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SEMIGROUPS OF HOLOMORPHIC FUNCTIONS IN THE POLYDISK

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ABSTRACT. In this paper we provide an easy-to-use characterization of infinitesimal generators of semigroups of holomorphic functions in the polydisk. We also present a number of examples related to that characterization.

1. Introduction and statement of the main result

A *(continuous) semigroup* (Φ_t) of holomorphic functions on a domain $D \subset \mathbb{C}^n$ is a continuous homomorphism from the additive semigroup of non-negative real numbers into the composition semigroup of all holomorphic self-maps of D endowed with the compact-open topology. Namely, the map $[0, +\infty) \ni t \mapsto (\Phi_t) \in \mathrm{Hol}(D, D)$ satisfies the following conditions:

(1) Φ_0 is the identity map id_D in D,
(2) $\Phi_{t+s} = \Phi_t \circ \Phi_s$, for all $t, s \geq 0$,
(3) Φ_t tends to id_D as t tends to 0 uniformly on compacta of D.

Given a semigroup, it is well-known (see, *e.g.*, [1, Section 2.5.3]) that there is a holomorphic map $F : D \to \mathbb{C}^n$ such that

$$\frac{\partial \Phi_t}{\partial t} = F(\Phi_t).$$

This vector field F is called the *infinitesimal generator* of the semigroup (Φ_t). Indeed, $F(z) = \frac{d\Phi_t(z)}{dt}|_{t=0}$ for all $z \in D$. It is worth remarking here that the key property for a holomorphic vector field $F : D \to \mathbb{C}^n$ to be an infinitesimal generator is *semicompleteness*.

It is clear that the analytical properties of an infinitesimal generator are strictly related to the dynamical and geometrical properties of its semigroup. For instance, any zero of F in D corresponds to a common fixed point for (Φ_t).

Therefore one of the main questions in the theory of semigroups of holomorphic functions is that of characterizing (in the most useful way) those holomorphic vector fields which are infinitesimal generators. On the unit disk and on the unit ball of

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$\mathbb{C}^n$ there are several characterizations of infinitesimal generators of a semigroup (see, *e.g.*, [2], [3], [4], [5], [6], [7] and [10], just to name a few).

On the polydisk, and more generally on the product of convex domains, much less is known; the main references are [2], whose very general statement of Theorem 8 can be applied also in the case of a product of two (or even more) domains in $\mathbb{C}^n$ or complex manifolds, and [9].

In order to state our result we need to introduce some notation: for any $j \in \{1, \dots, n\}$ set

$$S_j = \{(z, \zeta) \in \Delta^n \times \Delta^n : \quad k_\Delta(z_l, \zeta_l) < k_\Delta(z_j, \zeta_j) \ \forall l \neq j\},$$

where k_Δ is the Poincaré distance on the unit disk Δ. Notice that $\Delta^n \times \Delta^n = \bigcup_{j=1}^n \overline{S_j}$.

The major result of this paper is

Theorem 1.1. *Let $F = (F_1, F_2, \dots, F_n) : \Delta^n \to \mathbb{C}^n$ be a holomorphic map. Then F is the infinitesimal generator of a semigroup of holomorphic functions in Δ^n if and only if the following condition holds for any $j \in \{1, \dots, n\}$:*

(1.1)
$$\frac{1}{1-|z_j|^2} \mathsf{Re} \left[\frac{z_j - \zeta_j}{1 - \overline{z_j}\zeta_j} \overline{F_j(z)} \right] \leq \frac{1}{1-|\zeta_j|^2} \mathsf{Re} \left[\frac{z_j - \zeta_j}{1 - \overline{\zeta_j} z_j} \overline{F_j(\zeta)} \right] \quad \text{if } (z, \zeta) \in \overline{S_j}.$$

In particular, when applied to $n = 2$ the theorem reads as follows:

