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A NEW PROOF OF THE ORLICZ BUSEMANN-PETTY CENTROID INEQUALITY

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Abstract. Using shadow systems, we provide a new proof of the Orlicz Busemann-Petty centroid inequality, which was first obtained by Lutwak, Yang and Zhang.

1. Introduction

Recently, in three remarkable papers [12, 23, 24], an Orlicz Brunn-Minkowski theory which extends the L_p Brunn-Minkowski theory emerged. This extension is motivated by asymmetric concepts within the L_p Brunn-Minkowski theory developed by Ludwig [14], Haberl and Schuster [9, 11], and Ludwig and Reitzner [16]. As part of this new Orlicz Brunn-Minkowski theory, Lutwak, Yang and Zhang established two beautiful inequalities, the Orlicz Busemann-Petty centroid inequality [24] and the Orlicz Petty projection inequality [23]. It turns out that the objects of the Orlicz Brunn-Minkowski theory are much more general than those of the L_p Brunn-Minkowski theory. Fortunately, basic results in the L_p Brunn-Minkowski theory, such as affine isoperimetric inequalities, carry over to the general situation.

In this paper, inspired by the work of Campi and Gronchi [2, 3, 4], we will give an alternative proof of the Orlicz Busemann-Petty centroid inequality.

For more information on the L_p and Orlicz Brunn-Minkowski theory see, e.g., [1]–[5], [7]–[24], [29] and the references therein.

Let $\phi : \mathbb{R} \to [0, \infty)$ be an even strictly convex function such that $\phi(0) = 0$. The class of such a ϕ will be denoted by $\mathcal{C}$. Let K be a convex body (i.e., a compact, convex set with non-empty interior) in $\mathbb{R}^n$ that contains the origin in its interior. Denote by $|K|$ the volume of K. The *Orlicz centroid body* $\Gamma_\phi K$ of K, as defined in [24], is the convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h_{\Gamma_\phi K}(x) = \inf\Big\{\lambda > 0 : \frac{1}{|K|}\int_K \phi\Big(\frac{\langle x, z\rangle}{\lambda}\Big)dz \le 1\Big\},$$

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where $\langle x, z\rangle$ denotes the standard inner product of x and z, and the integration is with respect to Lebesgue measure in $\mathbb{R}^n$. In [24] it was shown that the function $h_{\Gamma_\phi K}$ is positively homogeneous and subadditive and hence a support function. Actually, the Orlicz centroid body can be defined on star bodies. It is clear that $|\Gamma_\phi K|/|K|$ is not translation invariant. A natural restriction which makes $|\Gamma_\phi K|/|K|$ bounded is to consider only convex bodies containing the origin in its interior.

Orlicz Busemann-Petty centroid inequality [24]**.** *If $\phi \in \mathcal{C}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized if and only if K is an ellipsoid centered at the origin.

The critical part of the proof in [24] is that the volume of the Orlicz centroid body is not increased after a Steiner symmetrization. It is well known that every convex body can be transformed into a ball by a sequence of suitable Steiner symmetrizations. Therefore the ratio $|\Gamma_\phi K|/|K|$ attains its minimum when K is a ball.

In this paper, we also follow this principle. The technique we will use is that of shadow systems developed by Rogers [26] and Shephard [28]. In fact, the technique of shadow systems has been applied by Campi and Gronchi [2] to recover the L_p Busemann-Petty centroid inequality, which was first obtained by Lutwak, Yang and Zhang [20]. So our work is a natural extension of the work of Campi and Gronchi [2]. It would be impossible to overstate our reliance on their work.

A *shadow system* along the unit direction v is a family of convex hulls in $\mathbb{R}^n$,

$$K_t = conv\{z + \alpha(z)tv : z \in A \subset \mathbb{R}^n\},$$

where A is an arbitrary bounded set of points, α is a real bounded function on A, and the parameter t runs in an interval of the real axis.

