LOJASIEWICZ EXPONENT NEAR THE FIBRE OF A MAPPING

TOMASZ RODAK AND STANISLAW SPODZIEJA

(Communicated by Ted Chinburg)

ABSTRACT. Let $g : X \to \mathbb{R}^k$ and $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$, be continuous semi-algebraic mappings, and $\lambda \in \mathbb{R}^m$. We describe the optimal exponent $\theta = L_{\infty,f \to \lambda}(g)$ for which the Lojasiewicz inequality $|g(x)| \geq C|x|^\theta$ holds with $C > 0$ as $|x| \to \infty$ and $f(x) \to \lambda$. We prove that there exists a semi-algebraic stratification $\mathbb{R}^m = S_1 \cup \cdots \cup S_j$ such that the function $\lambda \mapsto L_{\infty,f \to \lambda}(g)$ is constant on each stratum $S_i$. We apply this result to describe the set of generalized critical values of $f$.

INTRODUCTION

Let $M, N, L$ be finite-dimensional real vector spaces, $X \subset M$ be a closed semi-algebraic set, $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings (see [1]), and let $\lambda \in L$. The aim of this article is to describe the Lojasiewicz exponent at infinity of $g$ near the fibre $f^{-1}(\lambda)$, i.e. the supremum of the exponents $\theta$ for which the Lojasiewicz inequality

$$(L) \quad |g(x)| \geq C|x|^\theta \quad \text{as} \quad x \in X, \quad |x| \to \infty \quad \text{and} \quad f(x) \to \lambda$$

holds with $C > 0$ (cf. [12], [18]), where $|\cdot|$ is a norm. We denote this exponent by $L_{\infty,f \to \lambda}(g)$ (see Section 1 for details).

We prove that $L_{\infty,f \to \lambda}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$ for $\lambda \in L$ and that there exists a semi-algebraic stratification $L = S_1 \cup \cdots \cup S_j$ such that the function $\lambda \mapsto L_{\infty,f \to \lambda}(g)$ is constant on each stratum $S_i$ (Theorem 1.2). If $g$ and $f$ are complex regular mappings, the stratification is complex algebraic (Corollary 1.6). Note that if $\theta = L_{\infty,f \to \lambda}(g) \in \mathbb{Q}$, then (L) holds (Corollary 3.7). The key points in the proofs are Lipschitz stratifications ([13], [14], [20]) and properties of the set of points at which a mapping is not proper ([8]; see also Section 2).

If $f : M \to L$ is a semi-algebraic mapping of class $C^1$, we define the Lojasiewicz exponent of $df$ near the fibre $f^{-1}(\lambda)$ by

$$L_{\infty,\lambda}(f) = L_{\infty,f \to \lambda}(\nu(df)),$$

where $\nu$ is a function introduced by Rabier [17] (see Section 1). This notion was introduced by Ha [17] in the case of complex polynomial functions in two variables (see also [3], [5]).

Received by the editors May 19, 2009 and, in revised form, April 19, 2010.

2010 Mathematics Subject Classification. Primary 14R25; Secondary 58K55, 58K05.

Key words and phrases. Lojasiewicz exponent at infinity, generalized critical values, stratification.

This research was partially supported by the program POLONIUM.

©2010 American Mathematical Society
Reverts to public domain 28 years from publication
Let us recall that the exponent $\mathcal{L}_{\infty, \lambda}(f)$ is strongly related to the set of bifurcation points of $f$. Namely, one can define the set of generalized critical values of $f$ by

$$K_\infty(f) = \{ \lambda \in L : \mathcal{L}_{\infty, \lambda}(f) < -1 \}.$$  

It is a closed and semi-algebraic set. By Theorem 1.2 and Corollary 1.6, respectively.

If $f$ is of class $C^2$, then for any $\lambda \in L \setminus K_\infty(f)$ there exist a neighbourhood $U \subset L$ of $\lambda$ and a compact set $\Delta \subset M$ such that $f : f^{-1}(U) \setminus \Delta \to U$ is a trivial bundle (see [16], [17], [11]; see also [23], [21], [22], [7], [15] for polynomials and polynomial mappings). The smallest set $B \subset L$ such that $L \setminus B$ has the above property is called the bifurcation set at infinity of $f$ and is denoted by $B_\infty(f)$. Note that for a complex polynomial $f$ in two variables, $B_\infty(f) = K_\infty(f)$ (see [7], [15]).

Chadzyński and Krasinski ([3], Corollary 4.7) proved that for a complex polynomial $f$ in two variables with $\deg f > 0$ there exists $c_f \in \mathbb{Q}$ with $c_f \geq 0$ such that

$$\mathcal{L}_{\infty, \lambda}(f) = c_f \text{ for } \lambda \notin K_\infty(f) \quad \text{and} \quad \mathcal{L}_{\infty, \lambda}(f) < -1 \text{ for } \lambda \in K_\infty(f).$$

They also asked whether $\lambda \mapsto \mathcal{L}_{\infty, \lambda}(f)$ behaves similarly in the general case. Note that in the multi-dimensional case we cannot require $c_f \geq 0$. Indeed, for the polynomial $f(z_1, z_2, z_3) = (z_1^2 - 1)z_2z_3$ ([17], Remark 9.1) we have $c_f = -1$ (see [3], Proposition 6.4).

As a corollary from Theorem 1.2 we give a partial answer to the above-mentioned question. Namely, for a nonconstant polynomial $f : \mathbb{C}^n \to \mathbb{C}$ there exist a finite set $S \subset \mathbb{C}$ with $K_\infty(f) \subset S$ and $c_f \geq 0$ such that $\mathcal{L}_{\infty, \lambda}(f) = c_f$ for $\lambda \in \mathbb{C} \setminus S$ and $\mathcal{L}_{\infty, \lambda}(f) < c_f$ for $\lambda \in S$ (Corollary 1.7). It is not clear to the authors whether $S = K_\infty(f)$ in Corollary 1.7.

Section 2 has an auxiliary character and contains some results on semi-algebraic mappings, Lojasiewicz exponent and stratifications. In Sections 3 and 4 we prove Theorem 1.2 and Corollary 1.6 respectively.

1. Lojasiewicz exponent near the fibre of a mapping

Let $M, N, L$ be finite-dimensional real vector spaces, $X \subset M$ be a closed set, let $g : X \to N$ and $f : X \to L$, and let $\lambda \in L$.

