LOJASIEWICZ EXPONENT NEAR THE FIBRE OF A MAPPING

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(Communicated by Ted Chinburg)

Abstract. Let $g : X \to \mathbb{R}^k$ and $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$, be continuous semi-algebraic mappings, and $\lambda \in \mathbb{R}^m$. We describe the optimal exponent $\theta = L^\infty, f \to \lambda (g)$ for which the Lojasiewicz inequality $|g(x)| \geq C|x|^\theta$ holds with $C > 0$ as $|x| \to \infty$ and $f(x) \to \lambda$. We prove that there exists a semi-algebraic stratification $\mathbb{R}^m = S_1 \cup \cdots \cup S_j$ such that the function $\lambda \mapsto L^\infty, f \to \lambda (g)$ is constant on each stratum $S_i$. We apply this result to describe the set of generalized critical values of $f$.

Introduction

Let $M, N, L$ be finite-dimensional real vector spaces, $X \subset M$ be a closed semi-algebraic set, $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings (see [1]), and let $\lambda \in L$. The aim of this article is to describe the Lojasiewicz exponent at infinity of $g$ near the fibre $f^{-1}(\lambda)$, i.e. the supremum of the exponents $\theta$ for which the Lojasiewicz inequality

\[(L) \quad |g(x)| \geq C|x|^\theta \quad \text{as} \quad x \in X, \quad |x| \to \infty \quad \text{and} \quad f(x) \to \lambda \]

holds with $C > 0$ (cf. [12], [18]), where $|\cdot|$ is a norm. We denote this exponent by $L^\infty, f \to \lambda (g)$ (see Section 1 for details).

We prove that $L^\infty, f \to \lambda (g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$ for $\lambda \in L$ and that there exists a semi-algebraic stratification $\mathbb{R}^m = S_1 \cup \cdots \cup S_j$ such that the function $\lambda \mapsto L^\infty, f \to \lambda (g)$ is constant on each stratum $S_i$ (Theorem 1.2). If $g$ and $f$ are complex regular mappings, the stratification is complex algebraic (Corollary 1.6). Note that if $\theta = L^\infty, f \to \lambda (g) \in \mathbb{Q}$, then (L) holds (Corollary 1.3). The key points in the proofs are Lipschitz stratifications ([13], [14], [20]) and properties of the set of points at which a mapping is not proper ([8]; see also Section 2).

If $f : M \to L$ is a semi-algebraic mapping of class $C^1$, we define the Lojasiewicz exponent of $df$ near the fibre $f^{-1}(\lambda)$ by

$$L^\infty, \lambda (f) = L^\infty, f \to \lambda (\nu(df)),$$

where $\nu$ is a function introduced by Rabier [17] (see Section 1). This notion was introduced by Ha [17] in the case of complex polynomial functions in two variables (see also [3], [5]).

Received by the editors May 19, 2009 and, in revised form, April 19, 2010.

2010 Mathematics Subject Classification. Primary 14R25; Secondary 58K55, 58K05.

Key words and phrases. Lojasiewicz exponent at infinity, generalized critical values, stratification.

This research was partially supported by the program POLONIUM.

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Let us recall that the exponent $\mathcal{L}_{\infty, \lambda}(f)$ is strongly related to the set of bifurcation points of $f$. Namely, one can define the set of generalized critical values of $f$ by

$$K_\infty(f) = \{ \lambda \in L : \mathcal{L}_{\infty, \lambda}(f) < -1 \}.$$

It is a closed and semi-algebraic set. By Theorem 1.2, the mapping $L \ni \lambda \mapsto \mathcal{L}_{\infty, \lambda}(f)$ has a finite number of values (Corollary 1.5); hence there exists $\alpha > 0$ such that

$$K_\infty(f) = \{ \lambda \in L : \mathcal{L}_{\infty, \lambda}(f) < -1 - \alpha \}.$$

If $f$ is of class $\mathcal{C}^2$, then for any $\lambda \in L \setminus K_\infty(f)$ there exist a neighbourhood $U \subset L$ of $\lambda$ and a compact set $\Delta \subset M$ such that $f : f^{-1}(U) \setminus \Delta \to U$ is a trivial bundle (see [16], [17], [11]; see also [23], [21], [22], [7], [15] for polynomials and polynomial mappings). The smallest set $B \subset L$ such that $L \setminus B$ has the above property is called the bifurcation set at infinity of $f$ and is denoted by $B_\infty(f)$. Note that for a complex polynomial $f$ in two variables, $B_\infty(f) = K_\infty(f)$ (see [7], [15]).

Chadzyński and Krasinski ([3], Corollary 4.7) proved that for a complex polynomial $f$ in two variables, $\mathcal{L}_{\infty, \lambda}(f) = c_f$ and $\mathcal{L}_{\infty, \lambda}(f) < -1$ for $\lambda \in K_\infty(f)$. They also asked whether $\lambda \mapsto \mathcal{L}_{\infty, \lambda}(f)$ behaves similarly in the general case. Note that in the multi-dimensional case we cannot require $c_f \geq 0$. Indeed, for the polynomial $f(z_1, z_2, z_3) = (z_1z_2 - 1)z_2z_3$ ([17], Remark 9.1) we have $c_f = -1$ (see [3], Proposition 6.4).

As a corollary from Theorem 1.2 we give a partial answer to the above-mentioned question. Namely, for a nonconstant polynomial $f : \mathbb{C}^n \to \mathbb{C}$ there exist a finite set $S \subset \mathbb{C}$ with $K_\infty(f) \subset S$ and $c_f \geq -1$ such that $\mathcal{L}_{\infty, \lambda}(f) = c_f$ for $\lambda \in \mathbb{C} \setminus S$ and $\mathcal{L}_{\infty, \lambda}(f) < c_f$ for $\lambda \in S$ (Corollary 1.7). It is not clear to the authors whether $S = K_\infty(f)$ in Corollary 1.7.

Section 2 has an auxiliary character and contains some results on semi-algebraic mappings, Lojasiewicz exponent and stratifications. In Sections 3 and 4 we prove Theorem 1.2 and Corollary 1.6 respectively.

1. **Lojasiewicz exponent near the fibre of a mapping**

Let $M, N, L$ be finite-dimensional real vector spaces, $X \subset M$ be a closed set, let $g : X \to N$ and $f : X \to L$, and let $\lambda \in L$.

