A NON-RESIDUALLY SOLVABLE HYPERLINEAR ONE-RELATOR GROUP

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Abstract. In this short paper, we prove that the group \( \langle a, b \mid a^{-1}[a, a^b] \rangle \) is hyperlinear. Unlike the nonresidually finite Baumslag-Solitar groups, this group is not residually solvable.

1. Introduction

Let \( \Gamma \) denote the one-relator group \( \langle a, b \mid a^{-1}[a, a^b] \rangle \), where \( a^b = bab^{-1} \) and \([a, a^b] = a^{-1}(a^b)^{-1}aa^b\). This group was introduced by G. Baumslag in [Baum69] as an example of a noncyclic one-relator group with the property that all of its finite index quotients are cyclic. It follows that the group \( \Gamma \) is not residually finite. Also, \( \Gamma \) is not residually solvable, since \( a \) lies in every one of the derived subgroups of \( \Gamma \).

A countable discrete group \( G \) is hyperlinear if it can be embedded as a subgroup of the unitary group \( U(R^\omega) \) of an ultrapower \( R^\omega \) of the hyperfinite type \( I_1 \) factor \( R \) (cf. [Pest08]). Equivalently, \( G \) is hyperlinear if the group von Neumann algebra \( L(G) \) is embeddable into \( R^\omega \) (cf. [Pest08]). Proposition 4.14 of [Ueda09] establishes that every HNN extension of a \( R^\omega \)-embeddable type \( I_1 \) factor over a hyperfinite von Neumann subalgebra is also \( R^\omega \)-embeddable. We use this fact along with a now standard trick of McCool and Schupp for one-relator groups to prove that the group \( \Gamma \) above is hyperlinear. The main interest in this example is that it is an example of a nonresidually solvable hyperlinear one-relator group, and thus our result sheds a little light on the question of Nate Brown asking whether every one-relator group is hyperlinear. In [Rad00], Radulescu proved that the nonresidually finite Baumslag-Solitar group \( \langle a, b \mid ab^3a^{-1}b^{-2} \rangle \) is hyperlinear. Radulescu’s result is shown in [Pest08] to follow more simply from the fact that these Baumslag-Solitar groups are residually solvable, and hence sofic.

2. Main result

Theorem 2.1. The group \( \Gamma = \langle a, b \mid a^{-1}[a, a^b] \rangle \) is hyperlinear.

Proof. We apply a rewriting process due to McCool and Schupp (cf. [McCSch73]). Let \( a_0 = a \) and \( a_{-1} = bab^{-1} \). Note that the word

\[
a^{-1}[a, a^b] = a^{-2}ba^{-1}b^{-1}abab^{-1}
\]

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when rewritten in terms of $a_0$ and $a_{-1}$ becomes
\[ a_0^{-2}(a_{-1})^{-1}a_0(a_{-1}). \]
The group $H = \langle a_0, a_{-1} | a_0^{-2}(a_{-1})^{-1}a_0(a_{-1}) \rangle$ is amenable, essentially by the Tits alternative. Or, we may appeal to Theorem 1.2 of [CeGrig97] and note that $a_0^{-2}(a_{-1})^{-1}a_0(a_{-1})$ has exponent sum zero on $a_{-1}$ and can be obtained from $(a_{-1})a_0(a_{-1})^{-1}a_0^{-2}$ by inverting $a_{-1}$ and cyclically shifting, and hence $H$ is amenable. We then note that the group $\Gamma$ is isomorphic to the HNN extension
\[ H*_{\varphi} = \langle t, H | t^{-1}a_{-1}t = a_0 \rangle. \]

Now, consider the group von Neumann algebra $L(H*_{\varphi})$. By Corollary 3.5 of [Ueda05], this is isomorphic to a reduced HNN extension of the hyperfinite $II_1$ factor $\mathcal{R}$ over $L(\mathbb{Z})$. Therefore, by Proposition 4.14 of [Ueda09], $L(H*_{\varphi})$ is embeddable into $\mathcal{R}$ and, therefore $\Gamma$ is hyperlinear.

\textit{Remark 2.2.} We wish to thank the referee for pointing out that recently it has been shown that any HNN extension of a sofic group over an amenable subgroup is sofic. Precisely, this is Corollary 3.4 of [DykCol10]. We may, in the above proof, replace Ueda’s result by this one and obtain that $\Gamma$ is, in fact, a sofic group.

\textbf{References}


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