A *parallel chord movement* along the unit direction v, a particular type of a shadow system, is a family of convex bodies K_t in $\mathbb{R}^n$ defined by

$$K_t = \{z + \beta(z|v^\perp)tv : z \in K, 0 \le t \le 1\}, \tag{1.1}$$

where K is a convex body in $\mathbb{R}^n$ and β is a continuous real function on $v^\perp = \{z \in \mathbb{R}^n : \langle v, z\rangle = 0\}$. Notice that $|K_t|$ and the orthogonal projection $K_t|v^\perp$ of K_t are independent of t.

For a direction v, define a convex body by

$$K = \{x + yv : x \in K|v^\perp, y \in \mathbb{R}, f(x) \le y \le g(x)\}.$$

Then the parallel chord movement with speed function $\beta(x) = -(f(x) + g(x))$ is such that $K_0 = K$, $K_1 = K^v$, the reflection of K in the hyperplane is $v^\perp$, and $K_{1/2}$ is the Steiner symmetral of K with respect to $v^\perp$.

Theorem 1.1. *If $\{K_t : 0 \le t \le 1\}$ is a parallel chord movement along the unit direction v, then $\Gamma_\phi K_t$ is a shadow system along the same direction v.*

In order to deduce the Orlicz Busemann-Petty centroid inequality from Theorem 1.1, the following facts will be needed.

Fact 1 (Shephard [28]): *The volume of a shadow system is a convex function of the parameter t.*

Fact 2 (Lutwak, Yang and Zhang [24]): *Let $\phi \in \mathcal{C}$. For a convex body K in $\mathbb{R}^n$ and $T \in GL(n)$, $\Gamma_\phi(TK) = T(\Gamma_\phi K)$.*

Fact 3 (Lutwak, Yang and Zhang [24]): *The Orlicz centroid operator Γ_ϕ is continuous in the Hausdorff metric.*

Theorem 1.1 and Fact 1 imply that the volume of $\Gamma_\phi K_t$ is a convex function of t. From Fact 2 we get that $\Gamma_\phi(K^v) = (\Gamma_\phi K)^v$. Thus

$$|\Gamma_\phi K_{1/2}| \leq \frac{1}{2}|\Gamma_\phi K_0| + \frac{1}{2}|\Gamma_\phi K_1| = |\Gamma_\phi K|;$$

that is, the volume of the Orlicz centroid body is not increased after a Steiner symmetrization. The continuity of the Orlicz centroid operator implies the continuity of the ratio $|\Gamma_\phi K|/|K|$ in the Hausdorff metric. It follows that the ratio attains its minimum value when K is a ball.

Theorem 1.2. *If $\{K_t : 0 \leq t \leq 1\}$ is a parallel chord movement with speed function β, then the volume of $\Gamma_\phi K_t$ is a strictly convex function of t unless β is linear.*

If the speed function β of the parallel chord movement is linear, then it is easy to see that K_t is a linear image of K, for every t in the range of the movement. It is well known, see [25], that if K is not an origin symmetric ellipsoid, then there exists a direction v such that for the Steiner symmetral $S_v K$ of K,

$$S_v K \neq AK,$$

for all $A \in GL(n)$. Therefore, $|\Gamma_\phi K|/|K|$ is minimized if and only if K is an ellipsoid centered at the origin. The Orlicz Busemann-Petty centroid inequality is established.

2. Proofs of the main results

Since ϕ is strictly convex on $\mathbb{R}$ such that $\phi(0) = 0$, it follows that the function

$$\lambda \mapsto \int_K \phi\Big(\frac{\langle x, z\rangle}{\lambda}\Big) dz$$

is strictly decreasing in $(0, \infty)$. It is also continuous. Thus, we have for $x \in \mathbb{R}^n \setminus \{0\}$,

$$h_{\Gamma_\phi K}(x) = \lambda \quad \Leftrightarrow \quad \frac{1}{|K|}\int_K \phi\Big(\frac{\langle x, z\rangle}{\lambda}\Big) dz = 1. \tag{2.1}$$

Lemma 2.1. *If $\{K_t : 0 \leq t \leq 1\}$ is a parallel chord movement along the unit direction v, then the orthogonal projection of $\Gamma_\phi K_t$ onto $v^\perp$ is independent of t.*