**Definition 1.1.** By the **Lojasiewicz exponent at infinity of $g$ near the fibre $f^{-1}(\lambda)$** we mean

$$\mathcal{L}_{\infty, f^{-1}(\lambda)}(g) := \sup \{ \mathcal{L}_\infty(g,f^{-1}(U)) : U \subset L \text{ is a neighbourhood of } \lambda \},$$

where

$$\mathcal{L}_\infty(g,S) := \sup \{ \theta \in \mathbb{R} : \exists C,R > 0 \forall x \in S (x \geq R \Rightarrow |g(x)| \geq C|x|^\theta) \}$$

is the **Lojasiewicz exponent at infinity of $g$ on a set $S \subset X$**.

Our main result is

**Theorem 1.2.** Let $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings.

(i) For any $\lambda \in L$, $\mathcal{L}_{\infty, f^{-1}(\lambda)}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$. 

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(ii) The function 
\[ \vartheta_{g/f} : L \ni \lambda \mapsto \mathcal{L}_{\infty,f \rightarrow \lambda}(g) \]
is upper semi-continuous, and there exists a semi-algebraic stratification
\[ L = S_1 \cup \ldots \cup S_j \]
such that \( \vartheta_{g/f} \) is constant on each stratum \( S_i \), \( i = 1, \ldots, j \).

The proof of Theorem 1.2 is given in Section 3. Theorem 1.2(ii) was proved in [18] for complex polynomials, under the assumption (i).

Now let \( f : M \to L \) be a semi-algebraic mapping of class \( \mathcal{C}^1 \) and let \( df \) be the differential of \( f \). Let
\[ \nu(df) : M \ni x \mapsto \nu(df(x)) \in \mathbb{R}, \]
be the Rabier function, i.e. for \( A = df(x) : M \to L \),
\[ \nu(A) = \inf_{\|\phi\| = 1} \|A^*(\phi)\|, \]
where \( A^* : L^* \to M^* \) is the adjoint operator and \( \phi \in L^* \). For a semi-algebraic function \( f : M \to \mathbb{R} \) (or a complex polynomial) we have \( \nu(df) = |\nabla f| \), where \( \nabla f \) is the gradient of \( f \).

**Definition 1.3.** The Lojasiewicz exponent of \( df \) near a fibre \( f^{-1}(\lambda) \) is defined to be \( \mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,f \rightarrow \lambda}(\nu(df)) \).

**Remark 1.4.** Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a semi-algebraic mapping of class \( \mathcal{C}^1 \) and let \( \kappa(df) : \mathbb{R}^n \ni x \mapsto \kappa(df(x)) \in \mathbb{R} \) be the Kuo function [10]; i.e., for \( A = df(x) = (A_1, \ldots, A_m) : \mathbb{R}^n \to \mathbb{R}^m \),
\[ \kappa(A) = \min_{1 \leq i \leq m} \text{dist}(\nabla A_i, (\nabla A_j)_{j \neq i}), \]
where \( (a_j)_{j \neq i} \) is the vector space generated by the vectors \( (a_j)_{j \neq i} \). As \( \nu(A) \leq \kappa(A) \leq m \nu(A) \) ([11], Proposition 2.6), for any \( \lambda \in L \) we have
\[ \mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,f \rightarrow \lambda}(\kappa(df)). \]
An analogous result holds for the Gaffney function [4] (cf. [9], Proposition 2.3).

The function \( \nu(df) \) is continuous and semi-algebraic ([11], Proposition 2.4), so Theorem 1.2 implies:

**Corollary 1.5.** Let \( f : M \to L \) be a semi-algebraic mapping of class \( \mathcal{C}^1 \). Then \( \mathcal{L}_{\infty,\lambda}(f) \in \mathbb{Q} \cup \{-\infty, +\infty\} \) for any \( \lambda \in L \), and the function \( L \ni \lambda \mapsto \mathcal{L}_{\infty,\lambda}(f) \) is upper semi-continuous and has a finite number of values. In particular, there exists \( \alpha > 0 \) such that
\[ K_\alpha(f) = \{ \lambda \in L : \mathcal{L}_{\infty,\lambda}(f) < -1 - \alpha \}. \]

In the case of complex regular mappings, from Theorem 1.2 we obtain:

**Corollary 1.6.** Let \( X \subset \mathbb{C}^n \) be a complex algebraic set, and let \( g : X \to \mathbb{C}^m \) and \( f : X \to \mathbb{C}^k \) be complex regular mappings. Then there exists a complex algebraic stratification \( \mathbb{C}^k = S_1 \cup \ldots \cup S_j \) such that the function
\[ \vartheta_{g/f} : \mathbb{C}^k \ni \lambda \mapsto \mathcal{L}_{\infty,f \rightarrow \lambda}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\} \]
is constant on each stratum \( S_i \), \( i = 1, \ldots, j \). Moreover, \( \vartheta_{g/f} \) is upper semi-continuous.
The proof of the above corollary will be given in Section 4. The crucial fact in the proof is that $\partial g/f(C^n) = \partial g/f(R^{2n})$ and this set is finite (Theorem 1.2).

For a complex polynomial $f : C^n \to C$ the set $K_{\infty}(f)$ is finite (Proposition 2.4, see also [11], Theorem 3.1); hence Corollary 1.6 gives:

**Corollary 1.7.** Let $f : C^n \to C$ be a polynomial function with $\deg f > 0$. Then there exist a finite set $S \subset C$ with $K_{\infty}(f) \subset S$ and a constant $c_f \in \mathbb{Q}$ with $c_f \geq -1$ such that $\mathcal{L}_{\infty,\lambda}(f) = c_f$ for $\lambda \in \mathbb{C} \setminus S$, and $\mathcal{L}_{\infty,\lambda}(f) < c_f$ for $\lambda \in S$.

2. **Auxiliary results**

In what follows, $L$, $M$, $N$ are finite-dimensional real vector spaces. We will use the Euclidean norm $| \cdot |$ in $M$ (or in $N$, $L$). For $A \subset M$, let $g(\cdot, A)$ denote the distance function to $A$, i.e. $g(x, A) = \inf_{y \in A} |x - y|$ if $A \neq \emptyset$, and $g(x, \emptyset) = 1$.

2.1. **Semi-algebraic mappings.** A subset of $M$ is called semi-algebraic if it is defined by a finite alternative of finite systems of inequalities $P > 0$ or $P \geq 0$, where $P$ are polynomials on $M$ (see [1], [2]). A mapping $f : X \to N$, where $X \subset M$, is called semi-algebraic if the graph $\Gamma(f)$ of $f$ is a semi-algebraic set. For instance, the distance to a semi-algebraic set is a semi-algebraic function (cf. [2]):

**Proposition 2.1.** Let $V \subset M$ be a semi-algebraic set. Then the function $g_V : M \ni x \mapsto g(x, V) \in R$ is continuous and semi-algebraic.

