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UNIQUENESS OF TRAVELING WAVES FOR NONLOCAL LATTICE EQUATIONS

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Abstract. We establish the uniqueness (up to translation) of traveling waves for a nonlocal lattice equation with time delay. Our approach is based on exact a priori asymptotics of the wave profiles. This we accomplish by developing a structure theorem of entire solutions to a class of linear integro-differential equations.

1. Introduction

Chen and Guo [2] obtained the existence of traveling waves of the following local lattice system

$$\frac{dw_j}{dt} = g(w_{j+1}) + g(w_{j-1}) - 2g(w_j) + f(w_j),\ j \in \mathbb{Z}, \tag{1.1}$$

by the method of upper-lower solutions coupled with the technique presented in Zinner, Harris and Hudson [11]. They used solutions of an initial boundary value problem to approximate traveling waves for (1.1) and established the uniqueness (up to translation) by showing monotonicity of wave profiles and analyzing the asymptotic behavior of wave profiles. Ma and Zou [6] then extended this idea to a local and time-delayed lattice system

$$\frac{dw_j(t)}{dt} = D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + b(w_j(t-r)), \quad j \in \mathbb{Z}, \tag{1.2}$$

where $b(w)$ is monotone. Weng, Huang and Wu [9] derived the following nonlocal time-delayed lattice system to describe the mature population of a single species in a patch environment,

$$\frac{dw_j(t)}{dt} = D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + \sum_{k\in\mathbb{Z}} \beta(j-k)b(w_k(t-r)),\ j \in \mathbb{Z}, \tag{1.3}$$

where $\beta(k) \geq 0$ with $\sum_{k\in\mathbb{Z}} \beta(k) = 1$, $b(0) = 0$, $b(w) - dw = 0$ has a positive solution w^* such that $b(w) - dw > 0, \forall w \in (0, w^*)$. They also obtained the existence of traveling waves in the case where $b(u)$ is monotone in $u \in [0, w^*]$. The spreading speed and its coincidence with the minimal wave speed of traveling waves were

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established in [4] in the case where b is non-monotone. It seems very difficult to extend Chen and Guo's idea for the uniqueness of traveling waves to this nonlocal case.

The purpose of the current paper is to address the uniqueness of traveling wave solutions to nonlocal lattice equations. Our developed method also applies to more general nonlocal lattice systems such as

(1.4)
$$\frac{dw_j(t)}{dt}=\sum_{k\in\mathbb{Z}} D_k[w_{j+k}(t)-w_j(t)]-g(w_j(t))+\sum_{k\in\mathbb{Z}}\beta(j-k)\int_{-r}^{0} b(w_k(t+\theta))d\eta(\theta),\ j\in\mathbb{Z}.$$

Here we choose to consider (1.3) just for the sake of simplicity.

Throughout this paper, a traveling wave with speed c always refers to a pair (u,c), where $u\in C(\mathbb{R},\mathbb{R}_+)$ is a non-trivial and bounded solution of (1.3) having the form $w_j(t)=u(j+ct)$. We call u the wave profile. Substituting $w_j(t)=u(j+ct)$ into (1.3), we get the wave profile equation

$$cu'(x)=D[u(x+1)+u(x-1)-2u(x)]-du(x)+\sum_{k\in\mathbb{Z}}\beta(k)b(u(x-k-cr)). \tag{1.5}$$

Diekman and Kapper [3] studied the asymptotic behavior and the uniqueness of solutions for the integral equation $u(x)=\int_{\mathbb{R}} f(u(y))k(x-y)dy$. They employed the powerful Tauberian theorem when solutions are monotone (see also [1]). Further, they obtained a beautiful estimate of asymptotic behavior of non-monotone solutions with the help of the solution structure of the linear equation $u(x)=\int_{\mathbb{R}} f'(0)u(y)k(x-y)dy$. These earlier works ([2, 3, 1]) suggest that the asymptotic behavior of wave profiles when $x\to-\infty$ plays an important role in the study of uniqueness of traveling waves.

In order to obtain the exact asymptotic behavior of wave profiles, we explore a structure theorem of entire solutions to linear integro-differential equations in Section 2. This theorem is also of interest on its own. Because it is difficult (sometimes impossible; see, e.g., [5]) to prove the monotonicity of wave profiles for non-monotone systems, we directly study the asymptotic behavior of wave profiles without assuming their monotonicity by three steps. First we prove that every wave profile decays exponentially. Next we decompose each wave profile into two parts: one part decays rapidly, and the other is a solution of the linearized equation of the wave profile equation (1.5) at zero and dominates the asymptotic behavior. By applying the solution structure theorem, we get the expression of the dominating part. Finally, by the distribution of eigenvalues of the corresponding characteristic equation we obtain the exact asymptotic behavior. Consequently, we obtain the uniqueness (up to translation) of traveling waves.

2. A linear integro-differential equation

In this section, we use complex analysis to study the structure of entire solutions to the following linear differential-integral equation,

$$\sum_{k=0}^{m}\int_{\mathbb{R}} f^{(k)}(x+y)d\mu_k(y)=0,\quad x\in\mathbb{R}, \tag{2.1}$$

where $m\geq 1$ is an integer, $f^{(k)}(z)$ is the kth order derivative of $f(z)$ and each μ_k is a σ-finite real-value measure on $\mathbb{R}$. Note that the linearized wave profile equations at zero of the lattice equation (1.3), of the integral equation $u(t,z) = \int_0^\infty \int_{\mathbb{R}} G(u(t-s,z-y),s,y)dyds$ and of the nonlocal reaction-diffusion equation $\frac{\partial u}{\partial t}(t,z) = D\frac{\partial^2 u}{\partial z^2}(t,z) - g(u(t,z)) + \int_0^\infty \int_{\mathbb{R}} G(u(t-s,z-y))J(s,y)dyds$ are all in form (2.1). Further, (2.1) includes the delay differential equation $\dot{u}(t) = \int_{-r}^0 u(t+\theta)d\eta(\theta)$. Of interest, we consider entire solutions of (2.1) with the property that there exists $\gamma > 0$ such that $f^{(k)}(x) = O(e^{\gamma|x|}), \forall 0 \le k \le m-1$.

