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THERE IS NO TAME AUTOMORPHISM OF $\mathbb{C}^3$ WITH MULTIDEGREE $(3, 4, 5)$

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Abstract. Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be any polynomial mapping. The multidegree of F, denoted $\operatorname{mdeg} F$, is the sequence of positive integers $(\deg F_1, \dots, \deg F_n)$. In this paper we address the following problem: *for which sequence* $(d_1, \dots, d_n)$ *is there an automorphism or a tame automorphism* $F : \mathbb{C}^n \to \mathbb{C}^n$ *with* $\operatorname{mdeg} F = (d_1, \dots, d_n)$? We prove, among other things, that there is no tame automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$ with $\operatorname{mdeg} F = (3, 4, 5)$.

1. Introduction

Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be any polynomial automorphism. By the Jung and van der Kulk theorem [1, 2] we know that $\deg F_1 | \deg F_2$ or $\deg F_2 | \deg F_1$. On the other hand, if d_1, d_2 are positive integers such that $d_1 | d_2$, then $F = \Phi_2 \circ \Phi_1$, where

$$\begin{aligned} \Phi_1 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2, \\ \Phi_2 &: \mathbb{C}^2 \ni (u, w) \mapsto (u, w + u^{\frac{d_2}{d_1}}) \in \mathbb{C}^2 \end{aligned}$$

is an automorphism of $\mathbb{C}^2$ such that $\operatorname{mdeg} F = (d_1, d_2)$. Similarly if $d_2 | d_1$ we can give an appropriate automorphism of $\mathbb{C}^2$. Thus a sequence of positive integers (d_1, d_2) is the multidegree of some polynomial automorphism of $\mathbb{C}^2$ if and only if $d_1 | d_2$ or $d_2 | d_1$.

It seems natural to ask for which sequence $(d_1, \dots, d_n)$ is there a polynomial automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \dots, d_n)$? Also, the question about existence of a tame automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \dots, d_n)$ is natural. Recall that a *tame automorphism* is, by definition, a composition of linear automorphisms and triangular automorphisms, where a *triangular automorphism* is a mapping of the form

$$T : \mathbb{C}^n \ni \left\{ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} \mapsto \left\{ \begin{array}{l} x_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \dots, x_{n-1}) \end{array} \right\} \in \mathbb{C}^n.$$

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We will denote by $\mathrm{Tame}\,(\mathbb{C}^n)$ the group of all tame automorphisms of $\mathbb{C}^n$. This is, of course, a subgroup of the group $\mathrm{Aut}(\mathbb{C}^n)$ of all polynomial automorphisms of $\mathbb{C}^n$.

It is easy to see that if there is an automorphism (or tame automorphism) $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $\operatorname{mdeg} F = (d_1, \ldots, d_n)$, then there is also an automorphism (or tame automorphism) $\widetilde{F} : \mathbb{C}^n \to \mathbb{C}^n$ such that $\operatorname{mdeg} \widetilde{F} = (d_{\sigma(1)}, \ldots, d_{\sigma(n)})$ for any permutation σ of the set $\{1, \ldots, n\}$. Thus without loss of generality, we can assume that $d_1 \leq d_2 \leq \ldots \leq d_n$.

2. Some simple remarks

In this section we make some simple but useful remarks about existence of automorphisms and tame automorphisms with a given multidegree.

Proposition 2.1. *If for $1 \leq d_1 \leq \ldots \leq d_n$ there is a sequence of integers $1 \leq i_1 < \ldots < i_m \leq n$, with $m < n$, such that there exists an automorphism G of $\mathbb{C}^m$ with $\operatorname{mdeg} G = (d_{i_1}, \ldots, d_{i_m})$, then there exists an automorphism F of $\mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \ldots, d_n)$. Moreover, if G is tame, then an F can also be found that is tame.*

Proof. Let $j_1, \ldots, j_{n-m} \in \mathbb{N}$ be such that $1 \leq j_1 < \ldots < j_{n-m} \leq n$ and $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\}$. Of course, $\{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_{n-m}\} = \emptyset$. Consider the mapping $h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k \in \{i_1, \ldots, i_m\}, \\ x_k + (x_{i_1})^{d_k} & \text{for } k \in \{j_1, \ldots, j_{n-m}\}. \end{cases}$$

Of course h is an automorphism of $\mathbb{C}^n$ and $\deg h_k = d_k$ for $k \in \{j_1, \ldots, j_{n-m}\}$.