Corollary 1.2. *Let $F = (F_1, F_2) : \Delta^2 \to \mathbb{C}^2$ be a holomorphic map. Then F is the infinitesimal generator of a semigroup of holomorphic functions in Δ^2 if and only if the following condition holds:*

$$\begin{cases} \dfrac{1}{1-|z_1|^2} \mathsf{Re} \left[\dfrac{z_1 - \zeta_1}{1 - \overline{z_1}\zeta_1} \overline{F_1(z)} \right] \leq \dfrac{1}{1-|\zeta_1|^2} \mathsf{Re} \left[\dfrac{z_1 - \zeta_1}{1 - \overline{\zeta_1} z_1} \overline{F_1(\zeta)} \right] \\ \hfill \textit{if } k_\Delta(z_2, \zeta_2) \leq k_\Delta(z_1, \zeta_1), \\ \dfrac{1}{1-|z_2|^2} \mathsf{Re} \left[\dfrac{z_2 - \zeta_2}{1 - \overline{z_2}\zeta_2} \overline{F_2(z)} \right] \leq \dfrac{1}{1-|\zeta_2|^2} \mathsf{Re} \left[\dfrac{z_2 - \zeta_2}{1 - \overline{\zeta_2} z_1} \overline{F_2(\zeta)} \right] \\ \hfill \textit{if } k_\Delta(z_1, \zeta_1) \leq k_\Delta(z_2, \zeta_2). \end{cases}$$

An immediate consequence of the above theorem is the following corollary (compare with [1, Corollary 2.5.29] for the ball).

Corollary 1.3. *The set of infinitesimal generators of semigroups of holomorphic functions in Δ^n is a cone in $\mathrm{Hol}(\Delta^n, \mathbb{C}^n)$ with vertex at 0.*

In fact, we can state the following useful characterization:

Corollary 1.4. *Let $F = (F_1, \dots, F_n) : \Delta^n \to \mathbb{C}^n$ be a holomorphic map. Then F is the infinitesimal generator of a semigroup of holomorphic functions in Δ^n if and only if the holomorphic functions $G_j = (0, \dots, F_j, 0 \dots, 0) : \Delta^n \to \mathbb{C}^n$ are infinitesimal generators of semigroups of holomorphic functions for all $j \in \{1, \dots, n\}$.*

The first, unexpected problem, when dealing with semigroups on the polydisk, is the difficulty in finding meaningful examples. Indeed, the usual techniques used in the case of the unit disc in $\mathbb{C}$ or the unit ball for the Euclidean metric do not give but a small class of examples. The above corollaries are the key to building non-trivial examples.

2. Examples of semigroups in the polydisk

We start with a short list of examples, since they will be the bricks used to build a much wider variety of situations. Since our attention will be mainly devoted to the infinitesimal generator of the semigroups, each example of a semigroup will be followed by the corresponding infinitesimal generator.

Example 2.1. Of course, if $\Phi^1, \ldots, \Phi^n : \mathbb{R}^+ \to \mathrm{Hol}\,(\Delta, \Delta)$ are semigroups of holomorphic functions on the unit disk, then $\Phi : \mathbb{R}^+ \to \mathrm{Hol}\,(\Delta^n, \Delta^n)$ given by $\Phi_t(z) = \left(\left(\Phi^1\right)_t (z_1), \ldots, \left(\Phi^n\right)_t (z_n)\right)$ is a semigroup on Δ^n. In particular, each of the components of Φ depends on one variable only and the same happens for the infinitesimal generator F which is given by $F(z) = (F_1(z_1), \ldots, F_n(z_n))$, where F_j is the infinitesimal generator of Φ^j for $j = 1, \ldots, n$. In particular, all the F_j's can be characterized in several ways, according to [2], [3] or [10].

Example 2.2. A different kind of example is the following: take any negative $\lambda \in \mathbb{R}$ and $g \in \mathrm{Hol}(\Delta, \overline{\Delta^{n-1}})$ and denote $(z_2, \ldots, z_n)$ by $\tilde{z}$. Now set $\Phi_t(z) = \left(z_1, e^{\lambda t}\tilde{z} + \left(1 - e^{\lambda t}\right) g(z_1)\right)$. Since $\lambda < 0$ and $g\,(\Delta) \subset \overline{\Delta^{n-1}}$, it is easily seen that Φ_t maps Δ^n into itself for any $t \geq 0$. Checking that Φ verifies the composition rule and that $\Phi_0 = \mathrm{id}_{\Delta^n}$ is almost immediate. A simple computation shows that the infinitesimal generator F of Φ is given by $F(z) = (0, \lambda(\tilde{z} - g(z_1)))$. In this case one of the components of the semigroup is the identity but the other ones all depend on two components (and the function g can be chosen freely in $\mathrm{Hol}(\Delta, \overline{\Delta^{n-1}})$).