Define

$$h_k(\lambda) := \int_{\mathbb{R}} e^{\lambda y} d\mu_k(y) \quad \text{and} \quad \mathcal{K}(\lambda) := \sum_{k=0}^{m} \lambda^k h_k(\lambda).$$

We call $\mathcal{K}(\lambda) = 0$ the characteristic equation and its solution the eigenvalue.

We make the following assumptions:

(A1) For each k, there exists $\rho_k > 0$ such that $|h_k(\lambda)| < \infty$ for $|\operatorname{Re}\lambda| < \rho_k$.

(A2) There exist m different eigenvalues in the strip $|\operatorname{Re}\lambda| < \gamma$ with $\gamma < \rho$, where $\rho := \min\{\rho_k : 0 \le k \le m\}$.

Assumption (A1) ensures that $\mathcal{K}(\lambda)$ is analytic for $|\operatorname{Re}\lambda| < \rho$. Assumption (A2) makes it possible to transfer a solution $f(x)$ to another solution $\hat{f}(x)$ such that $\hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(m-1)}(0) = 0$, which will be shown in Lemma 2.1.

Theorem 2.1. *Assume (A1) and (A2) hold. Let $f(x)$ be a solution of (2.1) and $f^{(k)}(x) = O(e^{\gamma|x|}), \forall 0 \le k \le m-1$. Then f has the following expression*

$$f(x) = \sum_l \sum_{p=1}^{k_l} M_{l,p} x^{p-1} e^{-iw_l x},$$

where $-iw_l$ runs through all zeros of $\mathcal{K}(\lambda)$ in the strip $|\operatorname{Re}\lambda| \le \gamma$, $M_{l,p}$ are constants and k_l is the order of the multiplicity of the eigenvalue $-iw_l$.

To prove this theorem, we need a series of lemmas.

In what follows, we always assume that $w = u + iv \in \mathbb{C}$ with $u, v \in \mathbb{R}$. Define

$$F_+(w) := \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)e^{iwx}dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)e^{-vx}e^{iux}dx, \tag{2.2}$$

$$F_-(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(x)e^{iwx}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(x)e^{-vx}e^{iux}dx. \tag{2.3}$$

Note that $F_+(w)$ is well-defined for $v > \gamma$ and $F_-(w)$ is well-defined for $v < -\gamma$. The formula reciprocal to (2.2) is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F_+(u+iv)e^{-i(u+iv)x}du = \begin{cases} f(x), & x > 0 \\ 0, & x < 0. \end{cases}$$

There is a similar formula involving F_- (see, e.g., [8]). Adding, we may write

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} F_+(w)e^{-iwx}dw + \frac{1}{\sqrt{2\pi}} \int_{ib-\infty}^{ib+\infty} F_-(w)e^{-iwx}dw \tag{2.4}$$

for any $a > \gamma$ and $b < -\gamma$. In particular, we choose $a \in (\gamma, \rho)$ and $b \in (-\rho, -\gamma)$.

Lemma 2.1. *Assume (A2) holds. If $f(x)$ is solution with $f(x) = O(e^{\gamma|x|})$ as $|x| \to \infty$, then there exists another solution $\hat{f}(x)$ such that $\hat{f}(x) = O(e^{\gamma|x|})$ as $|x| \to \infty$ and $\hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(m-1)}(0) = 0$.*

Proof. Let $\lambda_1, \cdots, \lambda_m$ be m different eigenvalues in the strip $|\operatorname{Re}\lambda| < \gamma$. If $f(x)$ is a solution of (2.1), then so is $\hat{f}(x) := f(x) - \sum_{l=1}^m d_l e^{\lambda_l x}$ for any $d_l \in \mathbb{C}$. We claim that there exists d_l such that $\hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(m-1)}(0) = 0$. Indeed, note that $\hat{f}^{(k)}(0) = f^{(k)}(0) - \sum_{l=1}^m \lambda_l^k d_l$. It then suffices to prove that the linear system $Bx = f_0$ has a solution, where $f^0 = (f(0), f'(0), \cdots, f^{(m-1)}(0))^T$ and $B = (b_{ij})_{m\times m}$ with $b_{ij} = \lambda_j^{i-1}$. Obviously, $\det B$ is a Vandermonde determinant. This, together with the fact that λ_l are different, implies the existence of d_l. Clearly, $\hat{f}(x) = O(e^{\gamma|x|})$ as $|x| \to \infty$. ◻

By the above lemma, we may assume, without loss of generality, that $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$. Note that $f^{(k)}(x) = O(e^{\gamma|x|})$ when $|x| \to +\infty$. It then follows from the fact that $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$ that for any $0 \le k \le m-1$, $\int_0^\infty f^{(k)}(x)e^{iwx}dx = (-iw)^k \int_0^\infty f(x)e^{iwx}dx = (-iw)^k F_+(w)$ and similarly $\int_{-\infty}^0 f^{(k)}(x)e^{iwx}dx = (-iw)^k F_-(w)$. And hence,

(2.5)
$$f^{(k)}(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}(-iw)^k F_+(w)e^{-iwx}dw + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}(-iw)^k F_-(w)e^{-iwx}dw.$$

Besides, for any $\xi \in \mathbb{R}$,

(2.6)
$$f(x+\xi) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty} e^{-iw\xi}F_+(w)e^{-iwx}dw + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty} e^{-iw\xi}F_-(w)e^{-iwx}dw.$$

Lemma 2.2. *For each k and x, there holds*

(2.7)
$$\int_{\mathbb{R}} d\mu_k(y)\int_{ia-\infty}^{ia+\infty} w^k F_+(w)e^{-iw(x+y)}dw = \int_{ia-\infty}^{ia+\infty} dw \int_{\mathbb{R}} w^k F_+(w)e^{-iw(x+y)}d\mu_k(y)$$

and

(2.8)
$$\int_{\mathbb{R}} d\mu_k(y)\int_{ib-\infty}^{ib+\infty} w^k F_-(w)e^{-iw(x+y)}dw = \int_{ib-\infty}^{ib+\infty} dw \int_{\mathbb{R}} w^k F_-(w)e^{-iw(x+y)}d\mu_k(y).$$

Proof. We only prove (2.7) since the proof of (2.8) is similar. By Fubini's theorem, it suffices to prove

$$\hat{F_+}(k,x;y) := \int_{\mathbb{R}} \left|(u+ia)^k F_+(u+ia)e^{-i(u+ia)(x+y)}\right| du < \infty \tag{2.9}$$

and

$$\int_{\mathbb{R}} \hat{F_+}(k,x;y)d\mu_k(y) < \infty. \tag{2.10}$$

To this end, we first prove the following claim.

