ON SPACES OF COMPACT OPERATORS ON $C(K, X)$ SPACES

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Abstract. This paper concerns the spaces of compact operators $\mathcal{K}(E, F)$, where $E$ and $F$ are Banach spaces $C([1, \xi], X)$ of all continuous $X$-valued functions defined on the interval of ordinals $[1, \xi]$ and equipped with the supremum norm. We provide sufficient conditions on $X$, $Y$, $\alpha$, $\beta$, $\xi$, and $\eta$, with $\omega \leq \alpha \leq \beta < \omega_1$ for the following equivalence:

(a) $\mathcal{K}(C([1, \xi], X), C([1, \eta], Y))$ is isomorphic to $\mathcal{K}(C([1, \eta], X), C([1, \beta], Y))$,

(b) $\beta < \alpha^\omega$.

In this way, we unify and extend results due to Bessaga and Pełczyński (1960) and C. Samuel (2009). Our result covers the case of the classical spaces $X = \ell_p$ and $Y = \ell_q$, with $1 < p, q < \infty$.

1. Introduction

We use standard Banach space theory terminology and notions as can be found in [12] and [13]. Let $X$ be a Banach space and $K$ a compact Hausdorff space. By $C(K, X)$ we denote the Banach space of all continuous $X$-valued functions defined on $K$ and equipped with the supremum norm. This space will be denoted by $C(K)$ in the case $X = \mathbb{R}$. Given Banach spaces $X$ and $Y$, $\mathcal{K}(X, Y)$ denotes the Banach space of compact operators from $X$ to $Y$. We write $X \sim Y$ when the Banach spaces $X$ and $Y$ are isomorphic and $X \rightarrow Y$ when $Y$ has a subspace isomorphic to $X$. Let $\alpha$ be an ordinal number. By $[1, \alpha]$ we denote the interval of ordinals $\{\xi : 1 \leq \xi \leq \alpha\}$ endowed with the order topology.

The present paper is devoted to the isomorphic classifications of spaces of compact operators from $C([1, \xi], X)$ to $C([1, \eta], Y)$ spaces, with $\eta < \omega_1$. A fundamental result along these lines is the classical Bessaga-Pełczyński 1960 isomorphic classification of $C([1, \alpha])$ spaces, with $\alpha < \omega_1$ [1] Theorem 4.1, that is:

**Theorem 1.1.** Suppose that $\omega \leq \alpha \leq \beta < \omega_1$. Then we have

$$C([1, \alpha]) \sim C([1, \beta]) \iff \beta < \alpha^\omega.$$ 

Very recently C. Samuel classified the $\mathcal{K}(C([1, \alpha]), C([1, \alpha]))$ spaces, with $\alpha < \omega_1$ [15] Theorem 3.3 by proving:

**Theorem 1.2.** Suppose that $\omega \leq \alpha \leq \beta < \omega_1$. Then we have

$$\mathcal{K}(C([1, \alpha]), C([1, \alpha])) \sim \mathcal{K}(C([1, \beta]), C([1, \beta])) \iff \beta < \alpha^\omega.$$
The main aim of this short work is to unify and extend these results as follows.

**Theorem 1.3.** Let $\alpha, \beta, \xi$ and $\eta$ be ordinals with $\omega \leq \alpha \leq \beta < \omega_1$, $\xi \eta < \omega$ or else $\xi$ and $\eta$ of the same cardinality, $X$ a Banach space such that $X^*$ is weakly sequentially complete and has the approximation property and $Y$ a Banach space which has an unconditional basis and contains no subspace isomorphic to $l_1$. Then

$$K(C([1, \xi], X), C([1, \alpha], Y)) \sim K(C([1, \eta], X), C([1, \beta], Y)) \iff \beta < \alpha^\omega.$$ 

Observe that Theorem 1.1 is the case $\xi = \eta = 1$ and $X = Y = \mathbb{R}$ of Theorem 1.3. Moreover, Theorem 1.2 is the case $\alpha = \xi$, $\beta = \eta$ and $X = Y = \mathbb{R}$ of Theorem 1.3. Notice also that as an immediate consequence of Theorem 1.3 and Theorem 1.1 we get the following cancellation law:

**Corollary 1.4.** Let $\alpha, \beta, \xi$ and $\eta$ be ordinals with $\omega \leq \alpha \leq \beta < \omega_1$, $\xi \eta < \omega$ or else $\xi$ and $\eta$ of the same cardinality. Then for every $1 < p < q < \infty$ we have

$$K(C([1, \xi], l_p), C([1, \alpha], l_q)) \sim K(C([1, \eta], l_p), C([1, \beta], l_q)) \iff C([1, \alpha]) \sim C([1, \beta]).$$

This corollary gives a partial answer to [8, Problem 4.2.3]. We stress that the statement of Corollary 1.4 is also true for $1 < p < \infty$ and $q = 1$; see [8, Remark 4.1.3] for $\xi < \omega$ and [11, Remark 1.7] for $\xi \geq \omega$. Nevertheless the isomorphic classification of the $K(l_1, l_q^\infty)$ spaces, with $\omega \leq \alpha < \omega_1$ and $1 \leq q < \infty$, remains an open question; see [8, Problem 4.2.2]. In order to prove Theorem 1.3 we need some preliminary results.