**Definition 1.1.** By the Lojasiewicz exponent at infinity of $g$ near the fibre $f^{-1}(\lambda)$ we mean

$$\mathcal{L}_{\infty, f^{-1}(\lambda)}(g) := \sup \{ \mathcal{L}_\infty(g|f^{-1}(U)) : U \subset L \text{ is a neighbourhood of } \lambda \},$$

where

$$\mathcal{L}_\infty(g|S) := \sup \{ \theta \in \mathbb{R} : \exists \epsilon > 0, \forall x \in S \ (x \geq R \Rightarrow |g(x)| \geq \epsilon |x|^\theta) \}$$

is the Lojasiewicz exponent at infinity of $g$ on a set $S \subset X$.

Our main result is

**Theorem 1.2.** Let $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings.

(i) For any $\lambda \in L$, $\mathcal{L}_{\infty, f^{-1}(\lambda)}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(ii) The function
\[ \vartheta_{g/f} : L \ni \lambda \mapsto \mathcal{L}_{\infty, f \to \lambda} (g) \]
is upper semi-continuous, and there exists a semi-algebraic stratification
\[ L = S_1 \cup \cdots \cup S_j \]
such that \( \vartheta_{g/f} \) is constant on each stratum \( S_i \), \( i = 1, \ldots, j \).

The proof of Theorem 1.2 is given in Section 3. Theorem 1.2(ii) was proved in [18] for complex polynomials, under the assumption (i).

Now let \( f : M \to L \) be a semi-algebraic mapping of class \( \mathcal{C}^1 \) and let \( df \) be the differential of \( f \). Let
\[ \nu(df) : M \ni x \mapsto \nu(df(x)) \in \mathbb{R}, \]
be the Rabier function, i.e. for \( A = df(x) : M \to L \),
\[ \nu(A) = \inf_{||\phi|| = 1} ||A^*(\phi)||, \]
where \( A^* : L^* \to M^* \) is the adjoint operator and \( \phi \in L^* \). For a semi-algebraic function \( f : M \to \mathbb{R} \) (or a complex polynomial) we have \( \nu(df) = |\nabla f| \), where \( \nabla f \) is the gradient of \( f \).

**Definition 1.3.** The Lojasiewicz exponent of \( df \) near a fibre \( f^{-1}(\lambda) \) is defined to be
\[ \mathcal{L}_{\infty, \lambda} (f) = \mathcal{L}_{\infty, f \to \lambda} (\nu(df)). \]

**Remark 1.4.** Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a semi-algebraic mapping of class \( \mathcal{C}^1 \) and let \( \kappa(df) : \mathbb{R}^n \ni x \mapsto \kappa(df(x)) \in \mathbb{R} \) be the Kuo function [10]; i.e., for \( A = df(x) = (A_1, \ldots, A_m) : \mathbb{R}^n \to \mathbb{R}^m \),
\[ \kappa(A) = \min_{1 \leq i \leq m} \text{dist}(\nabla A_i, (\nabla A_j)_{j \neq i}), \]
where \( (a_j)_{j \neq i} \) is the vector space generated by the vectors \( (a_j)_{j \neq i} \). As \( \nu(A) \leq \kappa(A) \leq \sqrt{m} \nu(A) \) ([11], Proposition 2.6), for any \( \lambda \in L \) we have
\[ \mathcal{L}_{\infty, \lambda} (f) = \mathcal{L}_{\infty, f \to \lambda} (\kappa(df)). \]
An analogous result holds for the Gaffney function [4] (cf. [9], Proposition 2.3).

The function \( \nu(df) \) is continuous and semi-algebraic ([11], Proposition 2.4), so Theorem 1.2 implies:

**Corollary 1.5.** Let \( f : M \to L \) be a semi-algebraic mapping of class \( \mathcal{C}^1 \). Then \( \mathcal{L}_{\infty, \lambda} (f) \in \mathbb{Q} \cup \{-\infty, +\infty\} \) for any \( \lambda \in L \), and the function \( L \ni \lambda \mapsto \mathcal{L}_{\infty, \lambda} (f) \) is upper semi-continuous and has a finite number of values. In particular, there exists \( \alpha > 0 \) such that
\[ K_\infty (f) = \{ \lambda \in L : \mathcal{L}_{\infty, \lambda} (f) < -1 - \alpha \}. \]

In the case of complex regular mappings, from Theorem 1.2 we obtain:

**Corollary 1.6.** Let \( X \subset \mathbb{C}^n \) be a complex algebraic set, and let \( g : X \to \mathbb{C}^m \) and \( f : X \to \mathbb{C}^k \) be complex regular mappings. Then there exists a complex algebraic stratification \( \mathbb{C}^k = S_1 \cup \cdots \cup S_j \) such that the function
\[ \vartheta_{g/f} : \mathbb{C}^k \ni \lambda \mapsto \mathcal{L}_{\infty, f \to \lambda} (g) \in \mathbb{Q} \cup \{-\infty, +\infty\} \]
is constant on each stratum \( S_i \), \( i = 1, \ldots, j \). Moreover, \( \vartheta_{g/f} \) is upper semi-continuous.
The proof of the above corollary will be given in Section 4. The crucial fact in the proof is that \( \partial g/f(C^n) = \partial g/f(R^{2n}) \) and this set is finite (Theorem 1.2).

For a complex polynomial \( f : C^n \to \mathbb{C} \) the set \( K_\infty(f) \) is finite (Proposition 2.4, see also [11], Theorem 3.1); hence Corollary 1.6 gives:

**Corollary 1.7.** Let \( f : C^n \to \mathbb{C} \) be a polynomial function with \( \deg f > 0 \). Then there exist a finite set \( S \subset C \) with \( K_\infty(f) \subset S \) and a constant \( c_f \in \mathbb{Q} \) with \( c_f > -1 \) such that \( L_{\infty, \lambda}(f) = c_f \) for \( \lambda \in \mathbb{C} \setminus S \), and \( L_{\infty, \lambda}(f) < c_f \) for \( \lambda \in S \).

2. Auxiliary results

In what follows, \( L, M, N \) are finite-dimensional real vector spaces. We will use the Euclidean norm \(|\cdot|\) in \( M \) (or in \( N, L \)). For \( A \subset M \), let \( g(\cdot, A) \) denote the distance function to \( A \), i.e. \( g(x, A) = \inf_{y \in A} |x - y| \) if \( A \neq \emptyset \), and \( g(x, \emptyset) = 1 \).