Claim. For any $n \in \mathbb{N}$, $\left|\int_0^\infty f(x)e^{-ax}e^{iux}dx\right| = o(|u|^{-n})$, as $|u| \to \infty$.

Because $x^n f(x)e^{-ax}, \forall n \in \mathbb{N}$, is in $L^1([0,\infty))$, we have $\lim_{|u|\to\infty}[\int_0^\infty x^n f(x)e^{-ax} \times e^{iux}dx] = 0$ (see, e.g., [8, Theorem 1]). It then follows by l'Hospital's rule that

$$\lim_{|u|\to\infty}\frac{\int_0^\infty f(x)e^{-ax}e^{iux}dx}{u^n} = \lim_{|u|\to\infty}\frac{\int_0^\infty (ix)^n f(x)e^{-ax}e^{iux}dx}{n!} = 0.$$

By this claim, we then see that

$$(2.11)\quad \int_{\mathbb{R}} |(u+ia)^k F_+(u+ia)|du = \int_{\mathbb{R}} |u+ia|^k \left| \int_0^\infty f(x)e^{-ax}e^{-iux}dx \right| du < \infty,$$

and consequently, (2.9) holds. Note that

$$\int_{\mathbb{R}} \hat{F_+}(k,x;y)d\mu_k(y) \le e^{ax} \int_{\mathbb{R}} |(u+ia)^k F_+(u+ia)|du \int_{\mathbb{R}} e^{ay}d\mu_k(y) < \infty.$$

Again, (2.11), together with $a \in (\gamma, \rho)$, implies (2.10). $\square$

Lemma 2.3 ([8, Theorem 141])**.** *Let $\phi(w)$ be regular in the strip $a_1 \le v \le a_2$, and let $\phi(u+iv)$ be $L^1(\mathbb{R})$ (or $L^2(\mathbb{R})$) and tend to 0 as $u \to \pm\infty$, for v in the above interval. Let $\psi(w)$ have the similar properties in $b_1 \le v \le b_2$, where $b_2 < a_1$. Let*

$$\int_{ia-\infty}^{ia+\infty} \phi(w)e^{-iwx}dw + \int_{ib-\infty}^{ib+\infty} \psi(w)e^{-iwx}dw = 0$$

for all x, where $a_1 < a < a_2, b_1 < b < b_2$. Then ϕ and ψ are regular for $b_1 < v < a_2$, their sum is 0 in this strip, and they tend to 0, as $u \to \pm\infty$, uniformly in any interior strip.

Now we are ready to prove Theorem 2.1.

Proof. We use similar arguments as in the proof of [8, Theorem 146]. Substituting the expression of f in (2.4) into (2.1) and using identities (2.5)-(2.8) we obtain

$$(2.12)\qquad 0 = \int_{ia-\infty}^{ia+\infty} \mathcal{K}(-iw)F_+(w)e^{-iwx}dw + \int_{ib-\infty}^{ib+\infty} \mathcal{K}(-iw)F_-(w)e^{-iwx}dw.$$

By the definition of ρ, we see that $\mathcal{K}(-iw)F_+(w)$ is regular for $v \in (\gamma, \rho)$. Note that $|\mathcal{K}(-iw)| = O(|u|^m)$ as $|u| \to \infty$. It then follows from the claim in the proof of Lemma 2.2 that $F_+(w) = o(|u|^{-n}), \forall n > 0$ as $|u| \to \infty$, and from inequality (2.11) that $\mathcal{K}(-iw)F_+(w)$ is in $L^1(\mathbb{R})$ and tends to 0 as $|u| \to \infty$ for $v \in (\gamma, \rho)$. So does $\mathcal{K}(-iw)F_-(w)$ for $v \in (-\rho, -\gamma)$. Then, by Lemma 2.3, we obtain

$$(2.13)\qquad \mathcal{K}(-iw)F_+(w) = \chi(w), \quad \mathcal{K}(-iw)F_-(w) = -\chi(w),$$

where $\chi(w)$ is regular for $v \in (-\rho, \rho)$ and $\chi(w) \to 0$ as $|u| \to \infty$ uniformly in any interior strip. Hence, $F_+(w)$ and $F_-(w)$ are regular in this strip except possibly for poles at the zeros of $\mathcal{K}(-iw)$. Combining (2.4) and (2.13), we can write

$$(2.14)\quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} \frac{\chi(w)}{\mathcal{K}(-iw)} e^{-iwx}dw - \frac{1}{\sqrt{2\pi}} \int_{ib-\infty}^{ib+\infty} \frac{\chi(w)}{\mathcal{K}(-iw)} e^{-iwx}dw,$$

which is the sum of the residues at poles in the strip $b < \operatorname{Re}\lambda < a$. Since $a \in (\gamma, \rho)$ and $b \in (-\rho, -\gamma)$ are arbitrary, we know that the right hand side of equation (2.14) is also the sum of the residues at poles in the strip $|\operatorname{Re}\lambda| \le \gamma$. From the fact that $\frac{\chi(w)}{\mathcal{K}(-iw)} = F_+(w) \to 0$ as $|u| \to \infty$ and the calculation of the right hand side of (2.14) by the residue theorem (see, e.g., [7, Theorem 13.13]), we see that $f(x)$ is of the form given in the theorem. $\square$

3. Uniqueness of traveling waves

In this section, we first analyze the characteristic equation, then reduce the uniqueness problem to the study of asymptotic behavior under a Lipschitz condition, and finally obtain the exact asymptotic behavior with the help of Theorem 2.1.

Assume that the function $b(u)$ is differentiable at $u = 0$. Define

$$\Delta(c, \lambda) := c\lambda - D(e^{-\lambda} + e^{\lambda} - 2) + d - b'(0) \sum_{k\in\mathbb{Z}} \beta(k) e^{-\lambda(k+cr)},$$

where c is regarded as a parameter. We call $\Delta(c, \lambda) = 0$ the characteristic equation and its solutions eigenvalues. We impose the following assumptions on $\beta(k)$.