Let $X \subset M$ and let $f : X \to N$ be any mapping. We say (cf. [8]) that $f$ is proper at a point $y \in N$ if there exists an open neighbourhood $U$ of $y$ such that $f : f^{-1}(U) \to U$ is a proper map. The set of points at which $f$ is not proper is denoted by $\mathcal{S}_f$. It is obvious that the set $\mathcal{S}_f$ is closed. It is known that for a complex algebraic set $X \subset C^n$ and a complex regular mapping $f : X \to C^m$, the set $\mathcal{S}_f$ is complex algebraic.

**Proposition 2.2.** Let $X$ be a closed semi-algebraic set. If the mapping $f : X \to N$ is semi-algebraic, then the set $\mathcal{S}_f$ is also semi-algebraic.

**Proof.** Since $X$ is a closed set, we have

$$\mathcal{S}_f = \{ y \in N : \forall_{A, \varepsilon > 0} \exists_{x \in X} |x| > A \land |f(x) - y| < \varepsilon \}.$$ 

Then, by the Tarski-Seidenberg Theorem, we obtain the assertion. \qed

Let $f : X \to N$ with $X \subset M$. The degree of $f$ is defined by

$$\deg f = \inf \{ \theta \in \mathbb{R} : \exists_{C, R > 0} \forall_{x \in X} (|x| \geq R \Rightarrow |f(x)| \leq C|x|^\theta) \}.$$ 

Set $\text{supp } f = \{ x \in X : f(x) \neq 0 \}$.

A curve $\varphi : [r, +\infty) \to M$ is called meromorphic at $+\infty$ if $\varphi$ is the sum of a Laurent series of the form

$$\varphi(t) = a_p t^p + a_p - 1 t^{p - 1} + \cdots, \quad a_i \in M, \quad p \in \mathbb{Z}.$$ 

In the case of a polynomial function and the Laurent series at infinity, the above degree is the usual degree; that is, $\deg \varphi = p$ if $a_p \neq 0$, and $\deg \varphi = -\infty$ if $\varphi \equiv 0$.

**Proposition 2.3.** Let $X$ be a closed semi-algebraic set and let $f : X \to N$ be a semi-algebraic mapping. Then:

(i) $\deg f \in \mathbb{Q} \cup \{-\infty\}$.

(ii) $\deg f = -\infty$ if and only if $\text{supp } f$ is bounded.
In particular, the set 

\[ 0 \alpha< \text{unbounded}. \]

Then the set 

\[ \exists \psi \text{ is unbounded and semi-algebraic. So, by the Curve Selection Lemma at infinity,} \]

there exists a curve 

\[ \psi \text{ meromorphic at infinity such that} \]

Proof. If \( \supp f \) is bounded, then the assertion is obvious. Assume that \( \supp f \) is unbounded. Then the set 

\[ Y = \{(y, f(y)) \in X \times N : \forall x \in X \ |x| = |y| \Rightarrow 2|f(y)| \geq |f(x)|\} \]

is unbounded and semi-algebraic. So, by the Curve Selection Lemma at infinity, there exists a curve \( \psi = (\varphi, \eta) : [r, +\infty) \to Y \) meromorphic at \( +\infty \) such that \( \eta = f \circ \varphi, \deg \eta \in \mathbb{Z}, \) and \( \deg \varphi > 0. \) Let \( \theta = \deg \eta/\deg \varphi. \) Then \( \theta \in \mathbb{Q} \) and for some \( C, D, R > 0, \)

\[ C|\varphi(t)|^\theta \leq |f(\varphi(t))| \leq D|\varphi(t)|^\theta, \quad t > R. \]

The definition of \( Y \) now implies that for \( x \in X, \ |x| = |\varphi(t)|, \ t > R, \)

\[ |f(x)| \leq |f(\varphi(t))| \leq D|\varphi(t)|^\theta = D|x|^\theta. \]

So, \( \deg f \leq \theta. \) Since, by \( (2.2), \) \( \deg f \geq \theta, \) it follows that \( \deg f = \theta. \) This gives (i), (ii) and (iii). Part (iv) follows immediately from (iii). \( \square \)

2. \( C^1 \) semi-algebraic functions. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a semi-algebraic function of class \( C^1 \) in \( x = (x_1, \ldots, x_n). \) Then the gradient \( \nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n) : \mathbb{R}^n \to \mathbb{R}^n \)
is a semi-algebraic mapping.

Proposition 2.4. There exist \( C, \delta, R > 0 \) such that 

\[ |f(x)| \geq R \Rightarrow |x| |\nabla f(x)| \geq C|f(x)|, \]

\[ |f(x)| \leq \delta \Rightarrow |x| |\nabla f(x)| \geq C|f(x)|. \]

In particular, the set \( K_\infty(f) \) is finite. The assertion also holds for complex polynomials.

Proof. As in \cite{19} and \cite{6}, we use Hörmander’s method. To prove \( (2.3), \) assume the contrary. Then the semi-algebraic set 

\[ X = \{(x, y, z, \varepsilon) \in \mathbb{R}^{2n} \times \mathbb{R}^2 : y = \nabla f(x), z = f(x), |z| \geq \varepsilon, \varepsilon |y||x| < |z|\} \]

has an accumulation point of the form \( (x_0, y_0, z_0, +\infty). \) Thus, by the Curve Selection Lemma at infinity there exists a curve \( \psi = (\varphi, \tau, \eta_1, \eta_2) : [r, +\infty) \to X \)
meromorphic at infinity such that \( \psi(t) \to (x_0, y_0, z_0, +\infty) \) as \( t \to +\infty. \) Then \( \deg \eta_2 > 0, \deg \eta_1 > 0, \deg \varphi > 0, \) and 

\[ \deg \eta_2 + \deg \tau + \deg \varphi \leq \deg \eta_1. \]

On the other hand, 

\[ \deg \eta_1 = \deg \eta_1' + 1 = \deg(f \circ \varphi)' + 1 \leq \deg \tau + \deg \varphi, \]

and we obtain a contradiction. Analogously we prove \( (2.4) \) and the assertion in the complex case. \( \square \)
2.3. **Lojasiewicz exponent.** For three semi-algebraic sets $X, Y, Z \subset M$ such that $X \cap Y \subset Z$, we define a regular separation exponent of $Y$ and $Z$ on $X$ at a point $x_0 \in X \cap Y$ to be any real positive $\theta$ such that

$$g(x, Y) \geq Cg(x, Z)^\theta \quad \text{for } x \in X \cap \Omega,$$

where $C > 0$ and $\Omega$ is a neighbourhood of $x_0$. The infimum of all such exponents $\theta$ will be denoted by $L_{x_0}(X; Y, Z)$. By using the method of Lipschitz stratifications ([15], [14]), the following is proved in Theorem 1.5 of [20]:

**Proposition 2.5.** Let $X, Y, Z \subset M$ be closed semi-algebraic sets such that $X \cap Y \subset Z$, and let $x_0 \in X \cap Y$.