2.1. Semi-algebraic mappings. A subset of \( M \) is called semi-algebraic if it is defined by a finite alternative of finite systems of inequalities \( P > 0 \) or \( P \geq 0 \), where \( P \) are polynomials on \( M \) (see [1], [2]). A mapping \( f : X \to N \), where \( X \subset M \), is called semi-algebraic if the graph \( \Gamma(f) \) of \( f \) is a semi-algebraic set. For instance, the distance to a semi-algebraic set is a semi-algebraic function (cf. [2]):

**Proposition 2.1.** Let \( V \subset M \) be a semi-algebraic set. Then the function \( g_V : M \ni x \mapsto g(x, V) \in \mathbb{R} \) is continuous and semi-algebraic.

Let \( X \subset M \) and let \( f : X \to N \) be any mapping. We say (cf. [8]) that \( f \) is proper at a point \( y \in N \) if there exists an open neighbourhood \( U \) of \( y \) such that \( f : f^{-1}(U) \to U \) is a proper map. The set of points at which \( f \) is not proper is denoted by \( \mathcal{G}_f \). It is obvious that the set \( \mathcal{G}_f \) is closed. It is known that for a complex algebraic set \( X \subset C^n \) and a complex regular mapping \( f : X \to C^m \), the set \( \mathcal{G}_f \) is complex algebraic.

**Proposition 2.2.** Let \( X \) be a closed semi-algebraic set. If the mapping \( f : X \to N \) is semi-algebraic, then the set \( \mathcal{G}_f \) is also semi-algebraic.

**Proof.** Since \( X \) is a closed set, we have

\[
\mathcal{G}_f = \{ y \in N : \forall A, \varepsilon > 0 \exists x \in X \ |x| > A \ \wedge \ |f(x) - y| < \varepsilon \}.
\]

Then, by the Tarski-Seidenberg Theorem, we obtain the assertion. \( \square \)

Let \( f : X \to N \) with \( X \subset M \). The degree of \( f \) is defined by

\[
\deg f = \inf \{ \theta \in \mathbb{R} : \exists C, R > 0 \forall x \in X \ (|x| > R \Rightarrow |f(x)| \leq C|x|^\theta) \}.
\]

Set \( \text{supp} f = \{ x \in X : f(x) \neq 0 \} \).

A curve \( \varphi : [r, +\infty) \to M \) is called meromorphic at \( +\infty \) if \( \varphi \) is the sum of a Laurent series of the form

\[
\varphi(t) = a_p t^p + a_{p-1} t^{p-1} + \cdots, \quad a_i \in M, \quad p \in \mathbb{Z}.
\]

In the case of a polynomial function and the Laurent series at infinity, the above degree is the usual degree; that is, \( \deg \varphi = p \) if \( a_p \neq 0 \), and \( \deg \varphi = -\infty \) if \( \varphi \equiv 0 \).

**Proposition 2.3.** Let \( X \) be a closed semi-algebraic set and let \( f : X \to N \) be a semi-algebraic mapping. Then:

(i) \( \deg f \in \mathbb{Q} \cup \{ -\infty \} \).

(ii) \( \deg f = -\infty \) if and only if \( \text{supp} f \) is bounded.
(iii) If $\deg f \in \mathbb{Q}$, then there exist $C, R > 0$ such that

$$|f(x)| \leq C|x|^{\deg f} \quad \text{for } x \in X, \quad |x| \geq R.$$  

(iv) Let $\beta(f) = \min\{n \in \mathbb{Z} : n > 0, n \geq \deg f\}$. Then there exist $R > 0$ and $\alpha < 0$ such that

$$|f(x)| \leq (1 + |x|^2)^{\beta(f)}|x|^\alpha \quad \text{for } x \in X, \quad |x| > R.$$  

Proof. If $\text{supp } f$ is bounded, then the assertion is obvious. Assume that $\text{supp } f$ is unbounded. Then the set

$$Y = \{(y, f(y)) \in X \times N : \forall x \in X \ |x| = |y| \Rightarrow 2|f(y)| \geq |f(x)|\}$$

is unbounded and semi-algebraic. So, by the Curve Selection Lemma at infinity, there exists a curve $\psi = (\varphi, \eta) : [r, +\infty) \to Y$ meromorphic at $+\infty$ such that $\eta = f \circ \varphi$, $\deg \eta \in \mathbb{Z}$, and $\deg \varphi > 0$. Let $\theta = \deg \eta / \deg \varphi$. Then $\theta \in \mathbb{Q}$ and for some $C, D, R > 0$,

$$C|\varphi(t)|^\theta \leq |f(\varphi(t))| \leq D|\varphi(t)|^\theta, \quad t > R. \tag{2.2}$$

The definition of $Y$ now implies that for $x \in X, \ |x| = |\varphi(t)|, \ t > R$,

$$|f(x)| \leq |f(\varphi(t))| \leq D|\varphi(t)|^\theta = D|x|^\theta.$$ 

So, $\deg f \leq \theta$. Since, by (2.2), $\deg f \geq \theta$, it follows that $\deg f = \theta$. This gives (i), (ii) and (iii). Part (iv) follows immediately from (iii). \qed

2.2. $C^1$ semi-algebraic functions. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a semi-algebraic function of class $C^1$ in $x = (x_1, \ldots, x_n)$. Then the gradient $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) : \mathbb{R}^n \to \mathbb{R}^n$ is a semi-algebraic mapping.

Proposition 2.4. There exist $C, \delta, R > 0$ such that

$$|f(x)| \geq R \Rightarrow |x| \|\nabla f(x)\| \geq C|f(x)|, \tag{2.3}$$

$$|f(x)| \leq \delta \Rightarrow |x| \|\nabla f(x)\| \geq C|f(x)|. \tag{2.4}$$

In particular, the set $K_\infty(f)$ is finite. The assertion also holds for complex polynomials.

Proof. As in [19] and [6], we use Hörmander’s method. To prove (2.3), assume the contrary. Then the semi-algebraic set

$$X = \{(x, y, z, \varepsilon) \in \mathbb{R}^{2n} \times \mathbb{R}^2 : y = \nabla f(x), \ z = f(x), \ |z| \geq \varepsilon, \ \varepsilon|y||x| < |z|\}$$

has an accumulation point of the form $(x_0, y_0, z_0, +\infty)$. Thus, by the Curve Selection Lemma at infinity there exists a curve $\psi = (\varphi, \tau, \eta_1, \eta_2) : [r, +\infty) \to X$ meromorphic at infinity such that $\psi(t) \to (x_0, y_0, z_0, +\infty)$ as $t \to +\infty$. Then $\deg \eta_2 > 0, \ \deg \eta_1 > 0, \ \deg \varphi > 0$, and

