

METRIC CHARACTERIZATIONS OF ISOMETRIES AND OF UNITAL OPERATOR SPACES AND SYSTEMS

DAVID P. BLECHER AND MATTHEW NEAL

(Communicated by Marius Junge)

ABSTRACT. We give some new characterizations of unitaries, isometries, unital operator spaces, unital function spaces, operator systems, C^* -algebras, and related objects. These characterizations only employ the vector space and operator space structure (not mentioning products, involutions, or any kind of function on the space).

1. INTRODUCTION

We give some new characterizations of unitaries, isometries, unital operator spaces, operator systems, C^* -algebras, and related objects. Our results are all purely *linear-metric*, by which we mean that they only use the vector space structure of the space and its matrix norms, in the spirit of Ruan’s matrix norm characterization of operator spaces [18], not mentioning products, involutions, or any kind of function on the space. Our first main result characterizes unital operator spaces, that is, subspaces of a unital C^* -algebra containing the identity. More abstractly, a unital operator space is a pair (X, u) consisting of an operator space X containing a fixed element u such that there exists a Hilbert space H and a complete isometry $T : X \rightarrow B(H)$ with $T(u) = I_H$. Such spaces have played a significant role since the birth of operator space theory in [2]. Indeed, although the latter paper is mostly concerned with unital operator algebras, a good deal of it is presented in the setting of unital operator spaces. The text [6] also greatly emphasizes unital operator spaces, partly because this class includes several important objects such as operator systems and unital operator algebras. The following result complements Ruan’s characterization of operator spaces [18], the Blecher-Ruan-Sinclair abstract characterization of operator algebras [9], and a host of other theorems of this type (see e.g. [6, 16]). If $u \in X$ we write u_n for the diagonal matrix in $M_n(X)$ with u in each diagonal entry.

Theorem 1.1. *If u is an element in an operator space X , then (X, u) is a unital operator space if and only if*

$$\|[u_n \ x]\| = \left\| \left[\begin{array}{c} u_n \\ x \end{array} \right] \right\| = \sqrt{2},$$

Received by the editors November 30, 2009 and, in revised form, March 26, 2010.

2010 *Mathematics Subject Classification.* Primary 46L07, 47L25; Secondary 47B60, 47L07.

The first author was partially supported by grant DMS 0800674 from the National Science Foundation.

The second author was supported by Denison University.

for all $x \in M_n(X)$ of norm 1 and all $n \in \mathbb{N}$.

To explain the norms appearing in the theorem, recall that if X is an operator space, then there is a distinguished norm on $M_n(X)$, the $n \times n$ matrices with entries in X . Indeed if $X \subset B(H)$, then any element of $M_n(X)$ may be thought of as an operator on the Hilbert space $H^{(n)}$ and given the operator norm. The norm of a nonsquare matrix is obtained by adding rows or columns of zeroes to make the matrix square. In particular, $M_{2n,n}(X)$ and $M_{n,2n}(X)$ have canonical norms, which appear in the theorem statement.

This is the first linear-metric characterization of unital operator spaces that we are aware of. In [13] another description of unital operator spaces was given in terms of completely contractive unital matrix valued maps on the space, and their criterion is essentially that the direct sum of all these maps is a complete isometry. Some of the other topics we study are considered in [13] too. However, in our paper we avoid using criteria involving maps or functionals on the space, as we said.

We call the element u in the last theorem a *unitary in X* , or an *identity for X* . In addition to proving Theorem 1.1, in Section 2 we also characterize unitaries and isometries in a C^* -algebra, etc. In Section 3 we characterize operator systems, that is, selfadjoint subspaces of a unital C^* -algebra containing the identity. There is a famous abstract characterization of operator systems due to Choi and Effros (see [11, 16]) in terms of ‘matrix order’ and an ‘order-unit’. However, Theorem 1.1 immediately provides purely linear-metric characterizations of operator systems. Indeed, once one has characterized unital operator spaces metrically, operator systems come ‘for free’. This is because there are many ways, some well known and some described in Section 3, to obtain the involution, the selfadjoint part, or the positive cone of an operator system in terms of the identity and the norm (or matrix norms). We also give a criterion phrased only in terms of the identity and the matrix norms for when a unital operator space is a C^* -algebra. Section 4 discusses changing the identity in an operator system. Also, we present a connection between our theory and Choi and Effros’ order unit characterization mentioned above.

We now turn to precise definitions and notation. Any unexplained terms below can be found in [6] or in any of the other texts on operator spaces. All vector spaces are over the complex field \mathbb{C} . The letters H, K are usually reserved for Hilbert spaces. A given cone in a space X will sometimes be written as X_+ and $X_{sa} = \{x \in X : x = x^*\}$, assuming that there is an involution $*$ around. All normed (or operator) spaces are assumed to be complete. If $T : X \rightarrow Y$ and $n \in \mathbb{N}$ we have *amplifications* $T_n : M_n(X) \rightarrow M_n(Y) : [x_{ij}] \mapsto [T(x_{ij})]$. Then T is *completely isometric* (resp. *completely positive*, *completely contractive*) if every T_n is isometric (resp. positive, contractive). A (resp. complete) *order isomorphism* is a (resp. completely) positive linear bijection T such that T^{-1} is (resp. completely) positive. It is well known that a surjective complete isometry T between operator systems with $T(1) = 1$ is a complete order isomorphism (see e.g. 1.3.3 in [6]). Thus we will not be too concerned with positivity issues in this paper. Indeed, by the last fact, one could define an operator system to be a unital operator space (X, u) for which there exists a linear complete isometry $T : X \rightarrow B(H)$ with $T(u) = I_H$ and $T(X)$ selfadjoint.