(K) $\beta(k) = \beta(-k) \geq 0, \forall k \in \mathbb{Z}, \sum_{k\in\mathbb{Z}} \beta(k) = 1$, and there exists $\lambda^\sharp > 0$ such that $\sum_{k\in\mathbb{Z}} \beta(k)e^{-\lambda k}$ is convergent when $\lambda \in [0, \lambda^\sharp)$ and $\lim_{\lambda\uparrow\lambda^\sharp} \sum_{k\in\mathbb{Z}}[\beta(k) \times e^{-\lambda k}] = +\infty$.

The following lemma on the characteristic equation is quite useful.

Lemma 3.1. *Assume that $b'(0) > d$ and the assumption (K) holds. Then $\Delta(c, \lambda)$ has the following properties:*

(i) *The system $\Delta(c, \lambda) = 0$, $\frac{\partial \Delta}{\partial \lambda}(c, \lambda) = 0$ admits a unique positive solution (c^*, λ^*).*

(ii) *For each $c > c^*$, there are exactly two positive eigenvalues $\lambda_i = \lambda_i(c), i = 1, 2$ with $\lambda_1 < \lambda_2 < \lambda^\sharp$ and $\Delta(c, \lambda) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$.*

(iii) *For each $c \neq 0$, there are only finitely many eigenvalues in any vertical strip $|\operatorname{Re} \lambda| \leq \lambda^\diamond < \lambda^\sharp$.*

(iv) *For each $c \neq 0$, there exists $\delta > 0$ such that there are no eigenvalues in the strip $\lambda_1 - \delta \leq \operatorname{Re} \lambda \leq \lambda_1 + \delta$ other than λ_1.*

Proof. By direct computations, statements (i)-(ii) can be easily observed. To prove statement (iii), we first show that all eigenvalues in any vertical strip have a uniform bound. Assume, for the sake of contradiction, that $\{\lambda_j\}_{j=1}^\infty$ are eigenvalues with $|\lambda_j| \to \infty$ as $j \to \infty$. Substituting $\lambda_j := u_j + iv_j$ into $\Delta(c, \lambda_j) = 0$ and separating the real and imaginary parts, we obtain $cv_j - D[e^{u_j} \sin v_j - e^{-u_j} \sin v_j] + b'(0) \sum_{k\in\mathbb{Z}} \beta(k)e^{-u_j(k+cr)} \sin v_j(k + cr) = 0$. If $\{u_j\}_{j=1}^\infty$ is bounded, the left hand side of the above equality goes to infinity as $|v_j| \to \infty$ because $c \neq 0$. This is a contradiction. Since $\Delta(c, \lambda)$ is analytic in the strip $|\operatorname{Re} \lambda| \leq \lambda^\diamond$, the eigenvalues in this strip are isolated. So there are only finitely many eigenvalues in this strip.

Now we show statement (iv). By (iii), we see that there exists $\delta > 0$ such that there are no eigenvalues in the strip $\lambda_1 - \delta < \operatorname{Re} \lambda < \lambda_1 + \delta$ other than those with $\operatorname{Re} \lambda = \lambda_1$. Assume that $\lambda = \lambda_1 + iv$ is an eigenvalue. Then separating the real and imaginary parts of $\Delta(c, \lambda) = 0$ yields

(3.1)
$$\begin{cases} c\lambda_1 - D[e^{\lambda_1} \cos v + e^{-\lambda_1} \cos v - 2] + d - b'(0) \sum_{k\in\mathbb{Z}} \beta(k)e^{-\lambda_1(k+cr)} \cos v(k + cr) = 0, \\ cv - D[e^{\lambda_1} \sin v - e^{-\lambda_1} \sin v] + b'(0) \sum_{k\in\mathbb{Z}} \beta(k)e^{-\lambda_1(k+cr)} \sin v(k + cr) = 0. \end{cases}$$

Since $\Delta(c, \lambda_1) = 0$, we know from the first equation of (3.1) that

$$D(e^{\lambda_1} + e^{-\lambda_1})(1 - \cos v) + b'(0) \sum_{k\in\mathbb{Z}} \beta(k)e^{-\lambda_1(k+cr)}(1 - \cos v(k + cr)) = 0,$$

which implies $\cos v = 1$ and $\cos v(k+cr) = 1$ for those k such that $\beta(k) \neq 0$. Hence, $\sin v = 0$ and $\sin v(k+cr) = 0$ for those k such that $\beta(k) \neq 0$. Finally, from the second equation of (3.1), we have $v = 0$ because $c \neq 0$. $\square$

In order to reduce the uniqueness problem to the study of asymptotic behavior, we impose the Lipschitz condition (L) on the function b.

(L) For any $u, w \geq 0$, $|b(u) - b(w)| \leq b'(0)|u - w|$.

Since there is no traveling wave connecting zero with speed $c < c^*$ (see [4, Theorem 3.2(i)]), we always assume that wave speed c is greater than c^*.

Lemma 3.2. *Let (L) hold. Assume that (u_1, c) and (u_2, c) are two traveling waves of* (1.3). *If*

$$\lim_{x\to-\infty} u_1(x)e^{-\lambda_1 x} = \theta_1 \quad \textit{and} \quad \lim_{x\to-\infty} u_2(x)e^{-\lambda_1 x} = \theta_2$$

for some positive numbers θ_1 and θ_2, then u_1 is a translation of u_2; more precisely, $u_1(x) = u_2(x + \bar{x})$, where $\bar{x} = \frac{1}{\lambda_1} \ln \frac{\theta_1}{\theta_2}$.