## 2. Preliminary results

From now on following [11] the $C([1, \alpha], X)$ spaces will be denoted by $X^\alpha$. One of the key steps in proving Theorem 1.3 is the following proposition. We will denote by $X \hat{\otimes} Y$ the injective tensor product of the Banach spaces $X$ and $Y$.

**Proposition 2.1.** Let $X$ be a weakly sequentially complete Banach space and $Y$ a Banach space which has an unconditional basis and contains no subspace isomorphic to $l_1$. Then for every set $\Gamma$ we have

$$\mathbb{R}^\omega \not\hookrightarrow l_1(\Gamma, X) \hat{\otimes} Y^\omega.$$ 

**Proof:** Suppose for contradiction that $l_1(\Gamma, X) \hat{\otimes} Y^\omega$ has a subspace isomorphic to $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ is separable, it is easily seen that

$$\mathbb{R}^\omega \hookrightarrow l_1(\mathbb{N}, X) \hat{\otimes} Y^\omega. \tag{1}$$

Assume that $Y$ has an unconditional basis and contains no subspace isomorphic to $l_1$. Then $Y^\omega$ does not contain a subspace isomorphic to $l_1$ (see for instance [14, Theorem 2.3]) and has an unconditional basis. So by a well known result of James [12, Theorem 1.c.9] $Y^\omega$ has a shrinking unconditional basis. Moreover, by [5, Corollary 1] $l_1(\mathbb{N}, X)$ is weakly sequentially complete. Therefore according to [3, Theorem 3] $l_1(\mathbb{N}, X) \hat{\otimes} Y^\omega$ has the property (u) introduced by Pełczyński in [14, Definition 1]. Since this property is inherited by closed subspaces ([14, page 252] or [13, Proposition 1.c.3]), we would conclude by (1) that $\mathbb{R}^\omega$ has the property (u), which is absurd by an unpublished result of Pełczyński; see [10, pages 210-211] and [11, Proposition 5.3].

The next proposition is inspired by [15, Theorem 3.2].
Proposition 2.2. Let $\omega^\omega \leq \xi \leq \eta < \omega_1$ be two ordinals and $X$ a Banach space such that $X^\omega$ contains no subspace isomorphic to $R^{\omega^\omega}$. If $R^\eta \hookrightarrow X^\xi$, then $R^\eta \hookrightarrow R^\xi$.

Proof. We introduce two sets of ordinals:

$I_1 = \{ \theta : \omega^\omega \leq \theta < \omega_1, \ R^\theta \not\hookrightarrow R^\gamma, \ \forall \gamma < \theta \}$,

$I_2 = \{ \theta : \omega^\omega \leq \theta < \omega_1, \ R^\theta \not\hookrightarrow X^\gamma, \ \forall \gamma < \theta \}$.

First of all we will prove that $I_1 = I_2$. Clearly $I_2 \subset I_1$. Observe that by Theorem 1.1 and our hypothesis, we deduce that $\omega^\omega \in I_2$. Now, assume that $I_2$ is a proper subset of $I_1$. Let $\alpha_1$ be the least element of $I_1 \setminus I_2$. We have $\omega^\omega < \alpha_1$. Since $\alpha_1 \notin I_2$, there exists an ordinal $\gamma_1 < \alpha_1$ such that $R^{\alpha_1} \hookrightarrow X^{\gamma_1}$.

Let $\alpha_2 = \min\{ \gamma, \ \omega^\omega \leq \gamma < \alpha_1 : R^{\alpha_1} \hookrightarrow X^{\gamma} \}$. We have $\omega^\omega < \alpha_1$. Now, we will show that $\alpha_2 \in I_1$. If this is not the case, there exists an ordinal $\gamma_2 < \alpha_2$ such that $R^{\alpha_2} \hookrightarrow R^{\gamma_2}$. Therefore $X^{\alpha_2} \hookrightarrow X^{\gamma_2}$. Consequently, $R^{\alpha_1} \hookrightarrow X^{\gamma_2}$. This is a contradiction to the definition of $\alpha_2$.

So $\alpha_2 \in I_1$ and since $\alpha_2 < \alpha_1$, it follows from the definition of $\alpha_2$ that $\alpha_2 \in I_2$. That is, $R^{\alpha_2} \not\hookrightarrow X^\gamma$, $\forall \gamma < \alpha_2$. Thus by [6] Lemma 3.3, we conclude that $R^{\alpha_2} \not\hookrightarrow X^{\alpha_2}$.