$$\deg \eta_2 + \deg \tau + \deg \varphi \leq \deg \eta_1.$$ 

On the other hand,

$$\deg \eta_1 = \deg \eta_1' + 1 = \deg (f \circ \varphi)' + 1 \leq \deg \tau + \deg \varphi,$$

and we obtain a contradiction. Analogously we prove (2.4) and the assertion in the complex case. \qed
2.3. **Lojasiewicz exponent.** For three semi-algebraic sets $X, Y, Z \subset M$ such that $X \cap Y \subset Z$, we define a *regular separation exponent* of $Y$ and $Z$ on $X$ at a point $x_0 \in X \cap Y$ to be any real positive $\theta$ such that

$$g(x, Y) \geq C \varrho(x, Z)^\theta \quad \text{for } x \in X \cap \tilde{\Omega},$$

where $C > 0$ and $\tilde{\Omega}$ is a neighbourhood of $x_0$. The infimum of all such exponents $\theta$ will be denoted by $\mathcal{L}_{x_0}(X; Y, Z)$. By using the method of Lipschitz stratifications ([13, 14]), the following is proved in Theorem 1.5 of [20]:

**Proposition 2.5.** Let $X, Y, Z \subset M$ be closed semi-algebraic sets such that $X \cap Y \subset Z$, and let $x_0 \in X \cap Y$.

(i) Then $\mathcal{L}_{x_0}(X; Y, Z) \in \mathbb{Q}$, and (2.2) holds for $\theta = \mathcal{L}_{x_0}(X; Y, Z)$, some $C > 0$ and a neighbourhood $\Omega$ of $x_0$, provided $0^\theta = 0$.

(ii) If $x_0 \in X \setminus Z$, then $\mathcal{L}_{x_0}(X; Y, Z)$ is attained on an analytic curve, i.e. for any neighbourhood $\tilde{\Omega}$ of $x_0$ there exists an analytic curve $\varphi : [0, r) \to X \cap \tilde{\Omega}$ such that $\varphi((0, r)) \subset X \setminus Z$ and $\varphi(0) \in X \cap Y$, and for some constant $C_1 > 0$,

$$C g(\varphi(t), Z)^{\mathcal{L}_{x_0}(X; Y, Z)} \leq g(\varphi(t), Y) \leq C_1 g(\varphi(t), Z)^{\mathcal{L}_{x_0}(X; Y, Z)}, \quad t \in [0, r).$$

If $Z = X \cap Y$ and $x_0 \in \overline{X \setminus Y}$, then obviously $\mathcal{L}_{x_0}(X; Y, Z)$ is equal to the Lojasiewicz exponent $\mathcal{L}_{x_0}(X, Y)$ of $X$ and $Y$ at $x_0$, i.e. the optimum exponent $\theta$ in the following separation condition:

$$g(x, X) + g(x, Y) \geq C \varrho(x, X \cap Y)^\theta \quad \text{for } x \in \tilde{\Omega},$$

considered in a neighbourhood $\Omega \subset M$ of $x_0$ for some constant $C > 0$. Note that Proposition 2.5 also holds in the subanalytic case.

2.4. **Stratification.** By *stratification* of a subset $X \subset M$ we mean a decomposition of $X$ into a locally finite disjoint union

$$X = \bigcup S_\alpha,$$

where the subsets $S_\alpha$ are called *strata*, such that each $S_\alpha$ is a connected embedded submanifold of $M$, and each $(S_\alpha \setminus S_{\alpha'}) \cap X$ is the union of some strata of dimension smaller than $\dim S_\alpha$.

The *$i$-th skeleton* of the stratification (2.5) is

$$X^i = \bigcup_{\dim S_\alpha \leq i} S_\alpha.$$

The stratification (2.5) is called *semi-algebraic* if all the skeletons $X^i$ are semi-algebraic sets (or equivalently if the number of strata is finite and they are all semi-algebraic). The stratification (2.5) of a complex algebraic subset $X$ of a complex linear space $M$ is called *complex algebraic* if all the skeletons $X^i$ are complex algebraic subsets of $M$ and the number of strata is finite.

By Corollaries 2.6 and 2.7 in [20] we have:

**Proposition 2.6.** Let $X, Y, Z \subset M$ be closed semi-algebraic sets such that $X \cap Y \subset Z$. Then there exists a stratification

$$X \cap Y = S_1 \cup \cdots \cup S_k$$

der of $X \cap Y$ such that the function

$$X \cap Y \ni x \mapsto \mathcal{L}_x(X; Y, Z)$$
is constant on each stratum $S_i$. In particular, the function \(2.7\) is upper semi-continuous. If additionally $X_1, \ldots, X_n \subset X \cap Y$ are semi-algebraic sets, then one can require that the stratification \(2.6\) is compatible with any $X_j$, i.e. any $X_j$ is a union of some strata $S_i$.

3. Proof of Theorem 1.2

Let $X \subset M$ be a closed semi-algebraic set, and let $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings. The values $\vartheta_{g/f}(\lambda) \in \{-\infty, +\infty\}$ are characterised by the following:

Remark 3.1. (i) By Proposition \(2.3\) and the definition of $S_f$ we have:
\begin{equation}
\vartheta_{g/f}(\lambda) = +\infty \iff \lambda \in L \setminus S_f.
\end{equation}

(ii) Let $h = f|_g^{-1}(0)$. From the definition of $L_{\infty, f \to \lambda}(g)$ we have:
\begin{equation}
\vartheta_{g/f}(\lambda) = -\infty \iff \lambda \in S_h \iff L_{\infty}(f - \lambda|_g^{-1}(0)) < 0.
\end{equation}

Before the proof of Theorems 1.2 we give four lemmas and a proposition. Let $B = \{z \in M : |z| < 1\}$ and let $H : B \to M$ be of the form
\[
H(z) = \frac{z}{1 - |z|^2}.
\]

Lemma 3.2. The mapping $H$ is semi-algebraic and invertible with inverse
\[
H^{-1}(x) = \frac{2x}{1 + \sqrt{1 + 4|x|^2}}.
\]

Moreover, for any $R > 0$,
\begin{equation}
|H(z)| \geq R \iff \frac{2R}{1 + \sqrt{1 + 4R^2}} \leq |z| < 1.
\end{equation}

Proof. $H$ is a semi-algebraic mapping as the restriction of a rational mapping to the semi-algebraic set $B$. By an easy calculation we obtain \(3.3\) and the formula for $H^{-1}$. \hfill \square

By Lemma 3.2 we may define the following semi-algebraic sets:
\[
Y = \{(x, \lambda, \delta) \in X \times L \times \mathbb{R} : |f(x) - \lambda| \leq \delta\},
\]
\[
Z_1 = \{(z, \lambda, \delta) \in B \times L \times \mathbb{R} : (H(z), \lambda, \delta) \in Y\},
\]
\[
Z_2 = \partial B \times L \times \mathbb{R},
\]
\[
Z = Z_1 \cup Z_2.
\]

Let $V = g^{-1}(0)$, and let
\[
W = \{(z, \lambda, \delta) \in Z_1 : H(z) \in V\}.
\]

Define a mapping $F : Z \to \mathbb{R}$ by
\[
F(z, \lambda, \delta) = (1 - |z|^2)\vartheta((z, \lambda, \delta), W).
\]

Since $W$ is a semi-algebraic set, Proposition \(2.1\) implies that $F$ is a semi-algebraic mapping.