Proof. Let $v(x) = [u_1(x) - u_2(x+\bar{x})]e^{-\lambda_1 x}$. Then $v(\pm\infty) = 0$, and hence, $\max_{x\in\mathbb{R}} v(x)$ and $\min_{x\in\mathbb{R}} v(x)$ exist. Without loss of generality, we assume $\max_{x\in\mathbb{R}} v(x) \geq |\min_{x\in\mathbb{R}} v(x)|$ (otherwise, we may consider $v(x) = [u_2(x+\bar{x}) - u_1(x)]e^{-\lambda_1 x}$). So there exists $x_0 \in \mathbb{R}$ such that $v(x_0) = \max_{x\in\mathbb{R}} v(x) \geq 0$ and $v'(x_0) = 0$. We claim that $v(x_0 \pm 1) = v(x_0)$. Assume, for the sake of contradiction, that either $v(x_0 + 1) < v(x_0)$ or $v(x_0 - 1) < v(x_0)$. It then follows from (1.5) and (L) that

$$\begin{aligned}
0 = cv'(x_0) \quad &= \quad -c\lambda_1 v(x_0) + D[v(x_0+1)e^{\lambda_1} + v(x_0-1)e^{-\lambda_1} - 2v(x_0)] - dv(x_0) \\
&\quad\quad + \sum_{k\in\mathbb{Z}} \beta(k)[b(u_1(x_0 - k - cr)) - b(u_2(x_0 + \bar{x} - k - cr))]e^{-\lambda_1 x_0} \\
&\leq \quad -c\lambda_1 v(x_0) + D[v(x_0+1)e^{\lambda_1} + v(x_0-1)e^{-\lambda_1} - 2v(x_0)] - dv(x_0) \\
&\quad\quad + \sum_{k\in\mathbb{Z}} \beta(k)b'(0)|v(x_0 - k - cr)|e^{-\lambda_1(k+cr)} \\
&< \quad v(x_0)\left(-c\lambda_1 + D[e^{\lambda_1} + e^{-\lambda_1} - 2] - d + b'(0)\sum_{k\in\mathbb{Z}} \beta(k)e^{-\lambda_1(k+cr)}\right) \\
&= \quad -v(x_0)\Delta(c, \lambda_1) = 0,
\end{aligned}$$

which is a contradiction. Repeating the above argument, we further obtain $v(x_0 \pm n) = v(x_0), \forall n \in \mathbb{Z}$. This implies $v(x_0) = 0$ since $v(\pm\infty) = 0$. $\square$

In order to obtain the exact asymptotic decay rate of wave profiles as $x \to -\infty$, we impose the following condition on b:

(H) The function b is continuous from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $b'(0) > d$, and there exist $a > 0, \delta > 0$ and $\sigma > 1$ such that $b(w) \geq b'(0)w - aw^\sigma, \forall w \in [0, \delta]$.

It is easy to see that the assumption (H) implies the following two statements:

(S1) For each $\bar{w} > 0$, there exist $\bar{a} > 0$ and $\sigma > 1$ such that $b(w) \geq b'(0)w - \bar{a}w^\sigma, \forall w \in [0, \bar{w}]$, where $\bar{a} := \max\{a, \delta^{-\sigma} \max_{w\in[\delta,\bar{w}]} b'(0)w - b(w)\}$.

(S2) For any $\epsilon > 0$, there exists $\bar{\delta} > 0$ such that $b(w) \geq (1-\epsilon)b'(0)w, \forall w \in [0, \bar{\delta}]$.

In what follows, we concentrate on the asymptotic behavior of wave profiles and always assume that (K) and (H) hold. First, we show that wave profiles decay exponentially as $x \to -\infty$.

Lemma 3.3. *Suppose (u,c) is a traveling wave of (1.3) with $u(-\infty)=0$. Then there exists $\gamma=\gamma(c)>0$ such that $u(x)=O(e^{-\gamma x})$ as $x\to-\infty$.*

Proof. Since $b'(0)>d$ and $\sum_{k\in\mathbb{Z}}\beta(k)=1$, we may choose $\epsilon_0>0$ and $N>0$ such that $A:=(1-\epsilon_0)b'(0)\sum_{|k|\le N}\beta(k)-d>0$. By (S2), we see that for such ϵ_0, there exists $\delta_0>0$ such that $b(u)\ge(1-\epsilon_0)b'(0)u, \forall u\in[0,\delta_0]$. Since $u(-\infty)=0$, we may find $M>0$ such that $u(x)\le\delta_0, \forall x\le -M$. Define $q(u)(x):=u(x+1)+u(x-1)-2u(x)$. Then, integrating equation (1.5) from y to x with $x\le -M-N+cr$ gives

$$\begin{aligned}
&c[u(x)-u(y)]\\
&= D\int_y^x q(u)(\xi)d\xi - d\int_y^x u(\xi)d\xi + \sum_{k\in\mathbb{Z}}\beta(k)\int_y^x b(u(\xi-k-cr))d\xi\\
&\ge D\int_y^x q(u)(\xi)d\xi - d\int_y^x u(\xi)d\xi + b'(0)(1-\epsilon_0)\sum_{|k|\le N}\beta(k)\int_y^x u(\xi-k-cr)d\xi\\
&= D\int_y^x q(u)(\xi)d\xi + b'(0)(1-\epsilon_0)\sum_{|k|\le N}\beta(k)\int_y^x [u(\xi-k-cr)-u(\xi)]d\xi\\
&\quad +A\int_y^x u(\xi)d\xi. \qquad (3.2)
\end{aligned}$$

Since $\int_y^x q(u)(\xi)d\xi=\int_x^{x+1}u(\xi)d\xi-\int_{x-1}^x u(\xi)d\xi-\int_y^{y+1}u(\xi)d\xi+\int_{y-1}^y u(\xi)d\xi$ and $\int_y^x[u(\xi-k-cr)-u(\xi)]d\xi=-(k+cr)\int_0^1[u(x-t(k+cr))-u(y-t(k+cr))]dt$, letting $y\to-\infty$ in (3.2), we have

$$\begin{aligned}
A\int_{-\infty}^x u(\xi)d\xi \;&\le\; cu(x)-D\int_x^{x+1}u(\xi)d\xi+D\int_{x-1}^x u(\xi)d\xi\\
&\quad +b'(0)(1-\epsilon_0)\sum_{|k|\le N}|k+cr|\beta(k)\int_0^1 u(x-t(k+cr))dt. \qquad (3.3)
\end{aligned}$$