A *TRO* (ternary ring of operators) is a closed subspace Z of a C^* -algebra or of $B(K, H)$, such that $ZZ^*Z \subset Z$. We refer for example to [12, 6] for the basic

theory of TROs. A *ternary morphism* on a TRO Z is a linear map T such that $T(xy^*z) = T(x)T(y)^*T(z)$ for all $x, y, z \in Z$. We write ZZ^* for the closure of the linear span of products zw^* with $z, w \in Z$, and similarly for Z^*Z . These are C^* -algebras. The *ternary envelope* of an operator space X is a pair $(\mathcal{T}(X), j)$ consisting of a TRO $\mathcal{T}(X)$ and a completely isometric linear map $j : X \rightarrow \mathcal{T}(X)$, such that $\mathcal{T}(X)$ is generated by $j(X)$ as a TRO (that is, there is no closed subTRO containing $j(X)$) and which has the following property: given any completely isometric linear map i from X into a TRO Z which is generated by $i(X)$, there exists a (necessarily unique and surjective) ternary morphism $\theta : Z \rightarrow \mathcal{T}(X)$ such that $\theta \circ i = j$. If (X, u) is a unital operator space, then the ternary envelope may be taken to be the C^* -envelope of e.g. [6, Section 4.3]; this is a C^* -algebra $C_e^*(X)$ with identity u . If X is an operator system, then X is a selfadjoint subspace of $C_e^*(X)$.

2. CHARACTERIZATION OF ISOMETRIES AND UNITAL SPACES

Clearly, the definition of a unital operator space (X, u) above is unchanged if we replace $B(H)$ by a unital C^* -algebra or if we replace I_H with any unitary. Thus the element u is called a *unitary in X* . Similarly, we say that an element v in an operator space X is an *isometry* (resp. *coisometry*) *in X* if there exists a complete isometry T from X into $B(K, H)$, for Hilbert spaces H and K , with $T(v)$ an isometry (resp. coisometry).

A C^* -unitary in a TRO Z is an element $u \in Z$ with $uu^*z = z$ and $zu^*u = z$ for all $z \in Z$. We say that u is an C^* -coisometry (resp. C^* -isometry) if just the first (resp. second) condition holds. If Z is a C^* -algebra it is easy to see that these coincide with the usual definition of unitary, coisometry, and isometry. We will see in Lemma 2.3 that these also coincide with the operator space definitions above. Thus after that lemma, we can drop the ‘ C^* -’ from the notation just defined. The matrix norms appearing in the next theorem are simply the canonical ones on the space of 1×2 and 2×1 matrices with entries in A .

Theorem 2.1. *Let A be a C^* -algebra or TRO. An element $u \in A$ is a C^* -unitary if and only if $\|[u \ x]\|^2 = 1 + \|x\|^2$ and $\|[u \ x]^t\|^2 = 1 + \|x\|^2$, for all $x \in A$. Indeed, it suffices to consider norm one elements x here. Similarly, u is a C^* -coisometry (resp. C^* -isometry) iff the first (resp. second) of these norm conditions holds for all $x \in A$.*

Proof. We just prove the coisometry assertion, the others following by symmetry. By the C^* -identity, $\|[u \ x]\|^2 = \|uu^* + xx^*\|$. If u is a coisometry, then this equals $1 + \|x\|^2$. Conversely, suppose that $\|[u \ x]\|^2 = 2$ for all $x \in A$ of norm 1. It is easy to see that this implies $\|u\| = 1$. By the C^* -identity, the norm of the matrix with entries u^*u, u^*x, x^*u, x^*x is also 2. By writing this matrix as a diagonal matrix plus another matrix, the last norm is $\leq \max\{\|u^*u\|, \|x^*x\|\} + \|u^*x\| = 1 + \|u^*x\| \leq 2$. Hence $\|u^*x\| = 1$. The fact that $uu^*x = x$ can now be deduced from the well known functional calculus in JB^* -triples (see e.g. p. 238 in [8] for an exposition of the TRO case of this), which shows that u is a partial isometry, and so $\|uu^*x - x\| = \|u^*(uu^*x - x)\| = 0$. Instead, we will use some well known and elementary facts from C^* -module theory. By the above, the operator L_{u^*} of left multiplication by u^* is an isometric right module map from A onto the closed right ideal (submodule) u^*A of A^*A . However every isometric C^* -module map is ‘unitary’ (see e.g. [6,

Corollary 8.1.8]). That is, in C^* -module language,

$$x^*uu^*x = \langle uu^*x, x \rangle = \langle u^*x, u^*x \rangle = \langle x, x \rangle = x^*x, \quad x \in A.$$

Since $uu^* \leq 1$, we have $\|(1 - uu^*)x\|^2 \leq \|\sqrt{1 - uu^*}x\|^2 = \|x^*(1 - uu^*)x\| = 0$. Thus $(1 - uu^*)x = 0$ for all $x \in A$, as desired. \square

Remark 2.2. We are unaware of any other ‘linear-metric’ (in the sense of our introduction) characterizations of isometries. Unitaries in a C^* -algebra A have a nice characterization by Bohnenblust and Karlin (see e.g. [15, Theorem 9.5.16] and [1, 3] for related topics), but these characterizations are in terms of linear functionals on A .

Lemma 2.3. *Let u be an element in an operator space X . The following are equivalent:*

- (i) u is a unitary (resp. isometry, coisometry) in X .
- (ii) There exists a TRO Z containing X completely isometrically such that u is a C^* -unitary (resp. C^* -isometry, C^* -coisometry) in Z .
- (iii) The image of u in the ternary envelope $\mathcal{T}(X)$ is a C^* -unitary (resp. C^* -isometry, C^* -coisometry).

If u is both an isometry and a coisometry in X , then u is a unitary in X .

Proof. We focus mainly on the coisometry case, the others usually being similar. The fact that (iii) implies (ii) is obvious.

(i) \Rightarrow (ii) If $v = T(u)$ is a coisometry for a complete isometry $T : X \rightarrow B(K, H)$, then v is a C^* -coisometry in the TRO Z generated by $T(X)$ in $B(K, H)$.

(ii) \Rightarrow (iii) We may assume that Z is generated by X , and then this follows by the universal property of $\mathcal{T}(X)$.

(ii) \Rightarrow (i) If u is a C^* -coisometry in Z , then uu^* is the identity of the C^* -algebra ZZ^* . If we represent the ‘linking algebra’ of Z nondegenerately on a Hilbert space $H \oplus K$ in the usual way (see e.g. 8.2.22 in [6]), then $uu^* = I_H$, so that u is a coisometry from K to H .

Note that if u is a C^* -unitary in $Z = \mathcal{T}(X)$, then u^*Z is a C^* -algebra with identity u and $Tx = u^*x$ is a ‘unital complete isometry’.