Example 2.3. Denote by $\mathbb{H}$ the right half-plane and take $g_1, \ldots, g_{n-1} \in \mathrm{Hol}(\Delta, \mathbb{H})$. Then the function $\Phi_t(z) = (z_1 \exp(-g_1(z_n)t), \ldots, z_{n-1} \exp(-g_{n-1}(z_n)t), z_n)$ becomes a semigroup whose infinitesimal generator is given by $F(z) = (-z_1 g_1(z_n), \ldots, -z_{n-1} g_{n-1}(z_n), 0)$.

So far, we have shown some elementary examples. Now, we will apply our main result to obtain some non-trivial ones. In order to simplify notation, we give them in the two-dimensional case, but it is easily seen that Corollary 1.4 gives the possibility of building examples in which all components depend on all variables.

Example 2.4. Take two functions f_1, f_2 in $\mathrm{Hol}(\Delta, \overline{\Delta})$ and two negative real numbers λ_1 and λ_2. By Example 2.2, $G_1(z) = (\lambda_1(z_1 - f_1(z_2)), 0)$ and $G_2(z) = (0, \lambda_2(z_2 - f_2(z_1)))$ are infinitesimal generators of semigroups in Δ^2. Therefore, by Corollary 1.4, the holomorphic function $F(z) = (\lambda_1(z_1 - f_1(z_2)), \lambda_2(z_2 - f_2(z_1)))$ is the infinitesimal generator of a semigroup in Δ^2.

Example 2.5. Take two functions f_1, f_2 in $\mathrm{Hol}(\Delta, \mathbb{H})$. By Example 2.3, we know that the holomorphic functions $G_1(z) = (-z_1 f_1(z_2), 0)$ and $G_2(z) = (0, -z_2 f_2(z_1))$ are infinitesimal generators of semigroups in Δ^2. Therefore, again by Corollary 1.4, the function $F(z) = (-z_1 f_1(z_2), -z_2 f_2(z_1))$ is the infinitesimal generator of a semigroup in Δ^2.

Example 2.6. The following class of examples, though it contains very simple maps, shows another non-trivial application of Corollary 1.4. Take a matrix $M = (m_{i,j})_{i,j=1,2}$. It is easily seen that $\Phi_t(z) = e^{tM} z$ is a semigroup on $\mathbb{C}^2$ whose infinitesimal generator is $F(z) = Mz$. Therefore, (Φ_t) is a semigroup in Δ^2 if and only if F is the infinitesimal generator of a semigroup in Δ^2. Notice that if $\lambda \in \mathbb{C}$ is an eigenvalue of M with eigenvector $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfying that $\max\{|z_1|, |z_2|\} = 1$, then if (Φ_t) is a semigroup in Δ^2 we have that $\Phi_t(z) = e^{\lambda t} z \in \Delta^2$ and then $\mathsf{Re}\,\lambda \leq 0$. Moreover, by Corollary 1.4, $F(z) = Mz$ is the infinitesimal generator of a semigroup in Δ^2 if and only if

$$G_1(z) = \begin{pmatrix} m_{1,1} & m_{1,2} \\ 0 & 0 \end{pmatrix} z \text{ and } G_2(z) = \begin{pmatrix} 0 & 0 \\ m_{2,1} & m_{2,2} \end{pmatrix} z$$

are infinitesimal generators of semigroups in Δ^2. Let us characterize when G_1 and G_2 are infinitesimal generators of semigroups in Δ^2.

First notice that if $b_1, b_2 \in \mathbb{R}$, then the map $z \mapsto \begin{pmatrix} b_1 i & 0 \\ 0 & b_2 i \end{pmatrix} z$ is the infinitesimal generator of a group in Δ^2. Therefore, since the set of infinitesimal generators is a real cone, we have that G_1 and G_2 are infinitesimal generators of semigroups in Δ^2 if and only if

$$\tilde{G}_1(z) = \begin{pmatrix} \mathsf{Re}\,m_{1,1} & m_{1,2} \\ 0 & 0 \end{pmatrix} z \text{ and } \tilde{G}_2(z) = \begin{pmatrix} 0 & 0 \\ m_{2,1} & \mathsf{Re}\,m_{2,2} \end{pmatrix} z$$

are infinitesimal generators of semigroups in Δ^2.