Since $\sum_{k\in\mathbb{Z}}\beta(k)e^{\lambda k}$ is convergent for $\lambda\in(-\lambda^\sharp,\lambda^\sharp)$, we know that $\Delta(c,\lambda)$ is infinitely often differentiable for λ in some interval $[0,\delta]$ with $\delta>0$. Thus, it is easy to verify that $\frac{d^2}{d\lambda^2}\Delta(c,\lambda)\Big|_{\lambda=0}=-2D+b'(0)\sum_{k\in\mathbb{Z}}(k+cr)^2\beta(k)<+\infty$. Note that $|k+cr|\le(k+cr)^2$ when $|k|$ is sufficiently large. It then follows that $\int_{-\infty}^x u(\xi)d\xi<+\infty$. Letting $v(x):=\int_{-\infty}^x u(\xi)d\xi$ and integrating (3.3) from $-\infty$ to x, we obtain

$$\begin{aligned}
A\int_{-\infty}^x v(\xi)d\xi \;&\le\; cv(x)-D\int_x^{x+1}v(\xi)d\xi+D\int_{x-1}^x v(\xi)d\xi\\
&\quad +b'(0)(1-\epsilon_0)\sum_{|k|\le N}|k+cr|\beta(k)\int_0^1 v(x-t(k+cr)dt\\
&\le\; Kv(x+N_1) \qquad (3.4)
\end{aligned}$$

for some $K>0$ and $N_1>0$ since $v(x)$ is increasing. Choose $r_0>0$ such that $\mu:=K/r_0<1$. Then for $x\le M-N+cr$, we have

$$v(x-r_0)\le\frac{1}{r_0}\int_{x-r_0}^x v(\xi)d\xi\le\frac{1}{r_0}\int_{-\infty}^x v(\xi)d\xi\le\mu v(x+N_1). \qquad (3.5)$$

Define $h(x) = v(x)e^{-\gamma x}$, where $\gamma = \frac{1}{N_1+r_0}\ln\frac{1}{\mu}$. Then we have

$$h(x - r_0) = v(x - r_0)e^{-\gamma(x-r_0)} \le \mu e^{\gamma(N_1+r_0)}h(x + N_1) = h(x + N_1), \tag{3.6}$$

which shows h is bounded. Consequently, $v(x) = O(e^{\gamma x})$ when $x \to -\infty$. Now we claim $u(x) = O(e^{\gamma x})$ when $x \to -\infty$. Indeed, integrating the wave profile equation (1.5) from $-\infty$ to x and using the assumption that $b(u) \le b'(0)u, \forall u > 0$, we arrive at

$$\begin{aligned} cu(x) &= D[v(x+1) + v(x-1) - 2v(x)] - dv(x) + \int_{-\infty}^{0}\sum_{k\in\mathbb{Z}}\beta(k)b(u(\xi - k - cr))d\xi \\ &\le D[v(x+1) + v(x-1) - 2v(x)] - dv(x) + \sum_{k\in\mathbb{Z}} b'(0)\beta(k)v(x - k - cr). \end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $e^{-\gamma x}$ gives

$$\begin{aligned} cu(x)e^{-\gamma x} \le\ & D[h(x+1)e^{\gamma} + h(x-1)e^{-\gamma} - 2h(x)] - dh(x) \\ & + \sum_{k\in\mathbb{Z}} b'(0)\beta(k)e^{-\gamma(k+cr)}h(x - k - cr). \end{aligned} \tag{3.8}$$

Since the function h is bounded on $\mathbb{R}$ and the series $\sum_{k\in\mathbb{Z}}\beta(k)e^{-\gamma k}$ is convergent, we see from inequality (3.8) that $u(x)e^{-\gamma x}$ is bounded on $\mathbb{R}$. □

Let γ be defined as in Lemma 3.3. In what follows, the property established in Lemma 3.3 will be used once γ appears. For each λ satisfying $0 < \lambda < \gamma$, we can define the two-side Laplace transform

$$\mathcal{L}(\lambda) := \int_{\mathbb{R}} e^{-\lambda x}u(x)dx,$$

for which we have the following observation.

Lemma 3.4. *$\mathcal{L}(\lambda)$ is analytic for $\lambda \in (0, \lambda_1)$ and has a singularity at $\lambda = \lambda_1$.*

Proof. Rewrite the wave profile equation (1.5) as

$$cu'(x) - Dq(u)(x) + du(x) - b'(0)\sum_{k\in\mathbb{Z}}\beta(k)u(x - k - cr) = R(u)(x), \tag{3.9}$$

where $q(u)(x) = u(x+1) + u(x-1) - 2u(x)$ and $R(u)(x) = \sum_{k\in\mathbb{Z}}\beta(k)[b(u(x - k - cr)) - b'(0)u(x - k - cr)]$. Under Laplace transforms, (3.9) becomes

$$\Delta(c, \lambda)\mathcal{L}(\lambda) = \int_{\mathbb{R}} e^{-\lambda x}R(u)(x)dx. \tag{3.10}$$

We first claim that if the left hand side of (3.10) is analytic for $\lambda \in (0, \eta)$ with $\eta < \lambda^\sharp$, then there exists $\eta_1 > 0$ such that the right hand side of (3.10) is analytic for $\lambda \in (0, \eta + \eta_1)$. Indeed, it easily follows from (S1) that $0 \ge R(u)(x) \ge -\bar{a}\sum_{k\in\mathbb{Z}}\beta(k)u^\sigma(x - k - cr)$. Choose $\eta_1 > 0$ such that $\frac{\eta_1}{\sigma-1} < \gamma$ and $\eta + \eta_1 < \lambda^\sharp$. Then for any $\lambda \in (0, \eta + \eta_1)$, $\mathcal{L}(\lambda - \eta_1) < +\infty$, $a_1 := \sum_{k\in\mathbb{Z}}\beta(k)e^{-\lambda(k+cr)} < +\infty$

and $\sup_{x\in\mathbb{R}} u(x)e^{-\frac{\eta_1}{\sigma-1}x} < +\infty$, and hence,

$$\begin{aligned}\left|\int_{\mathbb{R}} e^{-\lambda x}R(u)(x)dx\right| &\leq \bar{a}\int_{\mathbb{R}} e^{-\lambda x}\sum_{k\in\mathbb{Z}}\beta(k)u^{\sigma}(x-k-cr)dx \\ &= \bar{a}\sum_{k\in\mathbb{Z}}\beta(k)e^{-\lambda(k+cr)}\int_{\mathbb{R}} e^{-\lambda x}u^{\sigma}(x)dx \\ &= \bar{a}a_1\int_{\mathbb{R}} e^{-(\lambda-\eta_1)x}u(x)\left(u(x)e^{-\frac{\eta_1}{\sigma-1}x}\right)^{\sigma-1}dx \\ &\leq \bar{a}a_1\mathcal{L}(\lambda-\eta_1)\left(\sup_{x\in\mathbb{R}} u(x)e^{-\frac{\eta_1}{\sigma-1}x}\right)^{\sigma-1} \\ &< +\infty.\end{aligned}$$