Proof. By (1.1) we have

$$\begin{aligned} h_{\Gamma_\phi K_t}(x) &= \inf\Big\{\lambda > 0 : \frac{1}{|K_t|}\int_{K_t} \phi\Big(\frac{\langle x, z\rangle}{\lambda}\Big) dz \leq 1\Big\} \\ &= \inf\Big\{\lambda > 0 : \frac{1}{|K_0|}\int_{K_0} \phi\Big(\frac{\langle x, z + \beta(z|v^\perp)tv\rangle}{\lambda}\Big) dz \leq 1\Big\} \\ &= \inf\Big\{\lambda > 0 : \frac{1}{|K|}\int_{K} \phi\Big(\frac{\langle x, z\rangle + \beta(z|v^\perp)t\langle x, v\rangle}{\lambda}\Big) dz \leq 1\Big\}. \end{aligned}$$

Then for $x \in v^\perp$, $h_{\Gamma_\phi K_t}(x) = h_{\Gamma_\phi K}(x)$. ∎

The following lemma shows that $h_{\Gamma_\phi K_t}(x)$ is a Lipschitz function of t, hence is continuous with respect to t.

Lemma 2.2. *If $\phi \in \mathcal{C}$, then for $t_1, t_2 \in [0,1]$ and $x \in \mathbb{R}^n \setminus \{0\}$,*

$$|h_{\Gamma_\phi K_{t_1}}(x) - h_{\Gamma_\phi K_{t_2}}(x)| \le |t_1 - t_2| \|\beta(\cdot|v^\perp)\langle x, v\rangle\|_\phi,$$

where $\|\cdot\|_\phi$ is defined for $f : K \to \mathbb{R}$ which is continuous and not constant to 0 as

$$\|f\|_\phi = \inf\Big\{\lambda > 0 : \frac{1}{|K|}\int_K \phi\Big(\frac{f(z)}{\lambda}\Big)dz \le 1\Big\}.$$

Proof. Let $f, g : K \to \mathbb{R}$ be continuous and not constant to 0. Then the strict convexity of ϕ on $\mathbb{R}$ implies that

$$\|f\|_\phi = \lambda_1 \quad \Leftrightarrow \quad \frac{1}{|K|}\int_K \phi\Big(\frac{f(z)}{\lambda_1}\Big)dz = 1 \tag{2.2}$$

and

$$\|g\|_\phi = \lambda_2 \quad \Leftrightarrow \quad \frac{1}{|K|}\int_K \phi\Big(\frac{g(z)}{\lambda_2}\Big)dz = 1. \tag{2.3}$$

The convexity of the function ϕ shows that

$$\phi\Big(\frac{f(z)+g(z)}{\lambda_1+\lambda_2}\Big) \le \frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{f(z)}{\lambda_1}\Big) + \frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{g(z)}{\lambda_2}\Big).$$

Integrating both sides with respect to the Lebesgue measure of K and using (2.2), (2.3) give

$$\frac{1}{|K|}\int_K \phi\Big(\frac{f(z)+g(z)}{\lambda_1+\lambda_2}\Big)dz \le 1.$$

From the definition of $\|\cdot\|_\phi$ we get

$$\|f+g\|_\phi \le \lambda_1 + \lambda_2 = \|f\|_\phi + \|g\|_\phi.$$

Thus

$$|\|f\|_\phi - \|g\|_\phi| \le \|f-g\|_\phi.$$

The facts that ϕ is even and

$$h_{\Gamma_\phi K_t}(x) = \|\langle x, \cdot\rangle + \beta(\cdot|v^\perp)t\langle x, v\rangle\|_\phi$$

conclude the proof. □

Since $\Gamma_\phi K_t$ is a convex body for every $0 \le t \le 1$, it can be represented by

$$\Gamma_\phi K_t = \{x + yv : x \in (\Gamma_\phi K_0)|v^\perp, f_t(x) \le y \le g_t(x)\}, \tag{2.4}$$

where f_t and $-g_t$ are convex functions defined on $(\Gamma_\phi K_0)|v^\perp$.