(i) Then $L_{x_0}(X; Y, Z) \in \mathbb{Q}$, and (2.7) holds for $\theta = L_{x_0}(X; Y, Z)$, some $C > 0$ and a neighbourhood $\Omega$ of $x_0$, provided $0^\theta = 0$.

(ii) If $x_0 \in X \setminus Z$, then $L_{x_0}(X; Y, Z)$ is attained on an analytic curve, i.e. for any neighbourhood $\Omega$ of $x_0$ there exists an analytic curve $\varphi : [0, r) \to X \cap \Omega$ such that $\varphi((0, r)) \subset X \setminus Z$ and $\varphi(0) \in X \cap Y$, and for some constant $C_1 > 0$,

$$Cg(\varphi(t), Z) L_{x_0}(X; Y, Z) \leq g(\varphi(t), Y) \leq C_1 g(\varphi(t), Z) L_{x_0}(X; Y, Z), \quad t \in [0, r).$$

If $Z = X \cap Y$ and $x_0 \in X \setminus Y$, then obviously $L_{x_0}(X; Y, Z)$ is equal to the Lojasiewicz exponent $L_{x_0}(X, Y)$ of $X$ and $Y$ at $x_0$, i.e. the optimum exponent $\theta$ in the following separation condition:

$$g(x, X) + g(x, Y) \geq Cg(x, X \cap Y)^\theta \quad \text{for } x \in \Omega,$$

considered in a neighbourhood $\Omega \subset M$ of $x_0$ for some constant $C > 0$. Note that Proposition 2.5 also holds in the subanalytic case.

2.4. **Stratification.** By stratification of a subset $X \subset M$ we mean a decomposition of $X$ into a locally finite disjoint union

$$X = \bigcup S_\alpha,$$

where the subsets $S_\alpha$ are called strata, such that each $S_\alpha$ is a connected embedded submanifold of $M$, and each $(S_\alpha \setminus S_{\alpha'}) \cap X$ is the union of some strata of dimension smaller than $\dim S_\alpha$.

The $i$-th skeleton of the stratification (2.5) is

$$X^i = \bigcup_{\dim S_\alpha \leq i} S_\alpha.$$

The stratification (2.5) is called semi-algebraic if all the skeletons $X^i$ are semi-algebraic sets (or equivalently if the number of strata is finite and they are all semi-algebraic). The stratification (2.5) of a complex algebraic subset $X$ of a complex linear space $M$ is called complex algebraic if all the skeletons $X^i$ are complex algebraic subsets of $M$ and the number of strata is finite.

By Corollaries 2.6 and 2.7 in [20] we have:

**Proposition 2.6.** Let $X, Y, Z \subset M$ be closed semi-algebraic sets such that $X \cap Y \subset Z$. Then there exists a stratification

$$X \cap Y = S_1 \cup \cdots \cup S_k$$

of $X \cap Y$ such that the function

$$X \cap Y \ni x \mapsto L_x(X; Y, Z)$$
is constant on each stratum \(S_i\). In particular, the function \(2.7\) is upper semi-
continuous. If additionally \(X_1, \ldots, X_n \subset X \cap Y\) are semi-algebraic sets, then one
can require that the stratification \(2.6\) is compatible with any \(X_j\), i.e. any \(X_j\) is a
union of some strata \(S_i\).

3. PROOF OF THEOREM 1.2

Let \(X \subset M\) be a closed semi-algebraic set, and let \(g : X \rightarrow N\) and \(f : X \rightarrow L\)
be continuous semi-algebraic mappings.

The values \(\vartheta_{g/f}(\lambda) \in \{-\infty, +\infty\}\) are characterised by the following:

**Remark 3.1.** (i) By Proposition 2.3 and the definition of \(S_f\) we have:

\[
\vartheta_{g/f}(\lambda) = +\infty \iff \lambda \in L \setminus \mathcal{S}_f.
\]

(ii) Let \(h = f|_{g^{-1}(0)}\). From the definition of \(\mathcal{L}_{\infty,f\rightarrow\lambda}(g)\) we have:

\[
\vartheta_{g/f}(\lambda) = -\infty \iff \lambda \in \mathcal{S}_h \iff \mathcal{L}_{\infty}(f - \lambda|_{g^{-1}(0)}) < 0.
\]

Before the proof of Theorems 1.2 we give four lemmas and a proposition. Let
\(B = \{z \in M : |z| < 1\}\) and let \(H : B \rightarrow M\) be of the form

\[
H(z) = \frac{z}{1 - |z|^2}.
\]

**Lemma 3.2.** The mapping \(H\) is semi-algebraic and invertible with inverse

\[
H^{-1}(x) = \frac{2x}{1 + \sqrt{1 + 4|x|^2}}.
\]

Moreover, for any \(R > 0\),

\[
|H(z)| \geq R \iff \frac{2R}{1 + \sqrt{1 + 4R^2}} \leq |z| < 1.
\]

**Proof.** \(H\) is a semi-algebraic mapping as the restriction of a rational mapping to
the semi-algebraic set \(B\). By an easy calculation we obtain (3.3) and the formula
for \(H^{-1}\). \(\square\)

By Lemma 3.2 we may define the following semi-algebraic sets:

\[
Y = \{(x, \lambda, \delta) \in X \times L \times \mathbb{R} : |f(x) - \lambda| \leq \delta\},
\]

\[
Z_1 = \{(z, \lambda, \delta) \in B \times L \times \mathbb{R} : (H(z), \lambda, \delta) \in Y\},
\]

\[
Z_2 = \partial B \times L \times \mathbb{R},
\]

\[
Z = Z_1 \cup Z_2.
\]

Let \(V = g^{-1}(0)\), and let

\[
W = \{(z, \lambda, \delta) \in Z_1 : H(z) \in V\}.
\]

Define a mapping \(F : Z \rightarrow \mathbb{R}\) by

\[
F(z, \lambda, \delta) = (1 - |z|^2)\vartheta((z, \lambda, \delta), W).
\]

Since \(W\) is a semi-algebraic set, Proposition 2.3 implies that \(F\) is a semi-algebraic
mapping.