Consider also the mapping $g = (g_1, \ldots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$g_k(u_1, \ldots, u_n) = \begin{cases} G_l(u_{i_1}, \ldots, u_{i_m}) & \text{for } k = i_l, \\ u_k & \text{for } k \in \{j_1, \ldots, j_{n-m}\}. \end{cases}$$

It is easy to see that g is an automorphism of $\mathbb{C}^n$ and $\deg g_k = d_k$ for $k \in \{i_1, \ldots, i_m\}$.

Now taking $F = g \circ h$ we obtain an automorphism of $\mathbb{C}^n$ such that $\deg F_i = d_i$ for all $i \in \{1, \ldots, n\}$. $\square$

Proposition 2.2. *If for a sequence of integers $1 \leq d_1 \leq \ldots \leq d_n$ there is an $i \in \{1, \ldots, n\}$ such that*

$$d_i = \sum_{j=1}^{i-1} k_j d_j \qquad \textit{with } k_j \in \mathbb{N},$$

then there exists a tame automorphism F of $\mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \ldots, d_n)$.

Proof. Define $h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n$ and $g = (g_1, \ldots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$ by

$$h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k = i, \\ x_k + x_i^{d_k} & \text{for } k \neq i \end{cases}$$

and

$$g_k(u_1, \ldots, u_n) = \begin{cases} u_k + u_1^{k_1} \cdots u_{i-1}^{k_{i-1}} & \text{for } k = i, \\ u_k & \text{for } k \neq i. \end{cases}$$

It is easy to see that h and g are automorphisms of $\mathbb{C}^n$ such that $\deg h_k = d_k$ for $k \neq i$ and $\deg g_i\,(h_1,\ldots,h_n) = d_i$. Since also $g_k(u_1,\ldots,u_n) = u_k$ for $k \neq i$, it is easy to see that the automorphism $F = g \circ h$ has $\deg F_k = d_k$ for all $k \in \{1,\ldots,n\}$. □

Let us notice that the condition on $(d_1,\ldots,d_n)$ given in the above proposition is not necessary for the existence of a tame automorphism F with $\operatorname{mdeg} F = (d_1,\ldots,d_n)$. For example, Shestakov and Umirbaev gave in [3] an example of $F \in \operatorname{Tame}\left(\mathbb{C}^3\right)$ with $\operatorname{mdeg} F = (6,8,9)$. See also Proposition 5.2 below.

Corollary 2.3. *If for a sequence of integers $1 \leq d_1 \leq \ldots \leq d_n$ we have $d_1 \leq n-1$, then there exists a tame automorphism F of $\mathbb{C}^n$ with* $\operatorname{mdeg} F = (d_1,\ldots,d_n)$.

Proof. Let $r_i \in \{0,1,\ldots,d_1-1\}$ be such that $d_i \equiv r_i (\operatorname{mod} d_1)$ for $i = 2,\ldots,n$. If there is an $i \in \{2,\ldots,n\}$ such that $r_i = 0$, then $d_i = kd_1$ for some $k \in \mathbb{N}\backslash\{0\}$ and by Proposition 2.2, there exists an automorphism F of $\mathbb{C}^n$ with the desired properties.

Thus we can assume that $r_i \neq 0$ for all $i = 2,\ldots,n$. Since $d_1 - 1 < n-1$, there are $i,j \in \{2,\ldots,n\}, i \neq j$, such that $r_i = r_j$. Without loss of generality we can assume that $i < j$, so $d_j = d_i + kd_1$ for some $k \in \mathbb{N}$. Then by Proposition 2.2 there exists an automorphism F of $\mathbb{C}^n$ with the desired properties. □

3. Examples

In this section we give some positive results about existence of tame automorphisms of $\mathbb{C}^3$ with a given multidegree (d_1,d_2,d_3). The first one is the following.