Lemma 2.3. *If $\{K_t : 0 \le t \le 1\}$ is a parallel chord movement along the unit direction v, then for every $x \in (\Gamma_\phi K_0)|v^\perp$,*

$$g_t(x) = \inf_{u\in v^\perp}\{h_{\Gamma_\phi K_t}(u+v) - \langle x, u\rangle\} \tag{2.5}$$

and

$$f_t(x) = \sup_{u\in v^\perp}\{\langle x, u\rangle - h_{\Gamma_\phi K_t}(u-v)\}. \tag{2.6}$$

Proof. Let $u \in v^{\perp}$. For $x \in (\Gamma_\phi K_0)|v^{\perp}$ we have

$$x + g_t(x)v \in \Gamma_\phi K_t, \quad x + f_t(x)v \in \Gamma_\phi K_t.$$

The definition of the support function shows that

$$\langle x + g_t(x)v, u + v\rangle \le h_{\Gamma_\phi K_t}(u+v),$$

$$\langle x + f_t(x)v, u - v\rangle \le h_{\Gamma_\phi K_t}(u-v).$$

Thus,

$$\langle x, u\rangle + g_t(x) \le h_{\Gamma_\phi K_t}(u+v), \quad \langle x, u\rangle - f_t(x) \le h_{\Gamma_\phi K_t}(u-v)$$

for all $u \in v^{\perp}$.

Since $\Gamma_\phi K_t$ has support hyperplanes at the two points $x + g_t(x)v, x + f_t(x)v \in \partial(\Gamma_\phi K_t)$, for $x \in \operatorname{relint}\big((\Gamma_\phi K_0)|v^{\perp}\big)$, there exist two vectors $u' + v$ and $u'' - v$ with $u', u'' \in v^{\perp}$ such that

$$\begin{aligned}\langle x + g_t(x)v, u' + v\rangle &= h_{\Gamma_\phi K_t}(u'+v),\\ \langle x + f_t(x)v, u'' - v\rangle &= h_{\Gamma_\phi K_t}(u''-v).\end{aligned}$$

If $x \notin \operatorname{relint}\big((\Gamma_\phi K_0)|v^{\perp}\big)$, it is possible that $g_t(x) = 0, f_t(x) = 0$. Then we cannot find $u', u'' \in v^{\perp}$ such that

$$\begin{aligned}\langle x + g_t(x)v, u' + v\rangle &= \langle x, u'\rangle = h_{\Gamma_\phi K_t}(u'+v),\\ \langle x + f_t(x)v, u'' - v\rangle &= \langle x, u''\rangle = h_{\Gamma_\phi K_t}(u''-v).\end{aligned}$$

The continuity of support functions ensures that we can take the infimum and supremum for all $u \in v^{\perp}$. Therefore, we get

$$g_t(x) = \inf_{u \in v^{\perp}} \{h_{\Gamma_\phi K_t}(u+v) - \langle x, u\rangle\}$$

and

$$f_t(x) = \sup_{u \in v^{\perp}} \{\langle x, u\rangle - h_{\Gamma_\phi K_t}(u-v)\}$$

for every $x \in (\Gamma_\phi K_0)|v^{\perp}$. □

Since $h_{\Gamma_\phi K_t}(x)$ is a Lipschitz function of t, with Lipschitz constant $\|\beta(\cdot|v^{\perp})\langle x, v\rangle\|_\phi$, from Lemma 2.3 we deduce that $g_t(x)$ and $f_t(x)$ are Lipschitz functions of t too. Hence $g_t(x)$ and $f_t(x)$ are continuous with respect to t. Moreover, the convexity of $g_t(x)$ and $-f_t(x)$ with respect to t can be stated as follows.

Lemma 2.4. *If $\{K_t : 0 \le t \le 1\}$ is a parallel chord movement along the unit direction v, then for every $x \in (\Gamma_\phi K_0)|v^{\perp}$, $g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in* $[0,1]$.