For any \(\lambda \in L, \delta \geq 0\) and \(S \subset X\) we set

\[
S_{\lambda,\delta} = \{x \in S : |f(x) - \lambda| \leq \delta\}.
\]
Lemma 3.3. Let $\lambda_0 \in L$ and $\delta_0 > 0$ be such that the set $V_{\lambda_0, \delta_0}$ is bounded, and suppose $X_{\lambda_0, \delta}$ is unbounded for any $\delta > 0$. Then there exist $C, D, R > 0$ such that for any $(x, \lambda, \delta) \in Y$, where $0 < \delta \leq \frac{\delta_0}{2}$ and $|\lambda - \lambda_0| \leq \delta$, we have

$$(3.4) \quad C|x|^{-\delta} \leq F(H^{-1}(x), \lambda, \delta) \leq D|x|^{-1}, \quad x \in X_{\lambda_0, \delta}, \quad |x| \geq R.$$ 

Proof. Let $Z^\delta = \{(z, \lambda, \delta) \in Z_1 : |\lambda - \lambda_0| \leq \delta\}$. Then $Z^{\delta'} \subset Z^{\delta''}$ if $\delta' \leq \delta''$. By the definition of $F$ we have

$$F(z, \lambda, \delta) = |H(z)|^{-1}|z|g((z, \lambda, \delta), W) \quad \text{for} \quad (z, \lambda, \delta) \in Z_1, \quad z \neq 0.$$ 

Hence, by (3.3), it suffices to prove that for some $c, d, r > 0$, with $r < 1$, and $\delta_1 = \frac{\delta_0}{2}$,

$$(3.5) \quad c \leq |z|g((z, \lambda, \delta), W) \leq d \quad \text{for} \quad (z, \lambda, \delta) \in Z^{\delta_1}, \quad r \leq |z| < 1.$$ 

Because $Z^{\delta_1}$ is bounded, the set $\{|z|g((z, \lambda, \delta), W) : (z, \lambda, \delta) \in Z_{\lambda_0, \delta_1}\}$ is also bounded. Hence the right-hand estimate in (3.5) holds. By (3.3) and the assumptions on $V_{\lambda_0, \delta_0}$ and $X_{\lambda_0, \delta}$, there exists $0 < r < 1$ for which the set $W$ has no accumulation points in $A = \{(z, \lambda, \delta) \in Z^{\delta_1} : r \leq |z|\}$. Moreover, $A$ is bounded, so $c = \inf\{|z|g((z, \lambda, \delta), W) : (z, \lambda, \delta) \in A\} > 0$. This gives the left-hand estimate in (3.5). 

Let $X_H = H^{-1}(X) \cup \partial B$ and $V_H = H^{-1}(V)$. Since $g$ and $H$ are semi-algebraic mappings the sets $V$, $X_H$, $V_H$ are semi-algebraic. Moreover, $X_H$ is closed and $V_H = (g \circ H)^{-1}(0)$. Define $g_H : X_H \to N$ by

$$g_H(z) = \begin{cases} 
  \frac{g \circ H(z)}{(1 + |H(z)|^2)^{\beta(g)}} & \text{for } z \in X_H \cap B, \\
  0 & \text{for } z \in \partial B,
\end{cases}$$

where $\beta(g)$ is defined in Proposition 2.3 (iv).

Lemma 3.4. The mapping $g_H$ is continuous, semi-algebraic and

$$(3.6) \quad (g_H)^{-1}(0) = V_H \cup \partial B.$$ 

Proof. By (2.4) in Proposition 2.3 $g_H$ is continuous. Since the mapping $g$ is semi-algebraic, so is $B \ni x \mapsto g \circ H(x)$, and hence also $h : (X_H \cap B) \ni z \mapsto (g(z), (1 + |H(z)|^2)^{\beta(g)}) \in N \times \mathbb{R}$. The graph of $g_H$ is the union of $\partial B \times \{0\}$ and the image of the graph $h$ under the semi-algebraic mapping $M \times \mathbb{N} \times (0, +\infty) \ni (z, y, t) \mapsto (z, \frac{1}{t}y) \in M \times N$, so the graph of $g_H$ is semi-algebraic. The equality (3.6) is obvious. 

The set $Z$ is semi-algebraic and $X_H$ is its image under the projection map $Z \ni (z, \lambda, \delta) \mapsto z \in M$. Hence, we may define a semi-algebraic mapping $G : Z \to N$ by

$$G(z, \lambda, \delta) = g_H(z).$$ 

Let $\Gamma$ be the graph of the semi-algebraic mapping $(G, F) : Z \to N \times \mathbb{R}$. Since $Z$ is a closed set, so is $\Gamma$.

Lemma 3.5. There exists a stratification

$$(3.7) \quad G^{-1}(0) = S_1 \cup \cdots \cup S_j$$

such that the function

$$(3.8) \quad \mathcal{L} : G^{-1}(0) \ni v \mapsto \mathcal{L}_{(v,0,0)}(\Gamma; Z \times \{0\} \times \mathbb{R}, Z \times N \times \{0\})$$

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is constant on each stratum $S_i$. In particular, the set of values of $\mathcal{L}$ is a finite subset of $\mathbb{Q}$.

**Proof.** By (3.6), $G^{-1}(0) = F^{-1}(0)$, so

$$G^{-1}(0) \times \{0\} \times \{0\} = \Gamma \cap (Z \times \{0\} \times \mathbb{R}) \subset Z \times N \times \{0\}.$$  

Proposition 2.5 now shows that the values of $\mathcal{L}$ are rational numbers. Moreover, from Proposition 2.6 we obtain a stratification (3.7) satisfying the assertion. □

Take any $\lambda_0 \in L$ and define

$$l_{\lambda_0}(g) = \max\{\mathcal{L}(z, \lambda_0, 0) : (z, \lambda_0, 0) \in Z_2\}.$$  

By Lemma 3.5, $l_{\lambda_0}(g) \in \mathbb{Q}$.