The final assertion follows from the equivalence with (iii), since a C^* -isometry which is also a C^* -coisometry is a C^* -unitary. \square

One may also phrase (iii) above in terms of the injective envelope $I(X)$.

We recall u_n is the diagonal matrix in $M_n(X)$ with u in every diagonal entry.

Theorem 2.4. *An element u in an operator space X is a unitary in X if and only if $\|[u_n \ x]\|^2 = 1 + \|x\|^2$ and $\|[u_n \ x]^t\|^2 = 1 + \|x\|^2$, for all $x \in M_n(X)$ and $n \in \mathbb{N}$. Indeed, it suffices to consider norm one matrices x here. Similarly, u is a coisometry (resp. isometry) in X iff the first (resp. second) of these norm conditions holds for all $x \in M_n(X)$.*

Proof. The proof is short because of our preparation above, but certainly nontrivial. We just prove the coisometry assertion, the others following by symmetry. The ‘easy direction’ is just as in Theorem 2.1 and is left as an exercise.

Conversely, given the $\|[u_n \ x]\|^2 = 1 + \|x\|^2$ condition, we consider $X \subset Z = \mathcal{T}(X)$. As in the proof of Theorem 2.1 we obtain $\|u^*x\| = \|x\|$, for all $x \in X$. This is equivalent to $\|uu^*x\| = \|x\|$ for every $x \in X$. Indeed if $\|u^*x\| = 1$ for x of norm 1, then $1 \geq \|uu^*x\| \geq \|x^*uu^*x\| = 1$ by the C^* -identity. Thus, the

operator $L : \mathcal{T}(X) \rightarrow \mathcal{T}(X)$ of left multiplication by uu^* is an isometry on X . For $x \in M_n(X)$ we have

$$\|L_n(x)\| = \|u_n u_n^* x\| = \|u_n^* x\| = \|x\|$$

by similar considerations. That is, L is a complete isometry on X .

By the ‘essential’ property of the ternary envelope (see e.g. [12] or [6, 8.3.12 (3) and 4.3.6]), L is an isometry on $\mathcal{T}(X)$. By the proof of Theorem 2.1, u is a C^* -coisometry in $\mathcal{T}(X)$. By Lemma 2.3, u is a coisometry in X . \square

Remark 2.5. Notice that in a selfadjoint operator space, if $u = u^*$, then the one condition $\|[u \ x]\|^2 = 1 + \|x\|^2$ in the characterization of unitaries above is equivalent to the other condition $\|[u \ x]^t\|^2 = 1 + \|x\|^2$, and similarly for the matricial version of these equalities.

Of course, Theorem 1.1 follows from Theorem 2.4.

It is natural to ask about the ‘commutative’ case of our results above, characterizing *unital function spaces*. By the latter term we mean a pair (X, g) , where X is a Banach space and $g \in X$, such that there exists a linear isometry $T : X \rightarrow C(K)$ for a compact Hausdorff space K , with $T(g) = 1$. We call such g a *function-unitary*. (This differs from the unitaries considered in [3].) By replacing $C(K)$ by the C^* -algebra generated by $T(X)$ we may assume that $T(X)$ separates points of K , a common assumption in the function theory literature. Indeed, unital function spaces may be taken as the ‘basic setting’ for the presentation of what some might call the ‘classical Shilov boundary’. Indeed, if K is the Shilov boundary of a unital function space X , then function-unitaries in X in our sense are just the elements in X which are unimodular on K . It is essentially an exercise to show that for an element g in a Banach space X , the following are equivalent:

- (i) g is a function-unitary in X in the sense above.
- (ii) g is a unitary in $\text{Min}(X)$ in the sense of the introduction of our paper.
- (iii) $\sup\{\|sf + tg\| : s, t \in \mathbb{C}, |s|^2 + |t|^2 = 1\} = \sqrt{2}$ for every $f \in X$ with $\|f\| = 1$.

Thus (X, g) is a unital function space if and only if (iii) holds. In (ii), the notation $\text{Min}(X)$ refers to the ‘minimal operator space structure’ on X (see e.g. 1.2.21 in [6]), namely the one associated with matrix norms

$$\|[x_{ij}]\| = \sup\{\|\varphi(x_{ij})\|_{M_n} : \varphi \in \text{Ball}(X^*)\}.$$

3. OPERATOR SYSTEMS

Once one has characterized unital operator spaces, operator systems come as an instant byproduct, by the following considerations. It is explained in 1.3.7 of [6] (relying on results from [2]) that every unital operator space (X, u) contains a canonical operator system $\Delta(X)$. Since this subset of X depends (only) on the unit u and the norm and vector space structure of X , as will be made very clear below, we will write it as Δ^u . If X is represented as a subspace of $B(H)$ via a complete isometry T taking u to I_H , then $\Delta^u = X \cap X^*$, the latter involution and intersection taken in $B(H)$. However the important point for us is that as a subspace of X , Δ^u does not depend on the particular H or T . Nor does its positive cone, which will be written as Δ^u_+ , nor does its involution; these depend only on the unit u and the normed vector space structure of X . Indeed, we now mention a recipe for describing these elements more explicitly in terms of the norm and linear structure of X .

Definition 3.1. Let u, x be elements of a Banach space X . We say that x is *u-hermitian* if there is a constant K such that $\|u + itx\|^2 \leq 1 + Kt^2$ for all $t \in \mathbb{R}$. We say that x is *u-positive* if it is *u-hermitian* and if $\|x\|u - x\| \leq \|x\|$.

It is well known (and is an easy exercise) in the theory of numerical ranges in Banach algebras that an element x in a C^* -algebra with identity u is selfadjoint iff it is *u-hermitian* (see e.g. [4, Lemma 5.2]). Similarly, in a Banach algebra with identity u , an element x is *u-hermitian* in the sense above iff it is hermitian in the usual sense. The one direction is as in [4, Lemma 5.2]. For the other direction, if x is hermitian, then a result of Sinclair states that norm equals the spectral radius on $1 + itx$. Thus

$$\|1 + itx\|^2 = \sup\{|1 + it \chi(x)|^2\} = \sup\{1 + t^2 |\chi(x)|^2\} = 1 + t^2 \|x\|^2,$$

where the supremum is over characters χ of the algebra generated by 1 and x .