Proof. We first show that if $u_1, u_2 \in v^{\perp}$, then

$$h_{\Gamma_\phi K_{\frac{t_1+t_2}{2}}}(u_1 + u_2 + 2v) \le h_{\Gamma_\phi K_{t_1}}(u_1 + v) + h_{\Gamma_\phi K_{t_2}}(u_2 + v). \tag{2.7}$$

In fact, let $h_{\Gamma_\phi K_{t_1}}(u_1+v)=\lambda_1, h_{\Gamma_\phi K_{t_2}}(u_2+v)=\lambda_2$. The convexity of ϕ gives that

$$\begin{aligned}
&\phi\Big(\frac{\langle u_1+u_2+2v,z\rangle+\beta(z|v^\perp)\frac{t_1+t_2}{2}\langle u_1+u_2+2v,v\rangle}{\lambda_1+\lambda_2}\Big)\\
&=\phi\Big(\frac{\langle u_1+v,z\rangle+\beta(z|v^\perp)t_1+\langle u_2+v,z\rangle+\beta(z|v^\perp)t_2}{\lambda_1+\lambda_2}\Big)\\
&=\phi\Big(\frac{\langle u_1+v,z\rangle+\beta(z|v^\perp)t_1\langle u_1+v,v\rangle+\langle u_2+v,z\rangle+\beta(z|v^\perp)t_2\langle u_2+v,v\rangle}{\lambda_1+\lambda_2}\Big)\\
&\le\frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{\langle u_1+v,z\rangle+\beta(z|v^\perp)t_1\langle u_1+v,v\rangle}{\lambda_1}\Big)
\end{aligned}$$

(2.8)

$$+\frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{\langle u_2+v,z\rangle+\beta(z|v^\perp)t_2\langle u_2+v,v\rangle}{\lambda_2}\Big).$$

Integrating both sides and using (2.1), we obtain (2.7).

By Lemma 2.3 and (2.7), we obtain

$$\begin{aligned}
2g_{\frac{t_1+t_2}{2}}(x)&=\inf_{u\in v^\perp}\{h_{\Gamma_\phi K_{\frac{t_1+t_2}{2}}}(2(u+v))-\langle x,2u\rangle\}\\
&=\inf_{u_1,u_2\in v^\perp}\{h_{\Gamma_\phi K_{\frac{t_1+t_2}{2}}}(u_1+u_2+2v)-\langle x,u_1+u_2\rangle\}\\
&\le\inf_{u_1,u_2\in v^\perp}\{h_{\Gamma_\phi K_{t_1}}(u_1+v)+h_{\Gamma_\phi K_{t_2}}(u_2+v)-\langle x,u_1+u_2\rangle\}\\
&=\inf_{u_1\in v^\perp}\{h_{\Gamma_\phi K_{t_1}}(u_1+v)-\langle x,u_1\rangle\}+\inf_{u_2\in v^\perp}\{h_{\Gamma_\phi K_{t_1}}(u_2+v)-\langle x,u_2\rangle\}\\
&=g_{t_1}(x)+g_{t_2}(x).
\end{aligned}$$

The convexity of the function $-f_t$ of t can be proved in the same way. □

Lemma 2.5. *If $\{K_t : 0\le t\le 1\}$ is a parallel chord movement along the unit direction v, then for every $x\in(\Gamma_\phi K_0)|v^\perp$ and $t_1,t_2,\theta\in[0,1]$,*

$$f_{\theta t_1+(1-\theta)t_2}(x)\le\theta g_{t_1}(x)+(1-\theta)f_{t_2}(x)\le g_{\theta t_1+(1-\theta)t_2}(x).$$

Proof. Let $u_1,u_2\in v^\perp$ and

$$h_{\Gamma_\phi K_{t_1}}(-\theta u_1+\theta v)=\lambda_1,\quad h_{\Gamma_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2-v)=\lambda_2.$$