**Proposition 3.6.** Let $\delta_0 > 0$ be such that the set $V_{\lambda_0, \delta_0}$ is bounded, and suppose the set $X_{\lambda_0, \delta}$ is unbounded for any $\delta > 0$. Then

$$(3.9) \quad L_{\infty, f \rightarrow \lambda_0}(g) = 2\beta(g) - l_{\lambda_0}(g)$$

and for any sufficiently small $0 < \delta \leq \frac{\delta_0}{2}$ there exist $C, C', R > 0$ such that

$$(3.10) \quad |g(x)| \geq C|x|^{2\beta(g) - l_{\lambda_0}(g)} \quad \text{for} \quad x \in X_{\lambda_0, \delta}, \quad |x| \geq R$$

and

$$(3.11) \quad C'|\varphi(t)|^{2\beta(g) - l_{\lambda_0}(g)} \geq |g(\varphi(t))| \geq C|\varphi(t)|^{2\beta(g) - l_{\lambda_0}(g)}, \quad t \in [r, +\infty),$$

for some curve $\varphi : [r, +\infty) \rightarrow X_{\lambda_0, \delta}$ meromorphic at $+\infty$, with $\deg \varphi > 0$.

**Proof.** Let $E = \{(z, \lambda, \delta) \in Z_2 : \lambda = \lambda_0, \delta = 0\}$ and $\alpha = l_{\lambda_0}(g)$. By the definition of $l_{\lambda_0}(g)$, for any $(z, \lambda_0, 0) \in E$ there exist a neighbourhood $\Omega_z \subset M \times L \times \mathbb{R}$ of $(z, \lambda_0, 0)$ and $C_z > 0$ such that

$$|G(y, \lambda, \delta)| \geq C_z|F(y, \lambda, \delta)|^\alpha, \quad (y, \lambda, \delta) \in \Omega_z \cap Z.$$  

Since the set $E$ is compact, there exists $\tilde{C} > 0$ such that $C_z \geq \tilde{C}$ for $(z, \lambda_0, 0) \in E$, and there exist $0 < r_1 < 1$ and $0 < \delta_1 \leq \frac{\delta_0}{2}$ such that

$$|G(y, \lambda, \delta_1)| \geq \tilde{C}|F(y, \lambda, \delta_1)|^\alpha, \quad |\lambda - \lambda_0| \leq \delta_1, \quad r_1 \leq |y| < 1,$$

where $(y, \lambda, \delta_1) \in Z$. Consequently,

$$\frac{|g(x)|}{(1 + |x|^2)^{\beta(g)}} \geq \tilde{C}|F(H^{-1}(x), \lambda_0, \delta_1)|^\alpha, \quad x \in X_{\lambda_0, \delta_1}, \quad |x| \geq R,$$

where $R > 0$ is the unique solution of the equation $r_1 = \frac{2R}{1 + \sqrt{1 + 4R^2}}$. Together with (3.4) this gives

$$|g(x)| \geq \tilde{C}C(1 + |x|^2)^{\beta(g)}|x|^{-\alpha} \quad \text{for} \quad x \in X_{\lambda_0, \delta_1}, \quad |x| \geq R.$$  

Hence for any $0 < \delta \leq \delta_1$, (3.10) follows.

Take any $0 < \delta \leq \delta_1$. Let $(z_0, \lambda_0, 0) \in Z_2$ be a point such that $\mathcal{L}(z_0, \lambda_0, 0) = l_{\lambda_0}(g)$. By the assumption on $V_{\lambda_0, \delta_0}$ we have

$$(3.12) \quad (z_0, \lambda_0, 0, 0, 0) \in \Gamma \setminus (Z \times N \times \{0\}), \quad (z_0, \lambda_0, 0) \notin \mathcal{W},$$

and $\mathcal{L}(z_0, \lambda_0, 0) > 0$. Thus, by Proposition 2.5 for any sufficiently small neighbourhood $\tilde{\Omega}$ of $\omega = (z_0, \lambda_0, 0, G(z_0, \lambda_0, 0), F(z_0, \lambda_0, 0)) = (z_0, \lambda_0, 0, 0, 0)$ there exists an analytic curve

$$\psi = (\psi_1, \psi_2, \psi_3) : [0, r) \rightarrow \Gamma \cap \tilde{\Omega},$$

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where \( \psi_1 : [0, r) \to Z, \psi_2 = G \circ \psi_1 : [0, r) \to N, \psi_3 = F \circ \psi_1 : [0, r) \to R, \)
\( \psi((0, r)) \subset \Gamma \setminus \{ (Z \times N \times \{0\}) \) and \( \psi(0) \in \Gamma \cap (Z \times \{0\} \times R), \) such that for some constant \( C_1 > 0, \)
\[
(3.13) \quad \varrho(\psi(t), Z \times \{0\} \times R) \leq C_1 \varrho(\psi(t), Z \times M \times \{0\})^\alpha \quad \text{for } t \in [0, r).
\]
Let \( \varphi_1 : [0, r) \to M, \varphi_2 : [0, r) \to L, \varphi_3 : [0, r) \to R, \) and let \( \psi_1 = (\varphi_1, \varphi_2, \varphi_3). \) By the choice of \( \psi \) we have \( \varphi_1(t) \in B \) for \( t \in (0, r), \) and \( \varphi(0) \in \partial B \) by (3.12). Hence,

\[
(3.14) \quad |H(\varphi_1(t))| \to \infty \quad \text{as } t \to 0.
\]
Since the neighbourhood \( \hat{\Omega} \) of \( \omega \) can be small, one can assume that \( 0 \leq \varphi_3(t) < \delta \)
for \( t \in (0, r). \) Then, by the definition of \( Z, \) we have \( |H(\varphi_1(t)) - \lambda_0| \leq \varphi_3(t) < \delta \)
for \( t \in (0, r), \) and so

\[
(3.15) \quad H(\varphi_1(t)) \in X_{\lambda_0, \delta} \quad \text{for } t \in (0, r).
\]
By (3.13),

\[
|G(\psi_1(t))| \leq C_1 |F(\psi_1(t))|^{\alpha} \quad \text{for } t \in [0, r].
\]
Hence, from (3.3) and (3.14), for some \( 0 < r_1 < r, \)
\[
|g(H(\varphi_1(t)))| \leq C_1 D^{\lambda_0}(g)|H(\varphi_1(t))|^{-\alpha} \quad \text{for } t \in (0, r_1).
\]
Together with (3.14) and (3.15), this gives

\[
|g(H(\varphi_1(t)))| \leq C'|H(\varphi_1(t))|^{2\beta(g)-\alpha} \quad \text{for } t \in (0, r_1]
\]
for some \( C' > 0. \) Now setting \( \varphi(t) = H(\varphi_1(\frac{1}{t})) \) for \( t \in \left[\frac{1}{r_1}, +\infty\right) \) we obtain (3.11).