Note that the *u*-positives in $\text{Ball}(X)$ are the *u-hermitians* with $\|u - x\| \leq 1$.

The following is then obvious:

Proposition 3.2. *If (X, u) is a unital operator space, then Δ^u is the span of the u-hermitians in X , Δ_{sa}^u is the set of u-hermitians in X , and Δ_+^u is the set of u-positives in X .*

A unital operator space (X, u) is an operator system iff the u-hermitians span X and iff the u-positives span X .

Of course the involution on Δ^u is just $(h + ik)^* = h - ik$ if $h, k \in \Delta_{sa}^u$.

Corollary 3.3. *Let (X, u) be a unital operator space which also possesses a conjugate linear involution $*$. Then (X, u) is an operator system whose involution is $*$ if and only if $x = x^* \in X$ implies that x is u-hermitian.*

Proof. Suppose that $x = x^*$ implies that x is *u-hermitian*. Since the set of elements with $x = x^*$ spans X , so does the set of *u-hermitians*. Thus X is an operator system. The rest is obvious. \square

The following characterization of operator systems employs a direct linear-metric test to determine when the adjoint of an element of X lies in X :

Theorem 3.4. *A unital operator space (X, u) (characterized above) is an operator system iff for all $x \in \text{Ball}(X)$*

$$\inf\left\{\left\|\begin{bmatrix} tu & x \\ y & tu \end{bmatrix}\right\| : y \in \text{Ball}(X)\right\} \leq \sqrt{t^2 + 1}$$

for all $t \in \mathbb{R}$ and $x \in \text{Ball}(X)$. In this case, the infimum is achieved for all $t \in \mathbb{R}$ precisely when $y = -x^$.*

Proof. If $I_H \in X = X^* \subset B(H)$, then by the C^* -identity it is easy to see that $\left\|\begin{bmatrix} tI & x \\ -x^* & tI \end{bmatrix}\right\|^2 = t^2 + 1$ for every $t \in \mathbb{R}$ and $x \in X$ with $\|x\| = 1$. Conversely, if $I_H \in X \subset B(H)$ and if the condition involving the infimum holds, then for all $n \in \mathbb{N}$ there is an element $y_n \in X$ with $\left\|\begin{bmatrix} nI & x \\ y_n & nI \end{bmatrix}\right\|^2 \leq n^2 + 1 + \frac{1}{n}$. The norm of the last matrix is unchanged if we multiply the ‘diagonal entries’ by -1 . Using the C^* -identity, it follows that for every state φ on $M_2(B(H))$ we have

$$n^2 + \varphi\left(\begin{bmatrix} xx^* & 0 \\ 0 & y_n y_n^* \end{bmatrix}\right) \pm n\varphi\left(\begin{bmatrix} 0 & x + y_n^* \\ y_n + x^* & 0 \end{bmatrix}\right) \leq n^2 + 1 + \frac{1}{n}.$$

Taking the supremum over all states φ , we deduce that

$$\|y_n + x^*\| = \left\| \begin{bmatrix} 0 & x + y_n^* \\ y_n + x^* & 0 \end{bmatrix} \right\| \leq \frac{1}{n} + \frac{1}{n^2}.$$

Hence $y_n \rightarrow -x^*$, and so $X^* = X$. □

Remark 3.5. 1) In Theorem 3.4 it does not suffice to take $t = 1$, even in the case that $\|x\| = 1$. Indeed it is easy to argue that any nonselfadjoint unital function space will be a counterexample to this.

2) The 2×2 matrix trick involving the C^* -identity in the last proof can also be used to prove several related results. For example, we have:

(a) A matricial characterization of Δ_{sa}^u and Δ_+^u : if $x \in \text{Ball}(X)$, then $x \in \Delta_{sa}^u$ iff $\left\| \begin{bmatrix} tu & x \\ -x & tu \end{bmatrix} \right\| \leq \sqrt{t^2 + 1}$ for all $t \in \mathbb{R}$; also $x \in \Delta_+^u$ iff $\left\| \begin{bmatrix} tu & u - x \\ x - u & tu \end{bmatrix} \right\| \leq \sqrt{t^2 + 1}$ for all $t \in \mathbb{R}$.

(b) (X, u) in Corollary 3.3 is an operator system iff $\left\| \begin{bmatrix} tu & x \\ -x^* & tu \end{bmatrix} \right\| \leq \sqrt{t^2 + 1}$ for all $t \in \mathbb{R}$ and $x \in \text{Ball}(X)$.

(c) A simple linear-metric ‘method’ to retrieve a forgotten product of any two operators in $B(H)$, one of which is an isometry or coisometry: Suppose that $I_H \in X \subset B(H)$ and that v is a coisometry (resp. isometry) on H which lies in X . If $y, z \in \text{Ball}(X)$, then $z = -vy^*$ (resp. $y = -z^*v$) in $B(H)$ if and only if $\left\| \begin{bmatrix} t1 & y \\ z & tv \end{bmatrix} \right\| \leq \sqrt{1 + t^2}$ for all $t \in \mathbb{R}$.

The key point is that the condition on the right of the last ‘iff’ does not involve any product, whereas the condition on the left does involve a product. The condition does involve isometries or coisometries, however we already characterized these purely linear-metrically.

We can also use the 2×2 matrix trick in the last proof to characterize C^* -algebras among the unital operator spaces. We will write the identity u of our operator system X as 1 . This topic is very closely related to the question of recovering a forgotten product on a C^* -algebra, which was discussed e.g. on p. 316 of [4]. The route we take here is that since unitaries have been characterized in Section 2, to characterize C^* -algebras it suffices to 1) characterize when X is the span of the unitaries it contains, and 2) characterize when the product in a containing C^* -algebra of every two unitaries u and v in X is again in X . There seem to be many simple characterizations of these in the context of unital C^* -algebras. For example, X contains the product vu of any two unitaries u, v in X if and only if the matrix $\begin{bmatrix} 1 & u \\ v & x \end{bmatrix}$ is $\sqrt{2}$ times a unitary (characterized in Section 2) in $M_2(X)$ for some $x \in X$ or if there is a similar characterization using a 3×3 positivity condition as in [19]. We leave these to the interested reader. The best such condition we have found to date is the following result, whose proof we leave as an exercise using the idea in the proof of Theorem 3.4. Again, the key point is that the ‘right hand side’ of the ‘if and only if’ makes no mention of products or involutions.