For any $\lambda \in L$, $\delta \geq 0$ and $S \subset X$ we set
\[
S_{\lambda, \delta} = \{x \in S : |f(x) - \lambda| \leq \delta\}.
\]
Lemma 3.3. Let \( \lambda_0 \in L \) and \( \delta_0 > 0 \) be such that the set \( V_{\lambda_0,\delta_0} \) is bounded, and suppose \( X_{\lambda_0,\delta} \) is unbounded for any \( \delta > 0 \). Then there exist \( C, D, R > 0 \) such that for any \( (x, \lambda, \delta) \in Y \), where \( 0 < \delta \leq \frac{\delta_0}{2} \) and \( |\lambda - \lambda_0| \leq \delta \), we have
\[
(3.4) \quad C|x|^{-1} \leq F(H^{-1}(x), \lambda, \delta) \leq D|x|^{-1}, \quad x \in X_{\lambda_0,\delta}, \quad |x| \geq R.
\]

Proof. Let \( Z^\delta = \{(z, \lambda, \delta) \in Z_1 : |\lambda - \lambda_0| \leq \delta \} \). Then \( Z^\delta \subset Z^{\delta'} \) if \( \delta' \leq \delta'' \). By the definition of \( F \) we have
\[
F(z, \lambda, \delta) = |H(z)|^{-1}|z| \varrho((z, \lambda, \delta), W) \quad \text{for} \quad (z, \lambda, \delta) \in Z_1, \quad z \neq 0.
\]
Hence, by (3.3), it suffices to prove that for some \( c, d, r > 0 \), with \( r < 1 \), and \( \delta_1 = \frac{\delta_0}{2} \),
\[
(3.5) \quad c \leq |z| \varrho((z, \lambda, \delta), W) \leq d \quad \text{for} \quad (z, \lambda, \delta) \in Z^{\delta_1}, \quad r \leq |z| < 1.
\]
Because \( Z^{\delta_1} \) is bounded, the set \( \{|z| \varrho((z, \lambda, \delta), W) : (z, \lambda, \delta) \in Z_{\lambda_0,\delta_1} \} \) is also bounded. Hence the right-hand estimate in (3.5) holds. By (3.3) and the assumptions on \( V_{\lambda_0,\delta_0} \) and \( X_{\lambda_0,\delta} \), there exists \( 0 < r < 1 \) for which the set \( W \) has no accumulation points in \( A = \{(z, \lambda, \delta) \in Z^{\delta_1} : r \leq |z| \} \). Moreover, \( A \) is bounded, so \( c = \inf\{|z| \varrho((z, \lambda, \delta), W) : (z, \lambda, \delta) \in A\} > 0 \). This gives the left-hand estimate in (3.5).

Let \( X_H = H^{-1}(X) \cup \partial B \) and \( V_H = H^{-1}(V) \). Since \( g \) and \( H \) are semi-algebraic mappings the sets \( V, X_H, V_H \) are semi-algebraic. Moreover, \( X_H \) is closed and \( V_H = (g \circ H)^{-1}(0) \). Define \( g_H : X_H \to N \) by
\[
g_H(z) = \begin{cases} 
\frac{g \circ H(z)}{(1 + |H(z)|^2)^{\beta(g)}} & \text{for} \ z \in X_H \cap B, \\
0 & \text{for} \ z \in \partial B,
\end{cases}
\]
where \( \beta(g) \) is defined in Proposition 2.3 (iv).

Lemma 3.4. The mapping \( g_H \) is continuous, semi-algebraic and
\[
(3.6) \quad (g_H)^{-1}(0) = V_H \cup \partial B.
\]

Proof. By 2.1 in Proposition 2.3 \( g_H \) is continuous. Since the mapping \( g \) is semi-algebraic, so is \( B \ni x \mapsto g \circ H(x) \), and hence also \( h : (X_H \cap B) \ni z \mapsto (g(z), (1 + |H(z)|^2)^{\beta(g)}) \in N \times \mathbb{R} \). The graph of \( g_H \) is the union of \( \partial B \times \{0\} \) and the image of the graph \( h \) under the semi-algebraic mapping \( M \times N \times (0, +\infty) \ni (z, y, t) \mapsto (z, \frac{1}{t} y) \in M \times N \), so the graph of \( g_H \) is semi-algebraic. The equality (3.6) is obvious.

The set \( Z \) is semi-algebraic and \( X_H \) is its image under the projection map \( Z \ni (z, \lambda, \delta) \mapsto z \in M \). Hence, we may define a semi-algebraic mapping \( G : Z \to N \) by
\[
G(z, \lambda, \delta) = g_H(z).
\]
Let \( \Gamma \) be the graph of the semi-algebraic mapping \( (G, F) : Z \to N \times \mathbb{R} \). Since \( Z \) is a closed set, so is \( \Gamma \).

Lemma 3.5. There exists a stratification
\[
(3.7) \quad G^{-1}(0) = S_1 \cup \cdots \cup S_j
\]
such that the function
\[
(3.8) \quad \mathcal{L} : G^{-1}(0) \ni v \mapsto \mathcal{L}_{(v,0,0)}(\Gamma; Z \times \{0\} \times \mathbb{R}, Z \times N \times \{0\})
\]
is constant on each stratum $S_i$. In particular, the set of values of $\mathfrak{L}$ is a finite subset of $\mathbb{Q}$.

Proof. By (3.6), $G^{-1}(0) = F^{-1}(0)$, so
\[ G^{-1}(0) \times \{0\} \times \{0\} = \Gamma \cap (Z \times \{0\} \times \mathbb{R}) \subset Z \times N \times \{0\}. \]
Proposition 2.5 now shows that the values of $\mathfrak{L}$ are rational numbers. Moreover, from Proposition 2.6 we obtain a stratification (3.7) satisfying the assertion. \qed

Take any $\lambda_0 \in L$ and define
\[ l_{\lambda_0}(g) = \max\{\mathfrak{L}(z, \lambda_0, 0) : (z, \lambda_0, 0) \in Z_2\}. \]
By Lemma 3.5, $l_{\lambda_0}(g) \in \mathbb{Q}$.