Finally, (3.11) and (3.10) yield (3.9).

**Proof of Theorem 3.2.** Fix \( \lambda_0 \in L. \) First we prove (i). If for any \( \delta > 0 \) the set \( V_{\lambda_0, \delta} \) is unbounded, then \( L_{\infty, f} \rightarrow \lambda_0 \) \( (g) = -\infty. \) If for some \( \delta > 0 \) the set \( X_{\lambda_0, \delta} \) is bounded, then \( L_{\infty, f} \rightarrow \lambda_0 \) \( (g) = +\infty. \) The remaining case in (i) follows from the fact that \( \beta(g) \in Z \) (see Proposition 2.2) and from (3.9) in Proposition 3.6.

To prove (ii), we adopt the method of the proof of Theorem 3.2.2 in [18]. By Lemma 3.5, let

\[
\delta_{g/f}(L) = \{r_1, \ldots, r_s\} \subset \mathbb{Q} \cup \{-\infty, +\infty\}, \quad \text{where } r_1 \leq \cdots \leq r_s.
\]
Define \( \Lambda_x = \{\lambda \in L : L_{\infty, f} \rightarrow \lambda \}(g) \leq \xi \} \) for \( \xi \in \mathbb{R}. \)
Fix \( r_i. \) We now prove that the set \( \Lambda_{r_i} \) is closed and semi-algebraic. If \( r_i \in \{-\infty, +\infty\} \) this follows from Remark 3.1 and Proposition 2.2. So, let \( r_i = \frac{a}{b}, \)
where \( a, b \in \mathbb{Z} \) and \( b > 0. \) Define

\[
T = \{(x, c) \in X \times R : |g(x)|^{b} = c|x|^{a}\},
\]
and let \( p : T \ni (x, c) \mapsto (f(x), c) \in L \times R. \) Since the mapping \( p \) is semi-algebraic, Proposition 2.2 shows that the set \( \mathcal{S}_p \) is also semi-algebraic.

Let \( \pi : L \times R \ni (y, c) \mapsto y \in L \) and observe that

\[
(3.16) \quad \Lambda_{r_i} = \pi(\mathcal{S}_p).
\]

Indeed, let \( \lambda \in \Lambda_{r_i}, \) and let \( U \subset L \) be a neighbourhood of \( \lambda. \) Take a neighbourhood \( U_1 \subset L \) of \( \lambda \) such that \( U_1 \subset U. \) Then, by Proposition 3.6 there exist \( C' > 0 \) such that the set

\[
\{(x, y) \in f^{-1}(U_1) \times N : y = g(x), |y|^{b} \leq C'|x|^{a}\}
\]
is unbounded. Since it is semi-algebraic, there exists a curve \( \psi = (\varphi, \eta): [r, +\infty) \to f^{-1}(U_1) \times N \) meromorphic at infinity such that \( \text{deg} \varphi > 0 \), \( \eta = g \circ \varphi \) and
\[
|g(\varphi(t))|^b \leq C' |\varphi(t)|^a, \quad t \in [r, +\infty).
\]
Then, for some \( \lambda' \in U_1 \subset U \) and 0 \( \leq c \leq C' \),
\[
f \circ \varphi(t) \to \lambda' \quad \text{and} \quad \frac{|g(\varphi(t))|^b}{|\varphi(t)|^a} \to c \quad \text{as} \quad t \to \infty.
\]
Hence, \( \lambda' \in \pi(\mathcal{S}_p) \cap U \), and so \( \lambda \in \pi(\mathcal{S}_p) \).

Now let \( \lambda \in \pi(\mathcal{S}_p) \). Take any neighbourhood \( U \subset L \) of \( \lambda \), and let \( \lambda' \in U \) and \( c \in \mathbb{R} \) be such that \( (\lambda', c) \in \mathcal{S}_p \). Then for some sequence \( (x_n, c_n) \in T \), where \( x_n \in f^{-1}(U) \) and \( c_n \in \mathbb{R} \) for \( n \in \mathbb{N} \), we have
\[
|x_n| \to \infty, \quad f(x_n) \to \lambda' \quad \text{and} \quad c_n \to c \quad \text{as} \quad n \to \infty.
\]
Hence, there exists \( C > 0 \) such that \( |c_n| \leq C \) for \( n \in \mathbb{N} \), and so
\[
|g(x_n)|^b \leq C|x_n|^a, \quad n \in \mathbb{N}.
\]
This gives \( \mathcal{L}_\infty(g[f^{-1}(U)]) \leq r_i \), and hence \( \mathcal{L}_{\infty,f}^{\lambda}(g) \leq r_i \). Summing up, \( \lambda \in \Lambda_{r_i} \) and (3.16) is proved.

By Proposition 2.2, the set \( \mathcal{S}_p \) is semi-algebraic, so, by (3.10), \( \Lambda_{r_i} \) is closed and semi-algebraic. In particular, the function \( \vartheta_{g/f} \) is upper semi-continuous. From the definition of \( \Lambda_{r_i} \), we have \( \Lambda_{r_j} \varsubsetneq \cdots \varsubsetneq \Lambda_{r_r} = L \). Hence, \( \Lambda_{r} \) is semi-algebraic for any \( \xi \in \mathbb{R} \). Therefore there exists a semi-algebraic stratification of the form (1.1) compatible with any intersection \( X_1 \cap \cdots \cap X_j \), where \( X_1, \ldots, X_j \in \{ \Lambda_{r_1}, \ldots, \Lambda_{r_j} \} \). Thus, the function \( \vartheta_{g/f} \) is constant on each stratum \( S_i \), and Theorem 1.2 is proved.

\[\square\]

Corollary 3.7. If \( \theta = \mathcal{L}_{\infty,f}^{\lambda}(g) \in \mathbb{Q} \), then for some \( C, C', R, \delta > 0 \),
\[
(\text{3.17}) \quad |g(x)| \geq C|x|^\theta \quad \text{for} \quad x \in X, \quad |x| \geq R, \quad |f(x) - \lambda| < \delta,
\]
\[
(\text{3.18}) \quad C'|\varphi(t)|^\theta \geq |g(\varphi(t))| \geq C|\varphi(t)|^\theta \quad \text{for} \quad t \in [r, +\infty),
\]
where \( \varphi: [r, +\infty) \to X \) is a curve meromorphic at infinity such that \( \text{deg} \varphi > 0 \) and \( |f(\varphi(t)) - \lambda| < \delta \) for \( t \in [r, +\infty) \).