Theorem 3.6. *A unital operator space $(X, 1)$ (characterized above) possesses a product with respect to which it is isomorphic to a C^* -algebra via a unital complete isometry, if and only if X is spanned by the unitaries in X (characterized in*

Section 2), and for every unitary v in X we have

$$\inf \left\{ \left\| \begin{bmatrix} t1 & y \\ z & tv \end{bmatrix} \right\| : z \in \text{Ball}(X) \right\} \leq \sqrt{t^2 + 1}$$

for all $t \in \mathbb{R}$ and $y \in \text{Ball}(X)$.

Remark 3.7. We may replace “ $y \in \text{Ball}(X)$ ” in the last theorem with y in the set of unitaries in X .

We mention the interesting related topic of ‘dual operator systems’. We recall that an operator space X is a *dual operator space* if it is the operator space dual of another operator space. Also, it is a well known fact that this is essentially the same as saying that X is a weak* closed subspace of some $B(H)$. For this and for other aspects of the duality of operator spaces we refer the reader to e.g. [6, Section 1.4]. The analogous fact for operator systems, or for unital operator spaces, was conjectured in an earlier draft of the present paper and later solved in [7] using the following ‘first steps’:

Lemma 3.8. *Suppose that (X, u) is a unital operator space and suppose also that X is a dual Banach space. Then Δ^u and Δ_{sa}^u are weak* closed and the involution on Δ^u is weak* continuous.*

Proof. Suppose that (x_s) is a net in $\Delta_{\text{sa}}^u \cap \text{Ball}(X)$, with $x_s \rightarrow x$ weak* in X . Then $\|u + itx_s\| \leq \sqrt{1 + t^2}$. Taking a limit with s we see that $x \in \Delta_{\text{sa}}^u$. Thus by the Krein-Smulian theorem, Δ_{sa}^u is weak* closed. Next suppose that $(x_s + iy_s)$ is a bounded net in Δ^u , with limit z . Here $x_s, y_s \in \Delta_{\text{sa}}^u$. Then (x_s) and (y_s) are bounded nets. Suppose that a subnet (x_{s_λ}) converges weak* to x , say. Then (y_{s_λ}) has a subnet converging weak* to y , say. Replacing the nets by subnets, it is now easy to see that $z = x + iy \in \Delta^u$. So Δ^u is weak* closed. Finally, suppose that $x_s + iy_s \rightarrow x + iy$ for $x, y \in \Delta_{\text{sa}}^u$. The argument above shows that every weak* convergent subnet of (x_s) converges to x . Thus $x_s \rightarrow x$ weak*. Similarly, $y_s \rightarrow y$ weak*, and so $(x_s + iy_s)^* = x_s - iy_s \rightarrow x - iy$. By a variant of the Krein-Smulian theorem, the involution is weak* continuous. \square

Corollary 3.9. *If X is an operator system which is also a dual Banach space, then the involution on X is weak* continuous.*

It is easy to see that if X is a dual operator space possessing a weak* continuous conjugate linear involution $*$ for which $\|[x_{ji}^*]\|_n = \|[x_{ij}]\|_n$ for all matrices $[x_{ij}]$ with entries in X , then there exists a weak* homeomorphic $*$ -linear complete isometry from X onto a weak* closed selfadjoint subspace W of some $B(H)$. Indeed if $\varphi : X \rightarrow B(H)$ is any weak* continuous complete isometry, then the function $x \mapsto \begin{bmatrix} 0 & \varphi(x) \\ \varphi(x^*)^* & 0 \end{bmatrix}$ does the trick. This immediately gives an abstract characterization of weak* closed selfadjoint subspaces of $B(H)$.

4. MORE ON OPERATOR SYSTEMS

In the first half of this section we are mainly interested in understanding what happens when the identity element in an operator system is replaced.

The following facts will be useful to us below. Given a unital operator space (X, u) , another way to recapture the involution which is sometimes useful is as ux^*u , the latter product and involution taken in a ternary envelope $\mathcal{T}(X)$. Claim:

a unital operator space (X, u) is an operator system iff $uX^*u \subset X$ within $\mathcal{T}(X)$, and in this case the expression uX^*u is independent of the particular ternary envelope of X chosen. To explain more carefully what is meant by these statements, as well as to prove them, first suppose that (X, u) is an operator system and $(C_e^*(X), i)$ is a C^* -envelope. Then $i(x^*) = i(u)i(x)^*i(u)$ for all $x \in X$, since $i(u) = 1$. By the universal property of any ternary envelope $(\mathcal{T}(X), j)$ (stated at the end of the introduction), there is a ternary morphism $\theta : C_e^*(X) \rightarrow \mathcal{T}(X)$ with

$$j(u)j(x)^*j(u) = \theta(i(u))\theta(i(x))^*\theta(i(u)) = \theta(i(x^*)) = j(x^*) \in j(X).$$

Conversely, if $j(u)j(x)^*j(u) \in j(X)$ for all $x \in X$, for some ternary envelope $(\mathcal{T}(X), j)$ of a unital operator space (X, u) , and if $(C_e^*(X), i)$ is a C^* -envelope, then an analogous argument shows that $i(x)^* = i(u)i(x)^*i(u) \in i(X)$ for all $x \in X$. Thus $i(X)$, and therefore X , is an operator system. Moreover, if $x \in X$ is fixed and if $i(u)i(x)^*i(u) = i(x')$ for some $x' \in X$, then by the argument above, $j(u)j(x)^*j(u) = j(x')$ for the same $x' \in X$. This is what is meant by saying ‘the expression uX^*u is independent of the particular ternary envelope’.

We will not need this until much later, but it is worth mentioning that the set of positive elements Δ_+^u in a unital operator space (X, u) is precisely $\mathfrak{d}_u \cap X$, in the notation of [8]. We leave it to the reader to check this useful alternative description of the positive elements in X .