Example 3.1. For every $d_2, d_3 \in \mathbb{N}$ with $2 \leq d_2 \leq d_3$, there is a tame automorphism F of $\mathbb{C}^3$ such that

$$\operatorname{mdeg} F = (2,d_2,d_3).$$

Proof. This is a consequence of Corollary 2.3. □

Example 3.2. For any $d_3 \geq 4$ such that $d_3 \neq 5$ there is a tame automorphism F of $\mathbb{C}^3$ such that

$$\operatorname{mdeg} F = (3,4,d_3).$$

Proof. We have

$$4 = 0 \cdot 3 + 1 \cdot 4$$

and

$$d_3 = \begin{cases} (2+k)\cdot 3 + 0 \cdot 4 & \text{for } d_3 = 6+3k, \\ (1+k)\cdot 3 + 1 \cdot 4 & \text{for } d_3 = 7+3k, \\ (0+k)\cdot 3 + 2 \cdot 4 & \text{for } d_3 = 8+3k. \end{cases}$$

Thus we can apply Proposition 2.2. □

Example 3.3. For any $d_3 \geq 5$ such that $d_3 \neq 7$ there is a tame automorphism F of $\mathbb{C}^3$ such that

$$\operatorname{mdeg} F = (3,5,d_3).$$

Proof. We have

$$5 = 0 \cdot 3 + 1 \cdot 5, \qquad 6 = 2 \cdot 3 + 0 \cdot 5$$

and

$$d_3 = \begin{cases} (1+k)\cdot 3 + 1 \cdot 5 & \text{for } d_3 = 8+3k, \\ (3+k)\cdot 3 + 0 \cdot 5 & \text{for } d_3 = 9+3k, \\ (0+k)\cdot 3 + 2 \cdot 5 & \text{for } d_3 = 10+3k. \end{cases}$$

Thus we can apply Proposition 2.2. □

Example 3.4. For any $d_3 \geq 5$ such that $d_3 \neq 6, 7, 11$, there is a tame automorphism F of $\mathbb{C}^3$ such that

$$\operatorname{mdeg} F = (4, 5, d_3).$$

Proof. We have

$$\begin{aligned} 5 &= 0 \cdot 4 + 1 \cdot 5, \quad 8 = 2 \cdot 4 + 0 \cdot 5, \\ 9 &= 1 \cdot 4 + 1 \cdot 5, \quad 10 = 0 \cdot 4 + 2 \cdot 5 \end{aligned}$$

and

$$d_3 = \begin{cases} (3+k) \cdot 4 + 0 \cdot 5 & \text{for } d_3 = 12 + 4k, \\ (2+k) \cdot 4 + 1 \cdot 5 & \text{for } d_3 = 13 + 4k, \\ (1+k) \cdot 4 + 2 \cdot 5 & \text{for } d_3 = 14 + 4k, \\ (0+k) \cdot 4 + 3 \cdot 5 & \text{for } d_3 = 15 + 4k. \end{cases}$$

Thus we can apply Proposition 2.2. □

The above examples justify the following question.

Question. Is there any automorphism (or tame automorphism) F of $\mathbb{C}^3$ such that

$$\operatorname{mdeg} F \in \{(3,4,5), (3,5,7), (4,5,6), (4,5,7), (4,5,11)\}?$$

4. Partial answer

In this section we give a partial answer to the question posed in the last section. Namely, we show the following.

Theorem 4.1. *There is no tame automorphism $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ such that*

$$\operatorname{mdeg} F = (3, 4, 5).$$

Before we give the proof of Theorem 4.1 we recall some results and notions from the papers of Shestakov and Umirbaev [3, 4].