**Proposition 3.6.** Let $\delta_0 > 0$ be such that the set $V_{\lambda_0, \delta_0}$ is bounded, and suppose the set $X_{\lambda_0, \delta}$ is unbounded for any $\delta > 0$. Then
\[ \mathcal{L}_{f \rightarrow \lambda_0}(g) = 2\beta(g) - l_{\lambda_0}(g) \]
and for any sufficiently small $0 < \delta \leq \frac{\delta_0}{2}$ there exist $C, C', R > 0$ such that
\[ |g(x)| \geq C|x|^{2\beta(g) - l_{\lambda_0}(g)} \text{ for } x \in X_{\lambda_0, \delta}, \quad |x| \geq R \]
and
\[ C'|\varphi(t)|^{2\beta(g) - l_{\lambda_0}(g)} \geq |g(\varphi(t))| \geq C|\varphi(t)|^{2\beta(g) - l_{\lambda_0}(g)}, \quad t \in [r, +\infty), \]
for some curve $\varphi : [r, +\infty) \rightarrow X_{\lambda_0, \delta}$ meromorphic at $+\infty$, with $\deg \varphi > 0$.

**Proof.** Let $E = \{(z, \lambda, \delta) \in Z_2 : \lambda = \lambda_0, \delta = 0\}$ and $\alpha = l_{\lambda_0}(g)$. By the definition of $l_{\lambda_0}(g)$, for any $(z, \lambda_0, 0) \in E$ there exist a neighbourhood $\Omega_\varepsilon \subset M \times L \times \mathbb{R}$ of $(z, \lambda_0, 0)$ and $C_\varepsilon > 0$ such that
\[ |G(y, \lambda, \delta)| \geq C_\varepsilon|F(y, \lambda, \delta)|^\alpha, \quad (y, \lambda, \delta) \in \Omega_\varepsilon \cap Z. \]
Since the set $E$ is compact, there exists $\tilde{C} > 0$ such that $C_\varepsilon \geq \tilde{C}$ for $(z, \lambda_0, 0) \in E$, and there exist $0 < r_1 < 1$ and $0 < \delta_0 \leq \frac{\delta_0}{2}$ such that
\[ |G(y, \lambda, \delta_1)| \geq \tilde{C}|F(y, \lambda, \delta_1)|^\alpha, \quad |\lambda - \lambda_0| \leq \delta_1, \quad r_1 \leq |y| < 1, \]
where $(y, \lambda, \delta_1) \in Z$. Consequentially,
\[ \frac{|g(x)|}{(1 + |x|^2)^{\beta(g)}} \geq \tilde{C}|F(H^{-1}(x), \lambda_0, \delta_1)|^\alpha, \quad x \in X_{\lambda_0, \delta_1}, \quad |x| \geq R, \]
where $R > 0$ is the unique solution of the equation $r_1 = \frac{2R}{1 + \sqrt{4 + 4R^2}}$. Together with (3.3) this gives
\[ |g(x)| \geq \tilde{C}C(1 + |x|^2)^{\beta(g)}|x|^{-\alpha} \text{ for } x \in X_{\lambda_0, \delta_1}, \quad |x| \geq R. \]
Hence for any $0 < \delta \leq \delta_1$, (3.10) follows.

Take any $0 < \delta \leq \delta_1$. Let $(z_0, \lambda_0, 0) \in Z_2$ be a point such that $\mathfrak{L}(z_0, \lambda_0, 0) = l_{\lambda_0}(g)$. By the assumption on $V_{\lambda_0, \delta_0}$ we have
\[ (z_0, \lambda_0, 0, 0, 0) \in \tilde{\Gamma} \setminus (Z \times N \times \{0\}), \quad (z_0, \lambda_0, 0) \notin \overline{W}, \]
and $\mathfrak{L}(z_0, \lambda_0, 0) > 0$. Thus, by Proposition 2.5 for any sufficiently small neighbourhood $\tilde{\Omega}$ of $\omega = (z_0, \lambda_0, 0, 0) \in (z_0, \lambda_0, 0, 0, 0)$ there exists an analytic curve
\[ \psi = (\psi_1, \psi_2, \psi_3) : [0, r) \rightarrow \Gamma \cap \tilde{\Omega}, \]
where \( \psi_1 : [0, r) \to Z, \psi_2 = G \circ \psi_1 : [0, r) \to N, \psi_3 = F \circ \psi_1 : [0, r) \to \mathbb{R}, \)
\( \psi((0, r)) \subset \Gamma \setminus (Z \times N \times \{0\}) \) and \( \psi(0) \in \Gamma \cap (Z \times \{0\} \times \mathbb{R}) \), such that for some constant \( C_1 > 0 \),
\begin{equation}
\rho(t, Z \times \{0\} \times \mathbb{R}) \leq C_1 \rho(t, Z \times M \times \{0\})^\alpha \quad \text{for} \ t \in [0, r).
\end{equation}
Let \( \varphi_1 : [0, r) \to M, \varphi_2 : [0, r) \to L, \varphi_3 : [0, r) \to \mathbb{R} \), and let \( \psi_1 = (\varphi_1, \varphi_2, \varphi_3) \). By
the choice of \( \psi \) we have \( \varphi_1(t) \in B \) for \( t \in (0, r) \), and \( \varphi(0) \in \partial B \) by (3.12). Hence,
\begin{equation}
|H(\varphi_1(t))| \to \infty \quad \text{as} \quad t \to 0.
\end{equation}
Since the neighbourhood \( \tilde{\Omega} \) of \( \omega \) can be small, one can assume that \( 0 \leq \varphi_3(t) < \delta \)
for \( t \in [0, r) \). Then, by the definition of \( Z \), we have \( |H(\varphi_1(t)) - \lambda_0| \leq \varphi_3(t) < \delta \)
for \( t \in (0, r) \), and so
\begin{equation}
H(\varphi_1(t)) \in X_{\lambda_0, \delta} \quad \text{for} \ t \in (0, r).
\end{equation}
By (3.13),
\begin{equation}
|G(\psi_1(t))| \leq C_1|F(\psi_1(t))|^\alpha \quad \text{for} \ t \in [0, r].
\end{equation}
Hence, from (3.3) and (3.14), for some \( 0 < r_1 < r \),
\begin{equation}
|g(H(\varphi_1(t)))| = \frac{|g(H(\varphi_1(t)))|}{(1 + |H(\varphi_1(t))|^2)^{\beta(g)}} \leq C_1 D_{\lambda_0} \rho(\varphi_1(t))^{1 - \alpha}, \quad t \in (0, r_1).
\end{equation}
Together with (3.11) and (3.15), this gives
\begin{equation}
|g(H(\varphi_1(t)))| \leq \frac{C_1 |H(\varphi_1(t))|^{2\beta(g)} - \alpha}{t \in (0, r_1)}
\end{equation}
for some \( C' > 0 \). Now setting \( \varphi(t) = H(\varphi_1(t)) \) for \( t \in \left[ \frac{1}{r_1}, +\infty \right) \) we obtain (3.11). Finally, (3.11) and (3.10) yield (3.9).

