar{x}(\omega)) \leq \kappa(\omega)$. This yields **(C2)** with $b = 1$ and $H(\omega) := L_1(\omega) \dots L_M(\omega)$.

Remark 3. Assume that X is a separable Banach space and for each ω , the mapping $C_t(\cdot, \omega)$ is differentiable at the point $\bar{x}_0(\omega)$ and continuous in a neighborhood of this point. Then the family of linear operators $F_t(\omega) = C'_t(\bar{x}_0(\omega), \omega)$ forms a cocycle (this follows from the chain rule of differentiation), called the linearization of the cocycle $C_t(x, \omega)$. Suppose that $E \ln^+ \|F_t(\omega)\| < +\infty$. In this case, the assumption that $E \ln \|F_M(\omega)\| < 0$ for some $M > 0$, needed for the application of Theorem 2 under hypothesis **(B)**, is equivalent to the assumption that the Furstenberg-Kesten constant [16, 32, 1]

$$\lim_{t \rightarrow \infty} \frac{1}{t} E \ln \|F_t(\omega)\| = \inf_{t > 0} \frac{1}{t} E \ln \|F_t(\omega)\|$$

is negative.

6. APPLICATION TO AN INVESTMENT MODEL

To describe an example to which we will apply Theorem 1, assume that together with the probability space (Ω, \mathcal{F}, P) and its endomorphism T , we are given a family of σ -algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}$ such that $T^{-1}(\Gamma) \in \mathcal{F}_{t+1}$ if and only if $\Gamma \in \mathcal{F}_t$ (\mathcal{F}_t contains events observable prior to time t). Denote by Δ the unit simplex $\{v = (v_1, \dots, v_K) \geq 0 : \sum v_k = 1\}$. Let $R_t(\omega) = R(T^t\omega)$, $\lambda_t(\omega) = \lambda(T^t\omega)$ and $\lambda_t^*(\omega) = \lambda^*(T^t\omega)$ be stationary processes with values in Δ adapted to the filtration (\mathcal{F}_t) and r a number in $(0, 1)$ such that

$$(6.1) \quad rE_t \lambda^*(T\omega) + (1 - r)E_t R(T\omega) = \lambda^*(\omega) \text{ (a.s.)},$$

where $E_t(\cdot) = E(\cdot | \mathcal{F}_t)$. The existence and uniqueness of the solution $\lambda^*(\cdot)$ to equation (6.1) follows from the Banach contraction principle and the fact that $r < 1$.

Consider the random dynamical system whose paths (x_t) are defined by

$$(6.2) \quad x_{t+1} = x_t \frac{\sum_{k=1}^K [r\lambda_{t+1,k}^* + (1-r)R_{t+1,k}] \frac{\lambda_{t,k}}{\lambda_{t,k}x_t + \lambda_{t,k}^*}}{\sum_{k=1}^K [r\lambda_{t+1,k} + (1-r)R_{t+1,k}] \frac{\lambda_{t,k}^*}{\lambda_{t,k}x_t + \lambda_{t,k}^*}},$$

where $R_{t,k}$, $\lambda_{t,k}$, and $\lambda_{t,k}^*$ are the coordinates of the vectors R_t , λ_t , and λ_t^* , respectively. In the evolutionary model of an asset market developed in [11, 12] (see these references for details), $R_{t,k}$ are *relative dividends* of K assets, the sequences of vectors $\lambda = (\lambda_t)$ and $\lambda^* = (\lambda_t^*)$ are *investment strategies (portfolio rules)* and r is the *investment rate*. The vectors $\lambda_t \in \Delta$ and $\lambda_t^* \in \Delta$ indicate proportions according to which investors using the strategies λ and λ^* allocate wealth across assets. The strategy λ^* defined by (6.1) is a generalization of the *Kelly portfolio rule*, well-known in mathematical finance (see, e.g., [28]). It is assumed that there are two groups of investors, one using the strategy λ^* and the other any strategy λ distinct from λ^* . The variable x_t represents the ratio w_t/w_t^* where w_t^* and w_t denote the total wealth of the former and the latter groups of investors, respectively. The local stability of the dynamical system under consideration at the fixed point $\bar{x} = 0$ means that the portfolio rule λ^* is *evolutionary stable*. If the initial relative wealth $x_0 = a$ of the λ -investors (“mutants” in the terminology borrowed from biology) is small enough, then they will be eventually driven out of the market by the λ^* -investors: their relative wealth x_t^a will tend to zero.