Suppose that $\tilde{G}_1$ is an infinitesimal generator of the semigroup $\Psi_t(z)$ in Δ^2. Since $\mathsf{Re}\,m_{1,1}$ is an eigenvalue of $\begin{pmatrix} \mathsf{Re}\,m_{1,1} & m_{1,2} \\ 0 & 0 \end{pmatrix}$, we have that $\mathsf{Re}\,m_{1,1} \leq 0$. If $\mathsf{Re}\,m_{1,1} = 0$, then $\Psi_t(z) = \begin{pmatrix} 1 & t m_{1,2} \\ 0 & 1 \end{pmatrix} z$ and one easily deduces that $\Psi_t(\Delta^2) \subset \Delta^2$ if and only if $m_{1,2} = 0$. If $\mathsf{Re}\,m_{1,1} \neq 0$, then

$$\Psi_t(z) = \begin{pmatrix} e^{\mathsf{Re}\,m_{1,1} t} & \frac{m_{1,2}}{\mathsf{Re}\,m_{1,1}}(e^{\mathsf{Re}\,m_{1,1} t} - 1) \\ 0 & 1 \end{pmatrix} z.$$

In this case, an easy estimation shows that $\Psi_t(z) \in \Delta^2$ for all $z \in \Delta^2$ if and only if $|m_{1,2}| \leq -\mathsf{Re}\,m_{1,1}$. That is, if $\tilde{G}_1$ is an infinitesimal generator of a semigroup in Δ^2, then $|m_{1,2}| \leq -\mathsf{Re}\,m_{1,1}$. The converse is also true and we have that $\tilde{G}_1$, and then G_1, is an infinitesimal generator in Δ^2 if and only if $|m_{1,2}| \leq -\mathsf{Re}\,m_{1,1}$. Similarly we can work with G_2.

Summing up, we have that $F(z) = Mz$ is the infinitesimal generator of a semigroup in Δ^2 if and only if $|m_{1,2}| \leq -\mathsf{Re}\,m_{1,1}$ and $|m_{2,1}| \leq -\mathsf{Re}\,m_{2,2}$.

3. Proof of Theorem 1.1

In this section we write ρ_Δ for the pseudo-hyperbolic distance on the unit disk, that is, $\rho_\Delta(u, v) = \frac{|u-v|}{|1-\overline{u}v|}$. We denote by k_{Δ^n} the Kobayashi distance on Δ^n (which coincides with the Carathéodory distance since Δ^n is the unit ball of the Banach space $\mathbb{C}^n$ endowed with the max norm and $(\mathrm{Aut}(\Delta)) \times \cdots \times (\mathrm{Aut}(\Delta)) \subset \mathrm{Aut}(\Delta^n)$ acts transitively on Δ^n).

Proof. We start by proving the necessity of condition (1.1). Let $\Phi : \mathbb{R}^+ \to \mathrm{Hol}\,(\Delta^n, \Delta^n)$ be a semigroup whose components we denote by $\Phi^1, \ldots, \Phi^n$. Since the holomorphic self-maps of Δ^n are contractive for the Kobayashi distance we have

$$k_{\Delta^n}\left(\Phi_t(z), \Phi_t(\zeta)\right) \leq k_{\Delta^n}\left(z, \zeta\right) = \max\{k_\Delta(z_1, \zeta_1), \ldots k_\Delta(z_n, \zeta_n)\}.$$

Now we claim that it is enough to prove that condition (1.1) holds for $(z, \zeta) \in S_j$ for any $j = 1, \ldots, n$. Indeed, suppose $(z, \zeta) \in \overline{S_j}$; then there exists a sequence $(z^{(m)}, \zeta^{(m)}) \in S_j$ converging to (z, ζ) when $m \to +\infty$. As $(z^{(m)}, \zeta^{(m)}) \in S_j$, then condition (1.1) holds at $(z^{(m)}, \zeta^{(m)})$ for any $m \in \mathbb{N}$. Since both terms of the inequality are continuous functions on $\Delta^n \times \Delta^n$, by continuity we obtain that condition (1.1) holds at (z, ζ), too.