**Proof of Theorem 3.2.** Fix \( \lambda_0 \in L \). First we prove (i). If for any \( \delta > 0 \) the set \( V_{\lambda_0, \delta} \) is unbounded, then \( L_{\infty, f - \lambda_0}(g) = -\infty \). If for some \( \delta > 0 \) the set \( X_{\lambda_0, \delta} \) is bounded, then \( L_{\infty, f - \lambda_0}(g) = +\infty \). The remaining case in (i) follows from the fact that \( \beta(g) \in Z \) (see Proposition 2.2) and from (3.9) in Proposition 3.6.

To prove (ii), we adopt the method of the proof of Theorem 3.2.2 in [18]. By Lemma 3.5 let
\( \partial g/f(L) = \{r_1, \ldots, r_s\} \subset \mathbb{Q} \cup \{-\infty, +\infty\} \), where \( r_1 \leq \cdots \leq r_s \).
Define \( \Lambda_\xi = \{\lambda \in L : L_{\infty, f - \lambda}(g) \leq \xi\} \) for \( \xi \in \mathbb{R} \).
Fix \( r_1 \). We now prove that the set \( \Lambda_{r_1} \) is closed and semi-algebraic. If \( r_1 \in (-\infty, +\infty) \) this follows from Remark 3.1 and Proposition 2.2. So, let \( r_1 = \frac{a}{b} \), where \( a, b \in \mathbb{Z} \) and \( b > 0 \). Define
\begin{equation}
T = \{(x, c) \in X \times \mathbb{R} : |g(x)|^b = c|x|^a\},
\end{equation}
and let \( p : T \ni (x, c) \mapsto (f(x), c) \in L \times \mathbb{R} \). Since the mapping \( p \) is semi-algebraic, Proposition 2.2 shows that the set \( \mathcal{E}_p \) is also semi-algebraic.
Let \( \pi : L \times \mathbb{R} \ni (y, c) \mapsto y \in L \) and observe that
\begin{equation}
\Lambda_{r_1} = \pi(\mathcal{E}_p).
\end{equation}
Indeed, let \( \lambda \in \Lambda_{r_1} \), and let \( U \subset L \) be a neighbourhood of \( \lambda \). Take a neighbourhood \( U_1 \subset L \) of \( \lambda \) such that \( U_1 \subset U \). Then, by Proposition 3.6 there exist \( C' > 0 \) such that the set
\begin{equation}
\{x, y) \in f^{-1}(U_1) \times N : \ y = g(x), |y|^b \leq C'|x|^a\}
\end{equation}

Proof. The assertion follows immediately from (3.10), (3.11) and Theorem 1.2.

Then, for some \( \lambda' \in U \subset U \) and \( 0 \leq c \leq C' \),
\[
f \circ \varphi(t) \to \lambda' \quad \text{and} \quad \frac{|g(\varphi(t))|_b}{|\varphi(t)|_a} \to c \quad \text{as} \ t \to \infty.
\]
Hence, \( \lambda' \in \pi(\mathcal{S}_p) \cap U \), and so \( \lambda \in \pi(\mathcal{S}_p) \).

Now let \( \lambda \in \pi(\mathcal{S}_p) \). Take any neighbourhood \( U \subset \lambda \) of \( \lambda \), and let \( \lambda' \in U \) and \( c \in \mathbb{R} \) be such that \( (\lambda', c) \in \mathcal{S}_p \). Then for some sequence \( (x_n, c_n) \in T \), where \( x_n \in f^{-1}(U) \) and \( c_n \in \mathbb{R} \) for \( n \in \mathbb{N} \), we have
\[
|x_n| \to \infty, \quad f(x_n) \to \lambda' \quad \text{and} \quad c_n \to c \quad \text{as} \ n \to \infty.
\]
Hence, there exists \( C > 0 \) such that \( |c_n| \leq C \) for \( n \in \mathbb{N} \), and so
\[
|g(x_n)|_b \leq C|x_n|_a, \quad n \in \mathbb{N}.
\]
This gives \( L_\infty(g|f^{-1}(U)) \leq r_i \), and hence \( L_\infty,g,\lambda (g) \leq r_i \). Summing up, \( \lambda \in \Lambda_i \), and (3.16) is proved.

By Proposition 2.2, the set \( \mathcal{S}_p \) is semi-algebraic, so, by (3.10), \( \Lambda_i \) is closed and semi-algebraic. In particular, the function \( \vartheta_{g/f} \) is upper semi-continuous. From the definition of \( \Lambda_i \), we have \( \Lambda_i \supset \cdots \supset \Lambda_{i-1} = L \). Hence, \( \Lambda_i \) is semi-algebraic for any \( \xi \in \mathbb{R} \). Therefore there exists a semi-algebraic stratification of the form (1.1) compatible with any intersection \( X_1 \cap \cdots \cap X_j \), where \( X_1, \ldots, X_j \in \{ \Lambda_{i_1}, \ldots, \Lambda_{i_r} \} \).

Thus, the function \( \vartheta_{g/f} \) is constant on each stratum \( S_i \), and Theorem 1.2 is proved.

\[\square\]

Corollary 3.7. If \( \theta = L_\infty,f,\lambda (g) \in \mathbb{Q} \), then for some \( C, C', R, \delta > 0 \),
\[
|g(x)| \geq C|x|^{\theta} \quad \text{for} \ x \in X, \ |x| \geq R, \ |f(x) - \lambda| < \delta,
\]
\[
|C'||\varphi(t)|^{\theta} \geq |g(\varphi(t))| \geq |C|\varphi(t)|^{\theta} \quad \text{for} \ t \in [r, +\infty),
\]
where \( \varphi : [r, +\infty) \to X \) is a curve meromorphic at infinity such that \( \deg \varphi > 0 \) and \( |f(\varphi(t)) - \lambda| < \delta \) for \( t \in [r, +\infty) \).