Define $\lambda_k^* := \lambda_{0,k}^*$ and assume that the following conditions hold.

(K1) $E \ln \min_k \lambda_k^* > -\infty$.

(K2) The random variables $\mu_k := r\lambda_{1,k}^* + (1-r)R_{1,k}$, $k = 1, \dots, K$, are conditionally linearly independent given \mathcal{F}_0 , i.e., the equality $\alpha_1\mu_1 + \dots + \alpha_K\mu_K = 0$ holding (a.s.) for some \mathcal{F}_0 -measurable random variables α_k implies $\alpha_1 = \dots = \alpha_K = 0$ (a.s.).

Theorem 3. *For any strategy $\lambda \neq \lambda^*$, there exist a random variable $\gamma(\omega) > 0$ and a constant $c < 0$ such that with probability one, $\limsup_{t \rightarrow \infty} t^{-1} \ln x_t^a \leq c$ for all $0 \leq a \leq \gamma(\omega)$.*

Proof. We apply Theorem 1 with $X = (-\infty, +\infty)$ and $X(\omega) = [0, \infty)$. We define the function $f(x, \omega)$ as the right-hand side of (6.2) with $t = 0$ if $x = x_t > -\zeta(\omega)$, where $\zeta(\omega) := \min_k \lambda_k^*(\omega)$. For $x \leq -\zeta(\omega)$, we can define $f(x, \omega)$, for example, as any (fixed) number u . Clearly $\bar{x} := 0$ is a fixed point of $f(x, \omega)$ for each ω . For $0 < x \leq 1$, we have $0 < x^{-1}f(x, \omega) \leq 2\zeta^{-2}$, so that (B1) holds with $\delta = 1$ and $L = 2\zeta^{-2}$. Further,

$$f'(0, \omega) = \sum_{k=1}^K \mu_k \frac{\lambda_k}{\lambda_k^*}.$$

By virtue of Jensen’s inequality, we have

$$E \ln \sum_{k=1}^K \mu_k \frac{\lambda_k}{\lambda_k^*} = E(E_0 \ln \sum_{k=1}^K \mu_k \frac{\lambda_k}{\lambda_k^*}) < E(\ln \sum_{k=1}^K E_0 \mu_k \frac{\lambda_k}{\lambda_k^*}) = E \ln \sum_{k=1}^K \lambda_k = 0.$$

To show that the above inequality is indeed strict, assume the contrary. Then the random variable $\sum_{k=1}^K \mu_k \lambda_k / \lambda_k^*$ coincides a.s. with an \mathcal{F}_0 -measurable random variable. Hence it coincides a.s. with its conditional expectation given \mathcal{F}_0 , which is

equal to 1 (this follows from (6.1)). Thus $\sum_{k=1}^K \mu_k \lambda_k / \lambda_k^* = 1$ (a.s.) or, equivalently, $\sum_{k=1}^K \mu_k [(\lambda_k / \lambda_k^*) - 1] = 0$ (a.s.), which implies by virtue of **(K2)** that $(\lambda_k / \lambda_k^*) - 1 = 0$ (a.s.) for all $k = 1, 2, \dots, K$. Consequently, $\lambda = \lambda^*$, which is a contradiction. \square

7. SUFFICIENT CONDITIONS IN THE MARKOVIAN CASE

We conclude with some comments regarding hypotheses **(K1)** and **(K2)**. These hypotheses are formulated in terms of the Kelly strategy λ^* , which is defined as the solution to equation (6.1). For the applications it is important to provide conditions sufficient for **(K1)** and **(K2)** that are formulated in terms of one of the primitives of the model, namely the dividend process R_t . The former condition holds if $E \ln \min_k E_0 R_{1,k} > -\infty$, which is clear from (6.1). The latter is satisfied, for example, if the following requirements are fulfilled:

- (i) the random variables $R_{1,k}$, $k = 1, \dots, K$, are conditionally independent given \mathcal{F}_0 (the absence of “redundant” assets);
- (ii) the probability space (Ω, \mathcal{F}, P) and the filtration (\mathcal{F}_t) are generated by a stationary Markov process $\dots, s_{-1}, s_0, s_1, \dots$ with values in a measurable space S , and the vector function $R(\omega)$ depends only on s_0 (we shall denote it as $R(s_0)$);
- (iii) the process s_t has a transition function $\pi(s, d\sigma)$ possessing a jointly measurable density $p(s, \sigma)$ with respect to a probability measure $\pi(d\sigma)$ such that $0 < v \leq p(s, \sigma) \leq V$ for some constants $v \leq V$.