Example 4.1. If u is any unitary in a C^* -algebra or TRO A , then (A, u) is an operator system. This follows by the facts presented in the last paragraph, since in this case $uA^*u \subset A$ and $\mathcal{T}(A) = A$. Moreover, any two unitaries $u, v \in A$ induce in some sense the same operator system structure, since the map $T(x) = vu^*x$ on A is a surjective complete isometry taking u to v , and hence T is a complete order isomorphism too, by e.g. 1.3.3 in [6].

Proposition 4.2. *Let u and v be two unitaries in an operator space X such that (X, u) is an operator system and such that there is a product making X an operator algebra with identity v . Then there is an involution on the latter operator algebra and a product on (X, u) , making these C^* -algebras.*

Proof. Let D be the C^* -envelope of (X, v) . We may identify X with a subalgebra of D , with $v = 1_D$ (this follows for example from [6, Proposition 4.3.5]). All products and involution below are in D . By Lemma 2.3, $u^*u = uu^* = v$. By the remarks at the start of this section, $uX^*u = X$. From this it follows first that $u(u^3)^*u = u^* \in X$ and second that $X^* = u^*Xu^* \subset X$. So $X = D$. Moreover, $x \mapsto xu^*$ is a unital complete isometry, hence a complete order isomorphism, from (X, u) onto $D = (X, v)$. The rest is clear. \square

The features in the last example and proposition fail badly for more general operator spaces. It is easy to find operator spaces X with unitaries u, v for which (X, u) is not an operator system but (X, v) is or for which they are both operator systems but there exists no surjective complete isometry taking u to v . Moreover, there exists no such surjective complete isometry even in the case that u and v induce the same involution on X . We will give some explicit examples of these phenomena.

Example 4.3. Let X be the span of $1, f$, and \bar{f} , in the continuous functions on the unit circle in the complex plane, where $f(z) = z$ on the circle. Then $(X, 1)$ is an

operator system, but (X, f) is not, although f is a unitary in X . To see the latter claim note first that the circle is the Shilov boundary, since for any nontrivial arc U on the circle there is a function in X that does not achieve its norm on U^c . Thus $C_e^*(X)$ is the space of continuous functions on the circle (see e.g. 4.3.4 in [6]), and this is a ternary envelope. However $fX^*f \neq X$, and so (X, f) is not an operator system by the facts presented above Example 4.1.

Example 4.4. We describe a selfadjoint space X of continuous functions on a compact topological space K (equal to the Shilov boundary of X), with X containing constant functions, and a unimodular continuous g on K , such that (X, g) is an operator system with unchanged involution, but there exists no surjective isometric isomorphism $T : X \rightarrow X$ with $T(1) = g$.

Let K be the topological disjoint union of two copies of the closed right half of the unit circle \mathbb{T} in \mathbb{C} . Let g be 1 on the first semicircle and -1 on the other, and let $f(z) = z$ for any z in either semicircle. Set $X = \text{Span}(\{1, g, f, \bar{f}\})$. Seeing that the Shilov boundary of X is K is similar to Example 4.3, but slightly more complicated. The point is that if w is a point on the first (resp. second) semicircle, then the function $\frac{1}{3}(1 + g + \bar{w}f)$ (resp. $\frac{1}{3}(1 - g + \bar{w}f)$) achieves its norm only at w . Hence $C_e^*(X)$ is the space of continuous functions on K (see e.g. 4.3.4 in [6]). Clearly g is unitary in $C(K)$. Since $gx^*g = \bar{x} \in X^*$ for all $x \in X$, we have that (X, g) is an operator system with unchanged involution (this also follows from Corollary 4.7 below).

To see that there exists no surjective isometric isomorphism $T : X \rightarrow X$ with $T(1) = g$, assume the contrary, and use Proposition 4.5 below, together with the fact that a $*$ -isomorphism of $C(K)$ is a ‘composition with a homeomorphism’ $\tau : K \rightarrow K$ (the Banach-Stone theorem). Obtaining a contradiction is now a pleasant exercise, which we sketch: In this case τ will consist of two homeomorphisms, each from one copy of the semicircle either to itself or onto the other semicircle. Proposition 4.5 implies that $f \circ \tau = \tau = gh$ for some $h \in X$. Thus, on each semicircle, the modulus of h is constant. This forces h to be either a constant or a constant times f or \bar{f} on each semicircle. The relation $\tau = gh$ tells us what τ must be, and then we obtain the contradiction $\tau \neq gh$.

Thus X has all the properties described in the first paragraph of the example.

Proposition 4.5. *Let X be a unital operator space viewed within its C^* -envelope $A = C_e^*(X)$. Suppose that v is a unitary in X . Then there exists a surjective complete isometry $T : X \rightarrow X$ with $T(1) = v$ if and only if there is a $*$ -isomorphism $\theta : A \rightarrow A$ such that $v^*X = \theta(X)$.*

Proof. For the one direction simply set $T = v\theta(\cdot)$. Conversely, suppose that $T : X \rightarrow X$ is as stated. Since the C^* -envelope is a ternary envelope, by universal properties of the ternary envelope we may extend T to a surjective complete isometry $\tilde{T} : C_e^*(X) \rightarrow C_e^*(X)$. Then \tilde{T} is a ternary morphism by [6, Corollary 4.4.6], and using this $\theta = v^*\tilde{T}(\cdot)$ is easily seen to be a $*$ -isomorphism of A onto itself. The rest is obvious. \square

Proposition 4.6. *If X is an operator space with unitaries u and v such that (X, u) and (X, v) are operator systems, then the involutions on these two systems are the same if and only if u^*v is in the center of Z^*Z and equals v^*u , where $Z = \mathcal{T}(X)$.*

Proof. To say that the involutions are the same is to say that $ux^*u = vx^*v$ for all $x \in X$. Setting $x = u$ gives $u = vu^*v$, so that $v^*u = u^*v$. Moreover, it is simple

algebra to check that $x^*yu^*v = u^*vx^*y$ for $x, y \in X$. Since spans of products of terms of the form x^*y for $x, y \in X$ are dense in Z^*Z , we deduce that u^*v is in the center of Z^*Z .