Proof. The assertion follows immediately from (3.10), (3.11) and Theorem 1.2 \( \square \)

4. Proof of Corollary 1.0

Let \( (z_1, \ldots, z_n) \), \( (y_1, \ldots, y_m) \) be the coordinates of \( z \in \mathbb{C}^n \), \( y \in \mathbb{C}^m \), respectively.

As in the proof of Theorem 1.2 we now show that for any \( \xi \in \mathbb{Q} \cup \{-\infty, +\infty\} \), the set \( \Lambda_\xi = \{ \lambda \in \mathbb{C}^k : \mathcal{L}_{\infty,f}^{\lambda}(g) \leq \xi \} \) is complex algebraic. For \( \xi \in \{-\infty, +\infty\} \), this is obvious. Fix \( \xi = \frac{a}{b} \), where \( a, b \in \mathbb{Z}, b > 0, (a, b) = 1 \).

Let \( g = (g_1, \ldots, g_m) \). For any \( i = 1 \ldots n \) we define algebraic sets
\[
T_\xi^i = \{ (z, y, u) \in X \times \mathbb{C}^m \times \mathbb{C} : z_i u = 1, \ g_j^b(z) = y_j z_i^a, \ j = 1, \ldots, m \}
\]
if \( \xi \geq 0 \),
\[
T_\xi^i = \{ (z, y, u) \in X \times \mathbb{C}^m \times \mathbb{C} : z_i u = 1, \ g_j^b(z) z_i^{-a} = y_j, \ j = 1, \ldots, m \}
\]
if \( \xi < 0 \), and mappings
\[
p_i : T_\xi^i \ni (z, y, u) \mapsto (f(z), y, u) \in \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C}.
\]
Denote by $\mathcal{S}_i$ the set of points at which $p_i$ is not proper, and
\[ A_i = \mathcal{S}_i \cap \{ (\lambda, y, u) \in \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C} : u = 0 \}, \quad i = 1, \ldots, n. \]
Since each $\mathcal{S}_i$ is algebraic, so is $A_i$.

Let $\pi : \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C} \ni (\lambda, y, u) \mapsto \lambda \in \mathbb{C}^k$ and observe that
\begin{equation}
\Lambda_\xi = \bigcup_{i=1}^n \pi(A_i).
\end{equation}

Indeed, let $\lambda \in \mathbb{C}^k$ satisfy $L_{\infty,f-\lambda}(g) \leq \xi$. Take any neighbourhoods $U, W \subset \mathbb{C}^k$ of $\lambda$ such that $W \subset U$. By Corollary [3.7] there exist $C > 0$ and a curve $\varphi = (\varphi_1, \ldots, \varphi_n) : [r, +\infty) \to f^{-1}(W)$ meromorphic at infinity with $\deg \varphi > 0$ such that
\begin{equation}
|g(\varphi(t))| \leq C|\varphi(t)|^\xi, \quad t \in [r, \infty).
\end{equation}
Let $\deg \varphi_i = \deg \varphi$. Then $\deg \varphi_i > 0$. By the definition of $\varphi$, there exists $\lambda' \in \overline{W}$ such that
\begin{equation}
f(\varphi(t)) \to \lambda' \quad \text{as} \quad t \to \infty.
\end{equation}
By (4.2), there exists $y \in \mathbb{C}^m$ such that
\[ \eta(t) := \left( \frac{g^1(\varphi(t))}{\varphi_1^1(t)}, \ldots, \frac{g^m(\varphi(t))}{\varphi_1^m(t)} \right) \to y \quad \text{as} \quad t \to \infty. \]

Since $\deg \varphi_i > 0$, we may assume that $\varphi_i(t) \neq 0$ for $t \in [r, +\infty)$. Putting $u(t) = \frac{1}{\varphi_i(t)}$ for $t \in [r, +\infty)$, we easily see that
\[ p_i(\varphi(t), \eta(t), u(t)) \to (\lambda', y, 0) \quad \text{as} \quad t \to \infty. \]
Hence $(\lambda', y, 0) \in \mathcal{S}_i$, so $\lambda' \in U \cap \pi(A_i)$, and thus $\lambda \in \overline{\pi(A_i)}$. This gives the inclusion “$\subset$” in (4.1).

We now prove “$\supset$”. Let $\lambda \in \overline{\pi(A_i)}$. Take any neighbourhood $U$ of $\lambda$. Then there exists $\lambda' \in U \cap \pi(A_i)$, and so $(\lambda', y, 0) \in \mathcal{S}_i$ for some $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$. The definitions of $A_i$ and $T_i^{\xi}$ now yield a sequence $x_l = (x_{1,l}, \ldots, x_{n,l}) \in f^{-1}(U)$, $l \in \mathbb{N}$, such that $f(x_l) \to \lambda'$ and
\[ |x_{i,l}| \to \infty, \quad \frac{g^j(x_l)}{x_{i,l}^{a_i}} \to y_j \quad \text{as} \quad l \to \infty, \quad j = 1, \ldots, m. \]
Consequently, there exists $C > |y|$ such that
\[ |g(x_l)| \leq C|x_l|^\xi \quad \text{for} \quad l \in \mathbb{N}. \]
Hence, $L_{\infty}(g|f^{-1}(U)) \leq \xi$. This gives $L_{\infty,f-\lambda}(g) \leq \xi$, and the inclusion “$\supset$” in (4.1) is proved.

By Theorem [1.2] the set $\partial_{g/f}(\mathbb{C}^k) \subset \mathbb{Q} \cup \{-\infty, +\infty\}$ is finite, say $\{r_1, \ldots, r_s\}$ with $r_1 < \cdots < r_s$. By (4.1), the sets $\Lambda_{r_i}$, $i = 1, \ldots, s$, are algebraic, and $\Lambda_{r_i} \subset \cdots \subset \Lambda_{r_1} = \mathbb{C}^k$. Then the function $\partial_{g/f}$ is upper semi-continuous. Hence the usual complex stratification of $\mathbb{C}^n$ compatible with complex constructible sets $\Lambda_{r_i} \setminus \Lambda_{r_{i-1}}$ is a desired stratification. This ends the proof.
References


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