Then we have

$$\begin{aligned}
&\phi\Big(\frac{\langle u_2-\theta u_1-(1-\theta)v,z\rangle+\beta(z|v^\perp)t_2\langle u_2-\theta u_1-(1-\theta)v,v\rangle}{\lambda_1+\lambda_2}\Big)\\
&=\phi\Big(\frac{\langle u_2-v,z\rangle+\langle-\theta u_1+\theta v,z\rangle-\beta(z|v^\perp)((1-\theta)t_2+\theta t_1-\theta t_1)}{\lambda_1+\lambda_2}\Big)\\
&\le\frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{\langle u_2-v,z\rangle+\beta(z|v^\perp)((1-\theta)t_2+\theta t_1)\langle u_2-v,v\rangle}{\lambda_2}\Big)\\
&\qquad+\frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{\langle-\theta u_1+\theta v,z\rangle+\beta(z|v^\perp)t_1\langle-\theta u_1+\theta v,v\rangle}{\lambda_1}\Big).
\end{aligned}$$

Integrating both sides and using (2.1) give

(2.9) $$h_{\Gamma_\phi K_{t_2}}(u_2-\theta u_1-(1-\theta)v)\le h_{\Gamma_\phi K_{t_1}}(-\theta u_1+\theta v)+h_{\Gamma_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2-v).$$

Thus, from (2.9), we get

$$
\begin{aligned}
&(1-\theta)f_{t_2}(x)\\
&= \sup_{u\in v^\perp}\{\langle x,(1-\theta)u\rangle - h_{\Gamma_\phi K_{t_2}}((1-\theta)(u-v))\}\\
&= \sup_{-u_1,u_2\in v^\perp}\{\langle x,u_2-\theta u_1\rangle - h_{\Gamma_\phi K_{t_2}}(u_2-\theta u_1-(1-\theta)v)\}\\
&\geq \sup_{-u_1,u_2\in v^\perp}\{\langle x,u_2-\theta u_1\rangle - h_{\Gamma_\phi K_{t_1}}(-\theta u_1+\theta v) - h_{\Gamma_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2-v)\}\\
&= \sup_{-u_1\in v^\perp}\{\langle x,-\theta u_1\rangle - h_{\Gamma_\phi K_{t_1}}(-\theta u_1+\theta v)\}\\
&\qquad\qquad + \sup_{u_2\in v^\perp}\{\langle x,u_2\rangle - h_{\Gamma_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2-v)\}\\
&= -\theta g_{t_1}(x) + f_{\theta t_1+(1-\theta)t_2}(x).
\end{aligned}
$$

This gives the first inequality. The second inequality follows by interchanging t_1 with t_2 and x with $-x$. □

In order to prove Theorem 1.1 we shall require the following crucial lemma, which was proved by Campi and Gronchi [2].

Lemma 2.6. *Let $\{H_t : 0 \leq t \leq 1\}$ be a one-parameter family of convex bodies such that $H_t|v^\perp$ is independent of t. Assume that the bodies H_t are defined by*

$$H_t = \{x + yv : x \in H_t|v^\perp, y \in \mathbb{R}, f_t(x) \leq y \leq g_t(x)\}, \quad 0 \leq t \leq 1,$$

for suitable functions g_t, f_t. Then $\{H_t : 0 \leq t \leq 1\}$ is a shadow system along the direction v if and only if for every $x \in H_0|v^\perp$,

1. *$g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in $[0,1]$,*
2. *$f_{\lambda t_1+(1-\lambda)t_2}(x) \leq \lambda g_{t_1}(x) + (1-\lambda)f_{t_2}(x) \leq g_{\lambda t_1+(1-\lambda)t_2}(x)$, for every $t_1, t_2, \lambda \in [0,1]$.*

Proof of Theorem 1.1. Let $\{K_t : 0 \leq t \leq 1\}$ be a parallel chord movement along the unit direction v. By Lemma 2.1 we obtain that the orthogonal projection of $\Gamma_\phi K_t$ onto $v^\perp$ is independent of t. Then from Lemma 2.6 it is sufficient to show that the family $\Gamma_\phi K_t$ satisfies conditions (1) and (2) of Lemma 2.6. Actually, Lemma 2.4 and Lemma 2.5 demonstrate these two conditions for $\Gamma_\phi K_t$. Therefore, we deduce that $\Gamma_\phi K_t$ is a shadow system along the direction v. □