On the other hand, note that if $\alpha_1 < \alpha_2^\omega$, then by Theorem 1.1, $R^{\alpha_1} \sim R^{\alpha_2}$, which is absurd by the definition of $\alpha_1$. Consequently $\alpha_2^\omega \leq \alpha_1$ and $R^{\alpha_2} \hookrightarrow R^{\alpha_1}$. Furthermore, by the definition of $\alpha_2$, $R^{\alpha_1} \hookrightarrow X^{\alpha_2}$. Therefore $R^{\alpha_2} \hookrightarrow X^{\alpha_2}$, in contradiction to what we have proved above. Hence $I_1 = I_2$.

Next, to complete the proof of the proposition, suppose that $R^\eta \not\hookrightarrow R^\xi$ and let $\xi_1 = \min\{ \theta : R^\eta \not\hookrightarrow R^\theta \}$. Hence $\xi < \xi_1 \leq \eta$ and $R^{\xi_1} \not\hookrightarrow R^\gamma$, $\forall \gamma < \xi_1$. In particular, $\xi_1 \in I_1 = I_2$, which is absurd, because $R^{\xi_1} \hookrightarrow R^\eta \not\hookrightarrow X^\xi$.

3. PROOF OF THEOREM 1.3

Initially notice that if $\gamma$ and $\theta$ are ordinals with $\gamma < \omega$ and $\omega \leq \theta < \omega_1$, then by Theorem 1.1, $R^{\gamma^\theta} \sim R^\theta$. Therefore by [3] Proposition 5.3, we deduce that

$$(2) \quad K(X^\gamma, Y^\theta) \sim (X^\gamma)^* \hat{\otimes} Y^\theta \sim X^\gamma \hat{\otimes} Y \hat{\otimes} R^\gamma \sim X^\gamma \hat{\otimes} Y \hat{\otimes} R^\gamma \sim X^\gamma \hat{\otimes} Y^\theta.$$  

On the other hand, if $\omega \leq \gamma$, since $l_1([1, \gamma], X^*)$ has the approximation property (see for instance [2] page 285), it follows from [3] Proposition 5.3 that

$$(3) \quad K(X^\gamma, Y^\theta) \sim l_1([1, \gamma], X^*) \hat{\otimes} Y^\theta.$$  

Now suppose that $\beta < \alpha^\omega$. We have by Theorem 1.1 that $R^\alpha \sim R^\beta$ and thus $Y^\alpha \sim Y^\beta$. Hence according to (2) and (3) and our hypothesis on the cardinalities of $\xi$ and $\eta$, we conclude that $K(X^\xi, Y^\alpha) \sim K(X^\eta, Y^\beta)$.

Conversely, assume that $\omega \leq \alpha \leq \beta < \omega_1$ and $K(X^\xi, Y^\alpha) \sim K(X^\eta, Y^\beta)$. Then bearing in mind (2) and (3) we conclude that

$$(4) \quad R^\beta \hookrightarrow Y^\beta \hookrightarrow l_1([1, \eta], X^*) \hat{\otimes} Y^\beta \hookrightarrow l_1([1, \xi], X^*) \hat{\otimes} Y^\alpha \sim (l_1([1, \xi], X^*) \hat{\otimes} Y)^\alpha.$$  

Next we distinguish two cases: $\alpha < \omega^\omega$ and $\omega^\omega \leq \alpha$.

Case 1. $\alpha < \omega^\omega$. We thus obtain from Theorem 1.1 that $R^\alpha \sim R^\omega$. So, by (4) we infer that

$$(\beta) \quad R^\beta \hookrightarrow (l_1([1, \xi], X^*) \hat{\otimes} Y)^\alpha \sim (l_1([1, \xi], X^*) \hat{\otimes} Y)^\omega \sim l_1([1, \xi], X^*) \hat{\otimes} Y^\omega.$$  

Then, it follows from Proposition 2.2 that $\beta < \omega^\omega$. Again, an appeal to Theorem 1.1 tells us that $R^\beta \sim R^\omega$ and $\beta < \alpha^\omega$. 


Case 2. $\omega^\nu \leq \alpha$. In this case, by Proposition 2.1, $l_1([1,\xi],X^*) \hat{\otimes} Y$ contains no subspace isomorphic to $\mathbb{R}^{\omega^\nu}$. Consequently, by (3) and Proposition 2.2, $\mathbb{R}^\beta \hookrightarrow \mathbb{R}^\alpha$. Once again by Theorem 1.1, we get that $\beta < \alpha^\omega$. Thus the theorem is established.

REFERENCES


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