Note that $\mathcal{L}(\lambda)$ has a singularity at $\lambda = \lambda_1$. Indeed, the left hand side of (3.10) is zero at $\lambda = \lambda_1$ if $\mathcal{L}(\lambda_1) < +\infty$, and so is the right hand side. However, the right hand side is negative unless $u \equiv 0$. Thus, $\mathcal{L}(\lambda_1) = +\infty$. Now we use a property of Laplace transforms (see page 58 in [10]). Since u is positive, there exists a real number $B > 0$ such that $\mathcal{L}(\lambda)$ is analytic for $\lambda \in (0, B)$ and has a singularity at $\lambda = B$. Next we show $B = \lambda_1$. First, $B \leq \lambda_1$; otherwise, taking $\lambda = \lambda_1$ in (3.10), we know $u \equiv 0$, a contradiction. Since the abscissa of convergence of $\mathcal{L}(\lambda)$ is different from that of the right hand side of (3.10), we see that B must be the smallest positive root of the characteristic equation $\Delta(c, \lambda) = 0$, and hence $B = \lambda_1$. □

The lemma above shows that wave profiles decay faster than $e^{\lambda x}$ for each $\lambda < \lambda_1$ but not faster than $e^{\lambda_1 x}$. We further prove that the wave profiles decay as fast as $e^{\lambda_1 x}$. To this end, we first establish the following lemma with the help of Theorem 2.1.

Lemma 3.5. *Let (u, c) be a traveling wave of (1.3) with $u(-\infty) = 0$. Then for each η sufficiently close to λ_1 from the right, there exists a nonpositive continuous and bounded function ψ on $\mathbb{R}$ such that*

$$u(x) = \sum_{l=1}^{n_\eta}\sum_{p=1}^{k_l} M_{l,p}x^{p-1}e^{-iw_l x} + \psi(x)e^{\eta x}, \tag{3.11}$$

where $-iw_l$ are eigenvalues in the strip $|Re\lambda| \leq \eta$, $n_\eta < \infty$ is the number of the eigenvalues in this strip, the $M_{l,p} \in \mathbb{C}$ are constants and k_l is the order of the multiplicity of the eigenvalue $-iw_l$.

Proof. Let γ be defined as in Lemma 3.3. Choose $\sigma, \bar{a}$ and $\epsilon > 0$ so that (S1), (S2) hold, $\lambda_1 + \gamma\epsilon < \lambda_2, 1 + \epsilon < \sigma$ and there is no eigenvalue in the strip $\lambda_1 - \gamma\epsilon < \operatorname{Re}\lambda < \lambda_1 + \gamma\epsilon$ other than λ_1 (see Lemma 3.1(iv)). Let $\eta \in (\lambda_1, \lambda_1 + \gamma\epsilon)$. Since u is positive and bounded, there exists $M_1 > 0$ such that $u^{\sigma}(x) \leq M_1 u^{1+\epsilon}(x), \forall x \in \mathbb{R}$. Therefore, the remainder $R(u)(x)$ in (3.9) has the following property:

$$0 \geq R(u)(x) \geq -\bar{a}\sum_{k\in\mathbb{Z}}\beta(k)u^{\sigma}(x-k-cr) \geq -\bar{a}M_1\sum_{k\in\mathbb{Z}}\beta(k)u^{1+\epsilon}(x-k-cr). \tag{3.12}$$

Define $I : (-\lambda^{\sharp}, \lambda^{\sharp}) \times C(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ by

$$I(\lambda, u)(y) = D[u(y+1)e^{\lambda} + u(y-1)e^{-\lambda}] + b'(0)\sum_{k\in\mathbb{Z}}\beta(k)e^{-\lambda(k+cr)}u(y-k-cr).$$

Obviously, $I(0,u)(y) = e^{\lambda y} I(\lambda, ue^{-\lambda\cdot}), \forall \lambda$ and u. Then we can rewrite (3.9) as the following integral equation by using the variation of constant formula:

$$\begin{aligned} u(x) &= \lim_{z\to-\infty} u(z)e^{-\frac{d+2D}{c}(x-z)} + \frac{1}{c}\int_{-\infty}^{x} e^{-\frac{d+2D}{c}(x-y)}[I(0,u)(y) + R(u)(y)]dy \\ &= \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y}[e^{\eta(x+y)} I(\eta, ue^{-\eta\cdot})(x+y) + R(u)(x+y)]dy. \end{aligned} \tag{3.13}$$

Multiplying by $e^{-\eta x}$ on both sides of (3.13), we obtain

$$\begin{aligned} u(x)e^{-\eta x} =& \frac{1}{c}e^{-\eta x}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} I(0,u)(x+y)dy \\ &+ \frac{1}{c}e^{-\eta x}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} R(u)(x+y)dy \\ =& \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D+c\eta}{c}y} I(\eta, ue^{-\eta\cdot})(x+y)dy \\ &+ \frac{1}{c}e^{-\eta x}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} R(u)(x+y)dy \end{aligned}$$

Consider the iterative scheme $\psi^{(0)}(x) := \frac{1}{c}e^{-\eta x}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} R(u)(x+y)dy$ and

$$\psi^{(n)}(x) := \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D+c\eta}{c}y} I(\eta, \psi^{(n-1)})(x+y)dy + \psi^{(0)}(x), \quad n \geq 1.$$