Definition 4.2 ([3], Definition 1)**.** A pair $f, g \in k[X_1, \ldots, X_n]$ is called *-reduced if

(i) f, g are algebraically independent;

(ii) $\overline{f}, \overline{g}$ are algebraically dependent, where $\overline{h}$ denotes the highest homogeneous part of h;

(iii) $\overline{f} \notin k[\overline{g}]$ and $\overline{g} \notin k[\overline{f}]$.

Definition 4.3 ([3], Definition 1)**.** Let $f, g \in k[X_1, \ldots, X_n]$ be a *-reduced pair with $\deg f < \deg g$. Put $p = \frac{\deg f}{\gcd(\deg f, \deg g)}$. Then f, g is called a p-reduced pair.

Theorem 4.4 ([3], Theorem 2)**.** *Let $f, g \in k[X_1, \ldots, X_n]$ be a p-reduced pair, and let $G(x, y) \in k[x, y]$ with $\deg_y G(x, y) = pq + r$, $0 \leq r < p$. Then*

$$\deg G(f, g) \geq q\,(p \deg g - \deg g - \deg f + \deg[f, g]) + r \deg g.$$

In the above theorem, $[f, g]$ denotes the Poisson bracket of f and g; for us it is only important that

$$\deg[f, g] = 2 + \max_{1 \leq i < j \leq n} \deg \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)$$

if f, g are algebraically independent, and $\deg[f, g] = 0$ otherwise.

Notice also that the estimate from Theorem 4.4 is true even if the condition (ii) of Definition 4.2 is not satisfied. Indeed, if $G(x,y) = \sum_{i,j} a_{i,j} x^i y^j$, then by the algebraic independence of $\overline{f}$ and $\overline{g}$ we have

$$\begin{aligned} \deg G(f,g) &= \max_{i,j} \deg(a_{i,j} f^i g^j) \geq \deg_y G(x,y) \cdot \deg g \\ &= (qp+r)\deg g \geq q(p\deg g - \deg f - \deg g + \deg[f,g]) + r\deg g. \end{aligned}$$

The last inequality is a consequence of the fact that $\deg[f,g] \leq \deg f + \deg g$.

We will also use the following theorem.

Theorem 4.5 ([3], Theorem 3)**.** *Let $F = (F_1, F_2, F_3)$ be a tame automorphism of $\mathbb{C}^3$. If $\deg F_1 + \deg F_2 + \deg F_3 > 3$ (in other words, if F is not a linear automorphism), then F admits either an elementary reduction or a reduction of one of the types I-IV (see* [3, Definitions 2-4]*).*

Let us recall that an automorphism $F = (F_1, F_2, F_3)$ admits an *elementary reduction* if there exists a polynomial $g \in \mathbb{C}[x,y]$ and a permutation σ of $\{1,2,3\}$ such that $\deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)}$.

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Assume that $F = (F_1, F_2, F_3)$ is an automorphism of $\mathbb{C}^3$ such that $\operatorname{mdeg} F = (3,4,5)$. We will show that this hypothetical automorphism cannot be tame. First of all, notice that any pair F_i, F_j with $i,j \in \{1,2,3\}, i \neq j$, satisfies conditions (i) and (iii) of Definition 4.2. Indeed, this follows from the fact that F_1, F_2, F_3 are algebraically independent and that $3,4 \notin 5\mathbb{N}, 3,5 \notin 4\mathbb{N}$ and $4,5 \notin 3\mathbb{N}$. By Theorem 4.5 it is enough to show that F admits neither reductions of types I-IV nor an elementary reduction.

Assume that (F_1, F_2, F_3) admits a reduction of type I or II. Then by the definition (see [3, Definitions 2 and 3]), for some $n \in \mathbb{N}\backslash\{0\}$ and some permutation σ of $\{1,2,3\}$ we have $\deg F_{\sigma(1)} = 2n$ and $\deg F_{\sigma(2)} = ns$, where $s \geq 3$ is odd. But in the sequence $3,4,5$ there is only one even number, namely 4. Thus $2n = 4, n = 2$ and then ns is also even, a contradiction.