Let us prove that conditions (i)–(iii) imply **(K2)**. We first observe that in the Markov case, $\lambda_k^* = \lambda_k^*(s_0)$ and $\mu_k = \mu_k(s_1)$ are functions of s_0 and s_1 , respectively. It follows from (6.1) that μ satisfies

$$(7.1) \quad rE_0\mu(s_1) + (1 - r)R(s_0) = \mu(s_0) \text{ (a.s.)}$$

In the present setting, \mathcal{F}_t -measurable functions can be identified with measurable functions $\alpha(s^t)$, where $s^t := (\dots, s_{t-1}, s_t)$. Let $\alpha_1(s^0), \dots, \alpha_K(s^0)$ be \mathcal{F}_0 -measurable vector functions satisfying

$$(7.2) \quad \langle \alpha(s^0), \mu(s_1) \rangle = 0 \text{ (a.s.)}$$

where $\alpha := (\alpha_1, \dots, \alpha_K)$ and $\mu = (\mu_1, \dots, \mu_K)$. We have to prove that $\alpha = 0$ (a.s.). From (7.1) we get

$$(7.3) \quad r\langle \alpha(s^{-1}), E_0\mu(s_1) \rangle + (1 - r)\langle \alpha(s^{-1}), R(s_0) \rangle = \langle \alpha(s^{-1}), \mu(s_0) \rangle \text{ (a.s.)}$$

Let us show that $E|\langle \alpha(s^{-1}), E_0\mu(s_1) \rangle| = 0$. We have

$$(7.4) \quad \begin{aligned} E|\langle \alpha(s^{-1}), E_0\mu(s_1) \rangle| &= E|E_0\langle \alpha(s^{-1}), \mu(s_1) \rangle| \leq EE_0|\langle \alpha(s^{-1}), \mu(s_1) \rangle| \\ &= E|\langle \alpha(s^{-1}), \mu(s_1) \rangle| = EE_{-1}|\langle \alpha(s^{-1}), \mu(s_1) \rangle|. \end{aligned}$$

From (iii) we get $p(s, \sigma) \leq Vv^{-1}p(s_{-1}, \sigma)$. By using this, we obtain

$$(7.5) \quad \begin{aligned} E_{-1}|\langle \alpha(s^{-1}), \mu(s_1) \rangle| &= \int \pi(s_{-1}, ds) \int \pi(s, d\sigma) |\langle \alpha(s^{-1}), \mu(\sigma) \rangle| \\ &= \int \pi(s_{-1}, ds) \int p(s, \sigma) \pi(d\sigma) |\langle \alpha(s^{-1}), \mu(\sigma) \rangle| \\ &\leq Vv^{-1} \int \pi(s_{-1}, ds) \int p(s_{-1}, \sigma) \pi(d\sigma) |\langle \alpha(s^{-1}), \mu(\sigma) \rangle| \\ &= Vv^{-1} \int \pi(s_{-1}, d\sigma) |\langle \alpha(s^{-1}), \mu(\sigma) \rangle| = Vv^{-1}E_{-1}|\langle \alpha(s^{-1}), \mu(s_0) \rangle|. \end{aligned}$$

By combining (7.5), (7.4) and (7.2), we obtain that $E|\langle \alpha(s^{-1}), E_0\mu(s_1) \rangle| = 0$. This, together with (7.2) and (7.3), implies the equality $\langle \alpha(s^{-1}), R(s_0) \rangle = 0$ (a.s.). By using (i) we conclude that $\alpha(s^{-1}) = 0$ (a.s.), or, equivalently, $\alpha(s^0) = 0$ (a.s.), which completes the proof.

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DEPARTMENT OF ECONOMICS, UNIVERSITY OF MANCHESTER, MANCHESTER M13 9PL, UNITED KINGDOM

E-mail address: igor.evstigneev@manchester.ac.uk

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, ACADEMY OF SCIENCES OF RUSSIA, GSP-4, MOSCOW, 101447, RUSSIA

E-mail address: pirogov@mail.ru

SCHOOL OF MATHEMATICS AND LEEDS UNIVERSITY BUSINESS SCHOOL, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, UNITED KINGDOM

E-mail address: k.r.schenk-hoppe@leeds.ac.uk