From inequality (3.12), we know that

$$\begin{aligned} &0 \geq \psi^{(0)}(x) \\ &\geq -\frac{\bar{a}M_1}{c}e^{-\eta x}\sum_{k\in\mathbb{Z}}\beta(k)\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} u^{1+\epsilon}(x+y-k-cr)dy \\ &= -\frac{\bar{a}M_1}{c}\sum_{k\in\mathbb{Z}}\beta(k)e^{-\eta(k+cr)}\int_{-\infty}^{0} e^{\frac{d+2D+c\eta}{c}y} u(x+y-k-cr)e^{-(\eta-\gamma\epsilon)(x+y-k-cr)} \\ &\quad \times \left\{u(x+y-k-cr)e^{-\gamma(x+y-k-cr)}\right\}^{\epsilon} dy \\ &\geq -\frac{\bar{a}M_1}{c}\sum_{k\in\mathbb{Z}}\beta(k)e^{-\eta(k+cr)}\mathcal{L}(\eta-\gamma\epsilon)\left\{\sup_{x\in\mathbb{R}} u(x)e^{-\gamma x}\right\}^{\epsilon}, \end{aligned} \tag{3.14}$$

which means that $\psi^{(0)}(x)$ is nonpositive and bounded below. Also, because $I(\eta,u)$ is increasing in u and $I(\eta,u)(x) \geq I(\eta,1)\inf_{x\in\mathbb{R}} u(x)$, we see that $\psi^{(n)} \leq \psi^{(n-1)} \leq \cdots \leq \psi^{(0)}$ and

$$\psi^{(n)}(x) \geq \inf_{x\in\mathbb{R}} \psi^{(0)}(x)[1 + k(c,\eta) + (k(c,\eta))^2 + \cdots (k(c,\eta))^n], \quad \forall n \geq 1, x \in \mathbb{R}$$

where $k(c,\eta) = \frac{I(\eta,1)}{d+2D+c\eta}$. Since $\Delta(c,\lambda) > 0, \forall \lambda \in (\lambda_1, \lambda_2)$ (see Lemma 3.1(ii)) and $I(\eta,1) = (d+2D+c\eta-\Delta(c,\eta))$, we have $0 < k(c,\eta) < 1$. These facts indicate that the sequence $\{\psi^n\}$, being bounded and monotone, converges to a limit function, ψ say, which satisfies the equation $\psi(x) = \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D+c\eta}{c}y} I(\eta,\psi)(x+y)dy + \psi^{(0)}(x)$.

Thus, the function $\phi(x) := \psi(x)e^{\eta x}$ satisfies

$$\phi(x) = \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} I(0,\phi)(x+y)dy + \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} R(u)(x+y)dy. \tag{3.15}$$

The same equation is satisfied by the wave profile u (see (3.13)). So the function $f(x) := u(x) - \psi(x)e^{\eta x}$ satisfies the linear equation

$$f(x) = \frac{1}{c}\int_{-\infty}^{0} e^{\frac{d+2D}{c}y} I(0,f)(x+y)dy,$$

which is equivalent to the linearized wave profile equation

$$cf'(x) = D[f(x+1)+f(x-1)-2f(x)] - df(x) + b'(0)\sum_{k\in\mathbb{Z}} \beta(k)f(x-k-cr). \tag{3.16}$$

Since u is nonnegative and ψ is nonpositive, the function f is not zero identically. Thus, the boundedness of u and ψ implies that $f(x) = O(e^{\eta|x|})$ as $|x| \to +\infty$. Further, there are only finitely many eigenvalues in the strip $|\operatorname{Re}\lambda| \le \eta$ (see Lemma 3.1(iii)). Then, by using solution structure Theorem 2.1 with $m = 1, \gamma = \eta$ and $\rho = \lambda^{\sharp}$, we complete the proof. $\square$

Define the sum in (3.11) by $f(x) := \sum_{l=1}^{n_\eta}\sum_{p=1}^{k_l} M_{l,p}x^{p-1}e^{-iw_l x}$. Next we show that the function $f(x)$ consists of a single term $\theta e^{\lambda_1 x}$ for some $\theta > 0$. Although f is nonnegative and real, it may be a sum of complex functions. In order to exclude complex terms except for the term $e^{\lambda_1 x}$, we need more analysis on wave profiles.

Lemma 3.6. *Let f be defined as above. Then $f(x) = \theta e^{\lambda_1 x}$ for some $\theta > 0$.*

Proof. From Lemma 3.5, we know for each η sufficiently close to λ_1 from the right that there exists a bounded function ψ such that $f(x) = u(x) - \psi(x)e^{\eta x}$. By Lemma 3.1(iv), we can choose $\lambda_0 < \lambda_1$ and $\eta > \lambda_1$ such that there are no eigenvalues except for λ_1 in the strip $\lambda_0 \le \operatorname{Re}\lambda \le \eta$. Because $\int_{-\infty}^{0} u(x)e^{-\lambda_0 x}dx < +\infty$, we have that $\int_{-\infty}^{0} f(x)e^{-\lambda_0 x}dx = \int_{-\infty}^{0} u(x)e^{-\lambda_0 x}dx - \int_{-\infty}^{0}\psi(x)e^{(\eta-\lambda_0)x}dx < +\infty$. On the other hand, from the expression of f, we have that

$$\int_{-\infty}^{0} f(x)e^{-\lambda_0 x}dx = \sum_{l=1}^{n_\eta}\sum_{p=1}^{k_l}\int_{-\infty}^{0} M_{l,p}x^{p-1}e^{(-iw_l-\lambda_0)x}dx. \tag{3.17}$$

Let $w_l = u_l + iv_l$ with $u_l, v_l \in \mathbb{R}$. Define $v_0 := \min_{1\le l\le n_\eta}\{v_l\}$. The right hand side of (3.17) is dominated by $\sum_{l\in\{l:v_l=v_0\}} M_{l,p}\int_{-\infty}^{0} x^{k_l-1}e^{(v_0-\lambda_0)x-iu_l x}dx$, which is finite if and only if $v_0 > \lambda_0$. Recall that $-iw_l$ in expression (3.11) are eigenvalues in the strip $|\operatorname{Re}\lambda| \le \eta$, and hence $\operatorname{Re}(-iw_l) \le \eta$. At the the same time, $\operatorname{Re}(-iw_l) = v_l \ge v_0 > \lambda_0$. These facts together imply that $f(x)$ consists of a single term $\theta e^{\lambda_1 x}$ with some $\theta > 0$. $\square$

Now we are ready to prove the main result of this section.

Theorem 3.1. *Suppose (K), (H) and (L) hold. Then for each $c > c^*$, there exists at most one (up to translation) traveling wave (u, c) with $u(-\infty) = 0$.*

Proof. Assume that (u_1, c) and (u_2, c) are two traveling waves of (1.3) with $u_1(-\infty) = 0$ and $u_2(-\infty) = 0$. By Lemmas 3.5 and 3.6, we have $\lim_{x\to-\infty} u_i(x)e^{\lambda_1 x} = \theta_i > 0, i = 1, 2$. Thus, Lemma 3.2 implies that u_1 is a translation of u_2. $\square$

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