Conversely, suppose that $u^*v = v^*u$ is in the center of Z^*Z . Then for all $x \in Z$, we have $ux^*u = vv^*ux^*u = vx^*uv^*u = vx^*uu^*v = vx^*v$. □

Corollary 4.7. *Let v be a unitary in an operator subsystem $X \subset B(H)$. Then $v \in X_{\text{sa}}$ and v is in the center of $C_e^*(X)$, if and only if (X, v) is an operator system, and the involution associated with v equals the original involution.*

Proof. The one direction follows immediately from Proposition 4.6. For the other, if $v = v^*$ and v is in the center of $C_e^*(X)$, then by the facts presented above in Example 4.1, it follows that (X, v) is an operator system and it has unchanged involution. □

Remark 4.8. Of course saying that the involutions associated with two unitaries u and v coincide is equivalent to saying that every u -hermitian is v -hermitian. This is assuming that (X, u) is an operator system.

By a *function system* we will mean a closed selfadjoint subspace of $C(K)$, for compact K , containing constant functions. There is an obvious ‘abstract definition’: (X, g) is a function system if there exists an isometry $T : X \rightarrow C(K)$ with $T(g) = 1$ and $T(X)$ selfadjoint. See [17] for some theory of complex function systems. It is easy to see that this is equivalent to saying that $(\text{Min}(X), g)$ is an operator system, where $\text{Min}(X)$ was defined at the end of Section 2. Indeed, one direction of this follows by applying the functorial property of Min (see e.g. 1.2.21 in [6]) to T above. The other direction follows from the fact that the injective envelope of an operator system which is a Min space is a C^* -algebra ternary isomorphic to a commutative C^* -algebra (see 4.2.11 in [6]), and hence is commutative. Hence the subalgebra $C_e^*(X)$ is commutative, and one can take T above to be the canonical embedding of X into $C_e^*(X)$.

Many of the earlier results concerning operator systems have ‘function system’ analogues. For example, function systems are obviously the unital function spaces (X, g) which are spanned by their g -hermitians. The following is another characterization of function systems, which improves on Corollary 4.7:

Theorem 4.9. *Let X be a closed selfadjoint subspace of $C(K)$ for a compact set K . If v is a function-unitary in X in the sense of Remark 2.5 in Section 2, with $v = v^*$, then (X, v) is a function system and the involution associated with v equals the original involution.*

Proof. Let X be a selfadjoint subspace of $C(K)$, for compact K . The TRO generated by X in $C(K)$ is a $*$ -subTRO of $C(K)$, and it follows that the ‘ternary $*$ -envelope’ Z of X (see [5]) is ‘commutative’; that is, $xy = yx$ for all $x, y \in Z$. Also, $C = v^*Z$ is a commutative unital C^* -algebra, and $T(x) = v^*x$ is an isometric ‘unital’ map into C . Since X is a selfadjoint subspace of Z , it is easy to see that T is ‘ $*$ -linear’, and the rest is obvious. Alternatively, note that $vx^*v = x^*vv^* = x^*$ for $x \in X$. □

Remark 4.10. Closed selfadjoint subspaces of $C(K)$ for compact K may be characterized abstractly as the Banach spaces X with an involution $*$ having the property

that every extreme point of $\text{Ball}(X^*)$ is a scalar multiple of a functional satisfying $\varphi(x^*) = \overline{\varphi(x)}$ for $x \in X$. This follows by a routine Krein-Milman argument.

We also remark that simple examples show that the obvious variant of the last result fails for general selfadjoint operator spaces. Indeed the results above in the present section seem at the present time to be ‘best possible’.

In the remainder of the section, we mention a connection to the famous characterization due to Choi and Effros of operator systems [11] in terms of a given cone in the space. We will assume throughout that we have a fixed cone \mathfrak{c} in X and that this cone spans X (although this also often follows as a consequence of some of the conditions imposed below). We allow two variants of the theory, depending on whether or not we are assuming the existence of a given fixed involution $*$ on X . If the latter holds, we will assume further that $x = x^*$ for all $x \in \mathfrak{c}$.

Definition 4.11. By an *ordered operator space* below we will mean a pair (X, \mathfrak{c}) consisting of an operator space and a cone \mathfrak{c} in X , such that there exists a complete isometry of X into a C^* -algebra A taking \mathfrak{c} into A_+ .

Proposition 4.12. *Suppose that (X, \mathfrak{c}) is an ordered operator space and that u is a unitary in X contained in \mathfrak{c} . If \mathfrak{c} spans X , then (X, u) is an operator system. Moreover, in the ‘involutive variant’ of the theory (mentioned above in Definition 4.11), the involution induced by u equals the original involution $*$.*

Proof. We use notation and facts from [8]. Consider the ordered ternary envelope of X whose positive cone is a natural cone \mathfrak{d}_v given by an open tripotent v . Also, $\mathfrak{c} \subset \mathfrak{d}_v$. Since $u \in \mathfrak{c} \subset \mathfrak{d}_v$ and u is unitary, it follows that $u = v$. Thus $\mathfrak{c} \subset \mathfrak{d}_u \cap X = \Delta_+^u$. Since \mathfrak{c} is spanning, this implies by Proposition 3.2 that (X, u) is an operator system. In the ‘involutive variant’, notice that $ux^*u = x = x^*$ for $x \in \mathfrak{c}$ and hence for $x \in X$ since \mathfrak{c} is spanning. This also shows that u is central in the sense of [10] in the ordered ternary envelope of X . \square

Next, we seek conditions which imply that (X, u) is an operator system whose cone is precisely \mathfrak{c} . If we also had cones in $M_n(X)$, then Choi and Effros’ famous characterization of operator systems provide necessary and sufficient conditions for this. The most prominent of these conditions is the existence of a ‘matricial order unit’. We show here that the following weaker condition suffices:

Definition 4.13. We say that u is a *norm-order unit* for \mathfrak{c} if $u \in \mathfrak{c}$ and for every $x \in X_{\text{sa}}$ we have $\|x\|u - x \in \mathfrak{c}$.

In the ‘noninvolutive space variant’ of the theory (see the discussion above in Definition 4.11), here we replace X_{sa} by the u -hermitians on X .