Proof of Theorem 1.2. By Fubini's theorem it is easy to see that

$$|\Gamma_\phi K_t| = \int_{(\Gamma_\phi K_0)|v^\perp} \big(g_t(x) - f_t(x)\big)dx. \tag{2.10}$$

That the volume of $\Gamma_\phi K_t$ is a convex function of t therefore follows from the convexity of $g_t(x)$ and $-f_t(x)$ with respect to t.

Suppose that

$$\big|\Gamma_\phi K_{\frac{t_1+t_2}{2}}\big| = \frac{1}{2}|\Gamma_\phi K_{t_1}| + \frac{1}{2}|\Gamma_\phi K_{t_1}|$$

for some $t_1, t_2 \in [0,1]$. From (2.10) and the continuity of g_t, f_t with respect to x, we obtain that

$$g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x) = \frac{1}{2}\big(g_{t_1}(x) + g_{t_2}(x)\big) - \frac{1}{2}\big(f_{t_1}(x) + f_{t_2}(x)\big) \tag{2.11}$$

for almost every $x \in (\Gamma_\phi K_0)|v^\perp$. Let $x \in \operatorname{relint}\big((\Gamma_\phi K_0)|v^\perp\big)$. Then there exist $u_1, u_2, u_3, u_4 \in v^\perp$ such that

$$\begin{aligned}&\frac{1}{2}\big(g_{t_1}(x)+g_{t_2}(x)\big)-\frac{1}{2}\big(f_{t_1}(x)+f_{t_2}(x)\big)\\&=\frac{1}{2}\big(h_{\Gamma_\phi K_{t_1}}(u_1+v)+h_{\Gamma_\phi K_{t_2}}(u_2+v)+h_{\Gamma_\phi K_{t_1}}(u_3-v)+h_{\Gamma_\phi K_{t_2}}(u_4-v)\\&\qquad-\langle x,u_1\rangle-\langle x,u_2\rangle-\langle x,u_3\rangle-\langle x,u_4\rangle\big).\end{aligned}$$

By (2.7) we get

$$\begin{aligned}&\frac{1}{2}\big(g_{t_1}(x)+g_{t_2}(x)\big)-\frac{1}{2}\big(f_{t_1}(x)+f_{t_2}(x)\big)\\&\qquad\ge h_{\Gamma_\phi K_{\frac{t_1+t_2}{2}}}\Big(\frac{u_1+u_2}{2}+v\Big)-\Big\langle x,\frac{u_1+u_2}{2}\Big\rangle\\&\qquad\quad+h_{\Gamma_\phi K_{\frac{t_1+t_2}{2}}}\Big(\frac{u_3+u_4}{2}-v\Big)-\Big\langle x,\frac{u_3+u_4}{2}\Big\rangle\\&\qquad\ge g_{\frac{t_1+t_2}{2}}(x)-f_{\frac{t_1+t_2}{2}}(x).\end{aligned}\tag{2.12}$$

The equality of (2.11) forces equality in (2.12) and equality in (2.8). Since ϕ is strictly convex, we have

$$\frac{\langle u_1+v,z\rangle+\beta(z|v^\perp)t_1}{\lambda_1}=\frac{\langle u_2+v,z\rangle+\beta(z|v^\perp)t_2}{\lambda_2}\tag{2.13}$$

for every $z \in K_0$, owing to the continuity of β.

Setting $z = z' + sv$, $z' \in K_0|v^\perp$, in (2.13) and differentiating with respect to the parameter s, it turns out that $\lambda_1/\lambda_2 = 1$, that is,

$$\langle u_1+v,z\rangle+\beta(z|v^\perp)t_1=\langle u_2+v,z\rangle+\beta(z|v^\perp)t_2.$$

So we conclude that $\beta(x) = \langle x, u\rangle$ for some vector u. This completes the proof. □

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