Now assume that (F_1, F_2, F_3) admits a reduction of type III or IV. Then by the definition (see [3, Definition 4]), for some $n \in \mathbb{N}\backslash\{0\}$ and some permutation σ of $\{1,2,3\}$ we have $\deg F_{\sigma(1)} = 2n$ and either

$$\deg F_{\sigma(2)} = 3n, \ n < \deg F_{\sigma(3)} \leq \tfrac{3}{2}n \tag{1}$$

or

$$\tfrac{5}{2}n < \deg F_{\sigma(2)} \leq 3n, \ \deg F_{\sigma(3)} = \tfrac{3}{2}n. \tag{2}$$

Of course, as before, we have $2n = 4, n = 2$. Since $3n = 6$, (1) is impossible, and since $\frac{5}{2}n = 5, 3n = 6$ and $\deg F_{\sigma(2)} \in \mathbb{N}$, (2) is also impossible. Thus we obtain a contradiction.

It remains to show that (F_1, F_2, F_3) does not admit an elementary reduction.

Assume that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where $g \in k[x,y]$, is an elementary reduction of (F_1, F_2, F_3). Thus, in particular, $\deg g(F_1, F_2) = 5$. But this is impossible. Indeed, by Theorem 4.4, we have

$$\deg g(F_1, F_2) \geq q(pm - m - n + \deg[F_1, F_2]) + mr, \tag{3}$$

where $n = \deg F_1$, $m = \deg F_2$, $p = n/\mathrm{GCD}(n,m)$ and $\deg_y g(x,y) = qp + r$ with $0 \le r < p$. In our case we have $n = 3$, $m = 4$, $p = 3$. Since F_1, F_2 are algebraically independent, $\deg[F_1, F_2] \ge 2$. Thus (3) can be rewritten as follows:

$$\deg g(F_1, F_2) \ge q(3 \cdot 4 - 4 - 3 + \deg[F_1, F_2]) + 4r.$$

Since also $3 \cdot 4 - 4 - 3 + \deg[F_1, F_2] = 5 + \deg[F_1, F_2] \ge 7 > 5$, q must be zero and r must be not greater than 1. This means that $g(F_1, F_2) = g_1(F_1) + g_2(F_1)F_2$ for some $g_1, g_2 \in k[x]$. Since $3\mathbb{N} \cap (4+3\mathbb{N}) = \emptyset$, we see that $\deg g(F_1, F_2) = \max\{3 \deg g_1, 4 + 3 \deg g_2\}$. Since $5 \notin 3\mathbb{N} \cup (4 + 3\mathbb{N})$, we obtain a contradiction.

Now, assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where $g \in k[x,y]$, is an elementary reduction of (F_1, F_2, F_3). In this case we have $\deg g(F_1, F_3) = 4$. But this is also impossible. Indeed, by Theorem 4.4, we have

$$\deg g(F_1, F_3) \ge q(pm - m - n + \deg[F_1, F_2]) + mr,$$

where $n = 3$, $m = 5$, $p = 3$ and $\deg_y g(x,y) = 3q + r$ with $0 \le r < 3$. Since $pm - m - n + \deg[F_1, F_2] = 7 + \deg[F_1, F_2] > 4$, we see that q must be zero. Also, r must be zero, because $m = 5 > 4$. Thus $g(F_1, F_3) = g(F_1)$, and then $\deg g(F_1, F_3) = 3 \deg g$. This contradicts $4 \notin 3\mathbb{N}$.

Finally, assume that

$$(F_1 - g(F_2, F_3), F_2, F_3),$$

where $g \in k[x,y]$, is an elementary reduction of (F_1, F_2, F_3). As before, we obtain

$$3 = \deg g(F_2, F_3) \ge q(4 \cdot 5 - 5 - 4 + \deg[F_2, F_3]) + 5r,$$

where $\deg_y g(x,y) = 4q + r$ with $0 \le r < 4$. Then q and r must be zero. Thus $g(F_2, F_3) = g(F_2)$, and then $\deg g(F_2, F_3) = 4 \deg g$. This contradicts $3 \notin 4\mathbb{N}$. $\square$

In a similar way we can show the following theorem.