Proposition 4.14. *Suppose that (X, \mathfrak{c}) is an ordered operator space and that u is a unitary in X which is a norm-order unit for \mathfrak{c} . Then (X, u) is an operator system whose positive cone Δ_+^u is \mathfrak{c} .*

Proof. We saw in Proposition 4.12 that (X, u) is an operator system, $\mathfrak{c} \subset \mathfrak{d}_u$, and that in the ‘involutive space variant’ of the theory the involution is unambiguous. If $x \in \mathfrak{d}_u \cap X = \Delta_+^u$, then x is u -positive. If $t = \|x\|$, then $tu - x \in \mathfrak{c}$. Hence $tu - x = c$ for some $c \in \mathfrak{c}$. Viewed within the C^* -algebra $Z_2''(u)$ (notation as in [8]), we have $\|c\| = \|tu - x\| \leq t$, since $tu - x \leq tu$ in $Z_2''(u)$. Thus $x = tu - c = (t - \|c\|)u + (\|c\|u - c) \in \mathfrak{c}$. So $\Delta_+^u = \mathfrak{c}$. \square

Corollary 4.15. *Suppose that (X, u) is an operator system and that \mathfrak{c} is a subcone of Δ_+^u , such that u is a norm-order unit for \mathfrak{c} . Then $\Delta_+^u = \mathfrak{c}$.*

Proof. Clearly (X, \mathfrak{c}) is an ordered operator space, and we are in the situation of Proposition 4.14. \square

Remark 4.16. In the previous context, the range tripotent of an order unit can be shown to be unitary. Hence an order unit which is a partial isometry is unitary.

Remark 4.17. We close with some general remarks on the subject of our paper:

Some interesting new examples of unital operator spaces are presented in [7]. For example, the Fourier algebra $B(G)$ of any amenable group, with its usual operator space structure as the dual of the group C^* -algebra of G , is a unital operator space. Some quantum group analogues of this are true too.

Having an abstract characterization of a class of objects is also often useful in order to show that the class is closed under the usual shopping list of ‘constructions’, such as direct sums, certain quotients, tensor products, ultraproducts, interpolation, etc. In our case one may certainly do this, but we will not do so here, for the reason that all of these can seemingly be done without appealing to our new characterizations. For example, as C. K. Ng has suggested to us, one may prove their result from [13] about quotients by M -ideals using our criteria too. However there is a third way to prove this result: if X is a unital operator space (resp. operator system) sitting in its C^* -envelope A , then by basic facts about M -summands from e.g. [4], [6, Section 4.8] and the references therein, we may view a complete M -projection on X as a projection p in the center of A which is also in X . Thus pX is a unital subspace (resp. operator subsystem) of pAp . This does the M -summand case, and the M -ideal case follows by the idea in [13] of going to the second dual.

ACKNOWLEDGMENTS

We thank L. A. Harris for some useful comments and the referee for suggesting that we add several details to improve the exposition. The first versions of our paper were distributed in May 2008.

REFERENCES

- [1] C. A. Akemann and N. Weaver, *Geometric characterizations of some classes of operators in C^* -algebras and von Neumann algebras*, Proc. Amer. Math. Soc. **130** (2002), 3033–3037. MR1908927 (2003e:46093)
- [2] W. B. Arveson, *Subalgebras of C^* -algebras*, Acta Math. **123** (1969), 141–224. MR0253059 (40:6274)
- [3] P. Bandyopadhyay, K. Jarosz, and T. S. S. R. K. Rao *Unitaries in Banach spaces*, Illinois J. Math. **48** (2004), 339–351. MR2048228 (2005b:46026)
- [4] D. P. Blecher, E. G. Effros, and V. Zarikian, *One-sided M -ideals and multipliers in operator spaces. I*, Pacific J. Math. **206** (2002), 287–319. MR1926779 (2003h:46084)
- [5] D. P. Blecher, K. Kirkpatrick, M. Neal, and W. Werner, *Ordered involutive operator spaces*, Positivity **11** (2007), 497–510. MR2336213 (2008k:46174)
- [6] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, Oxford Univ. Press, Oxford, 2004. MR2111973 (2006a:46070)
- [7] D. P. Blecher and B. Magajna, *Dual operator systems*, preprint, arXiv:0807.4250
- [8] D. P. Blecher and M. Neal, *Open partial isometries and positivity in operator spaces*, Studia Math **182** (2007), 227–262. MR2360629 (2009a:46114)
- [9] D. P. Blecher, Z. J. Ruan, and A. M. Sinclair, *A characterization of operator algebras*, J. Functional Anal. **89** (1990), 188–201. MR1040962 (91b:47098)

- [10] D. P. Blecher and W. Werner, *Ordered C^* -modules*, Proc. London Math. Soc. **92** (2006), 682–712. MR2223541 (2007d:47071)
- [11] M.-D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209. MR0430809 (55:3814)
- [12] M. Hamana, *Triple envelopes and Silov boundaries of operator spaces*, Math. J. Toyama University **22** (1999), 77–93. MR1744498 (2001a:46057)
- [13] X. J. Huang and C. K. Ng, *An abstract characterization of unital operator spaces*, J. Operator Theory, to appear, arXiv:0805.2447.
- [14] M. Neal and B. Russo, *Operator space characterizations of C^* -algebras and ternary rings*, Pacific J. Math. **209** (2003), 339–364. MR1978376 (2004b:46076)
- [15] T. W. Palmer, *Banach algebras and the general theory of $*$ -algebras, Vol. II. $*$ -algebras*, Encyclopedia of Math. and its Appl., 79, Cambridge University Press, Cambridge, 2001. MR1819503 (2002e:46002)
- [16] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Math., 78, Cambridge University Press, Cambridge, 2002. MR1976867 (2004c:46118)
- [17] V. I. Paulsen and M. Tomforde, *Vector spaces with an order unit*, Indiana Univ. Math. J. **58** (2009), no. 3, 1319–1359. MR2542089 (2010f:46008)
- [18] Z.-J. Ruan, *Subspaces of C^* -algebras*, J. Funct. Anal. **76** (1988), 217–230. MR923053 (89h:46082)
- [19] M. E. Walter, *Algebraic structures determined by 3 by 3 matrix geometry*, Proc. Amer. Math. Soc. **131** (2003), 2129–2131. MR1963763 (2004c:46137)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77204-3008
E-mail address: `dblechter@math.uh.edu`

DEPARTMENT OF MATHEMATICS, DENISON UNIVERSITY, GRANVILLE, OHIO 43023
E-mail address: `nealm@denison.edu`