Proof. The assertion follows immediately from (3.10), (3.11) and Theorem 1.2. \[\square\]

4. PROOF OF COROLLARY 1.0

Let \( (z_1, \ldots, z_n), (y_1, \ldots, y_m) \) be the coordinates of \( z \in \mathbb{C}^n, y \in \mathbb{C}^m \), respectively.

As in the proof of Theorem 1.2, we now show that for any \( \xi \in \mathbb{Q} \cup \{-\infty, +\infty\} \), the set \( \Lambda_\xi = \{ \lambda \in \mathbb{C}^k : L_\infty,f,\lambda (g) \leq \xi \} \) is complex algebraic. For \( \xi \in \{-\infty, +\infty\} \), this is obvious. Fix \( \xi = \frac{a}{b} \), where \( a, b \in \mathbb{Z}, b > 0, (a, b) = 1 \).

Let \( g = (g_1, \ldots, g_m) \). For any \( i = 1, \ldots, n \) we define algebraic sets
\[
T_\xi^j = \{(z, y, u) \in X \times \mathbb{C}^m \times \mathbb{C} : z_iu = 1, \ g^j(z) = y_jz^a, \ j = 1, \ldots, m \}
\]
if \( \xi \geq 0 \),
\[
T_\xi^j = \{(z, y, u) \in X \times \mathbb{C}^m \times \mathbb{C} : z_iu = 1, \ g^j(z)x^{-a} = y_j, \ j = 1, \ldots, m \}
\]
if \( \xi < 0 \), and mappings
\[
p_i : T_\xi^j \ni (z, y, u) \mapsto (f(z), y, u) \in \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C}.
\]
Denote by \( \mathcal{S}_i \) the set of points at which \( p_i \) is not proper, and
\[
A_i = \mathcal{S}_i \cap \{(\lambda, y, u) \in \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C} : u = 0\}, \quad i = 1, \ldots, n.
\]
Since each \( \mathcal{S}_i \) is algebraic, so is \( A_i \).

Let \( \pi : \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C} \ni (\lambda, y, u) \mapsto \lambda \in \mathbb{C}^k \) and observe that
\[
(4.1) \quad \Lambda_\xi = \bigcup_{i=1}^n \pi(A_i).
\]

Indeed, let \( \lambda \in \mathbb{C}^k \) satisfy \( \mathcal{L}_{\infty, f \to \lambda}(g) \leq \xi \). Take any neighbourhoods \( U, W \subset \mathbb{C}^k \) of \( \lambda \) such that \( W \subset U \). By Corollary 3.7 there exist \( C > 0 \) and a curve \( \varphi = (\varphi_1, \ldots, \varphi_n) : [r, +\infty) \to f^{-1}(W) \) meromorphic at infinity with \( \deg \varphi > 0 \) such that
\[
(4.2) \quad |g(\varphi(t))| \leq C|\varphi(t)|^\xi, \quad t \in [r, \infty).
\]
Let \( \deg \varphi_i = \deg \varphi \). Then \( \deg \varphi_i > 0 \). By the definition of \( \varphi \), there exists \( \lambda' \in W \) such that
\[
(4.3) \quad f(\varphi(t)) \to \lambda' \quad \text{as} \quad t \to \infty.
\]
By (4.2), there exists \( y \in \mathbb{C}^m \) such that
\[
\eta(t) := \left( \frac{g^1(\varphi(t))}{\varphi^1(t)}, \ldots, \frac{g^m(\varphi(t))}{\varphi^m(t)} \right) \to y \quad \text{as} \quad t \to \infty.
\]
Since \( \deg \varphi_i > 0 \), we may assume that \( \varphi_i(t) \neq 0 \) for \( t \in [r, +\infty) \). Putting \( u(t) = \frac{1}{\varphi_i(t)} \) for \( t \in [r, +\infty) \), we easily see that
\[
p_i(\varphi(t), \eta(t), u(t)) \to (\lambda', y, 0) \quad \text{as} \quad t \to \infty.
\]
Hence \( (\lambda', y, 0) \in \mathcal{S}_i \), so \( \lambda' \in U \cap \pi(A_i) \), and thus \( \lambda \in \overline{\pi(A_i)} \). This gives the inclusion “\( \supseteq \)” in (4.1).

We now prove “\( \subseteq \)”.

Let \( \varphi_i = (\varphi_{i,1}, \ldots, \varphi_{i,l}) \in \mathcal{S}_i \), for some \( y = (y_1, \ldots, y_m) \in \mathbb{C}^m \). The definitions of \( A_i \) and \( T_{\xi}^k \) now yield a sequence \( x_l = (x_{1,l}, \ldots, x_{n,l}) \in f^{-1}(U) \), \( l \in \mathbb{N} \), such that \( f(x_l) \to \lambda' \) and
\[
|x_{i,l}| \to \infty, \quad \frac{g^j(x_l)}{x_{i,l}^\alpha} \to y_j \quad \text{as} \quad l \to \infty, \quad j = 1, \ldots, m.
\]
Consequently, there exists \( C > |y| \) such that
\[
|g(x_l)| \leq C|x_l|^\xi \quad \text{for} \quad l \in \mathbb{N}.
\]
Hence, \( \mathcal{L}_{\infty}(g|f^{-1}(U)) \leq \xi \). This gives \( \mathcal{L}_{\infty, f \to \lambda}(g) \leq \xi \), and the inclusion “\( \subseteq \)” in (4.1) is proved.

By Theorem 1.2 the set \( \vartheta_{g/f}(\mathbb{C}^k) \subset \mathbb{Q} \cup \{-\infty, +\infty\} \) is finite, say \( \{r_1, \ldots, r_s\} \) with \( r_1 < \cdots < r_s \). By (4.1), the sets \( \Lambda_{r_i}, \quad i = 1, \ldots, s \), are algebraic, and \( \Lambda_{r_i} \varsubsetneq \cdots \varsubsetneq \Lambda_{r_{i-1}} = \mathbb{C}^k \). Then the function \( \vartheta_{g/f} \) is upper semi-continuous. Hence the usual complex stratification of \( \mathbb{C}^n \) compatible with complex constructible sets \( \Lambda_{r_i} \setminus \Lambda_{r_{i-1}} \) is a desired stratification. This ends the proof. \( \square \)
References


Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, Poland
E-mail address: rodakt@math.uni.lodz.pl

Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, Poland
E-mail address: spodziej@math.uni.lodz.pl