Theorem 4.6. *There is no tame automorphism F of $\mathbb{C}^3$ such that*

$$\mathrm{mdeg}\, F \in \{(3,5,7), (4,5,7), (4,5,11)\}.$$

By the above theorem, Theorem 4.1, Corollary 2.3 and examples from Section 3, we have the following theorem.

Theorem 4.7. *In the following statements* mdeg *is considered as a map from the set of all endomorphisms of $\mathbb{C}^n$ into the set $\mathbb{N}^n$.*

(i) For all integers $d_3 \ge d_2 \ge 2$, $(2, d_2, d_n) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.

(ii) If $d_3 \ge 4$, then $(3, 4, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ if and only if $d_3 \ne 5$.

(iii) If $d_3 \ge 5$, then $(3, 5, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ if and only if $d_3 \ne 7$.

(iv) If $d_3 \ge 5$ and $d_3 \ne 6, 7, 11$, then $(4, 5, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.

Notice that (iv) cannot yet be formulated with "if and only if", as (iii) can, because we do not know whether $(4,5,6) \in \mathrm{mdeg}(\mathrm{Tame}\,(\mathbb{C}^3))$ or not. The author believes that $(4,5,6) \notin \mathrm{mdeg}(\mathrm{Tame}\,(\mathbb{C}^3))$, which however seems to be hard to prove.

5. General conjecture

The author believes that for any prime number $p \geq 3$ and $d_3 \geq d_2 \geq p$ the following is true.

Conjecture 5.1. $(p, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $p|d_2$ *or* $d_3 \in p\mathbb{N} + d_2\mathbb{N}$.

When d_1 is a composite number one cannot expect a similar result, as the following proposition shows.

Proposition 5.2. *For any number* $d_3 \geq 6$ *we have* $(4, 6, d_3) \in \operatorname{mdeg}\left(\operatorname{Tame}\left(\mathbb{C}^3\right)\right)$.

Proof. If d_3 is even, then one can easily see that $d_3 \in 4\mathbb{N}+6\mathbb{N}$. Thus we can assume that d_3 is odd and use the following constructions.

Since $\left(x+z^4\right)^3 = z^{12}+3xz^8+3x^2z^4+x^3$ and $\left(y+z^6\right)^2 = z^{12}+2yz^6+y^2$, we see that $\deg\left[\left(y+z^6\right)^2 - \left(x+z^4\right)^3\right] = 9$ and so $\deg\left[\left(y+z^6\right)^2 - \left(x+z^4\right)^3\right]\left(x+z^4\right)^k = 9+4k$ for any $k \in \mathbb{N}$. This means that

$$\operatorname{mdeg}(F_2 \circ F_1) = (4, 6, 9+4k),$$

where

$$\begin{aligned} F_1(x,y,z) &= \left(x+z^4, y+z^6, z\right), \\ F_2(u,v,w) &= \left(u, v, w+\left(v^2-u^3\right)u^k\right). \end{aligned}$$

Since $\left(x+z^4\right)^3 = z^{12}+3xz^8+3x^2z^4+x^3$ and $\left(y+\frac{3}{2}xz^2+z^6\right)^2 = z^{12}+3xz^8+2yz^6+\frac{9}{4}x^2z^4+3yxz^2+y^2$, it follows that $\deg\left[\left(y+\frac{3}{2}xz^2+z^6\right)^2 - \left(x+z^4\right)^3\right] = 7$ and $\deg\left[\left(y+\frac{3}{2}xz^2+z^6\right)^2 - \left(x+z^4\right)^3\right]\left(x+z^4\right)^k = 7+4k$ for any $k \in \mathbb{N}$. Thus we have

$$\operatorname{mdeg}(F_2 \circ F_1) = (4, 6, 7+4k),$$

where

$$\begin{aligned} F_1(x,y,z) &= \left(x+z^4, y+\tfrac{3}{2}xz^2+z^6, z\right), \\ F_2(u,v,w) &= \left(u, v, w+\left(v^2-u^3\right)u^k\right). \end{aligned}$$

□

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