If $(z, \zeta) \in S_j$, then, by continuity of the semigroup and the fact that $\Phi_0 = \mathsf{id}_{\Delta^n}$, there exists $t_0 > 0$ such that $(\Phi_t(z), \Phi_t(\zeta)) \in S_j$; that is, $k_\Delta\left(\Phi_t^l(z), \Phi_t^l(\zeta)\right) < k_\Delta\left(\Phi_t^j(z), \Phi_t^j(\zeta)\right)$ for any $l \neq j$, for any $0 \leq t \leq t_0$.

Then

$$\begin{aligned} k_{\Delta^n}\left(\Phi_t(z), \Phi_t(\zeta)\right) &= k_\Delta\left(\Phi_t^j(z), \Phi_t^j(\zeta)\right) \\ &\leq k_{\Delta^n}\left(z, \zeta\right) \\ &= k_\Delta(z_j, \zeta_j) = k_\Delta\left(\Phi_0^j(z), \Phi_0^j(\zeta)\right). \end{aligned}$$

Since $k_\Delta\left(\Phi_t^j(z), \Phi_t^j(\zeta)\right) > 0$, setting $h(t) = \rho_\Delta\left(\Phi_t^j(z), \Phi_t^j(\zeta)\right)$ we obtain that the C^1 function $t \mapsto h(t)$ has a non-positive derivative at 0 and therefore

$$\begin{aligned} h'(0) &= \frac{\partial \rho_\Delta}{\partial u}(z_j, \zeta_j) F_j(z) + \frac{\partial \rho_\Delta}{\partial \overline{u}}(z_j, \zeta_j) \overline{F_j(z)} \\ &\quad + \frac{\partial \rho_\Delta}{\partial v}(z_j, \zeta_j) F_j(\zeta) + \frac{\partial \rho_\Delta}{\partial \overline{v}}(z_j, \zeta_j) \overline{F_j(\zeta)} \\ &= \frac{\left(1 - |z_j|^2\right)\left(1 - |\zeta_j|^2\right)}{\rho_\Delta(z_j, \zeta_j) |1 - \overline{z_j}\zeta_j|^2} \mathsf{Re}\left[\frac{z_j - \zeta_j}{1 - \overline{z_j}\zeta_j} \frac{\overline{F_j(z)}}{1 - |z_j|^2} - \frac{z_j - \zeta_j}{1 - \overline{\zeta_j} z_j} \frac{\overline{F_j(\zeta)}}{1 - |\zeta_j|^2}\right] \leq 0. \end{aligned}$$

That is,

$$\frac{1}{1 - |z_j|^2} \mathsf{Re}\left[\frac{z_j - \zeta_j}{1 - \overline{z_j}\zeta_j} \overline{F_j(z)}\right] \leq \frac{1}{1 - |\zeta_j|^2} \mathsf{Re}\left[\frac{z_j - \zeta_j}{1 - \overline{\zeta_j} z_j} \overline{F_j(\zeta)}\right].$$

Now, let us prove the sufficiency of condition (1.1). For any $z \in \Delta^n$ consider the Cauchy problem

$$(C) \qquad \begin{cases} \dot{u_1} = F_1(u), \\ \vdots \\ \dot{u_n} = F_n(u), \\ u_1(0) = z_1, \\ \vdots \\ u_n(0) = z_n. \end{cases}$$

Denote by $\Phi_t(z) = (u_1(t), \ldots, u_n(t))$ its maximal solution defined on the interval $[0, I(z))$. In order to prove that Φ_t is a semigroup it is enough to show that $I(z) = +\infty$ for any $z \in \Delta^n$ and that the function $z \in \Delta^n \mapsto \Phi_t(z)$ is holomorphic. The holomorphicity is just a consequence of [1, Theorem 1.4.9]. To obtain the semicompleteness, since Δ^n is complete hyperbolic and the vector field $(F_1, \ldots, F_n)$ is autonomous, it is enough to show that $I(z)$ does not depend on the point z, *i.e.* is a constant.

We prove this assertion by contradiction. Let $z, \zeta \in \Delta^n$ be such that $T := I(z) < I(\zeta)$. Now for any $j = 1, \ldots, n$, set $h_j(t) = \rho_\Delta(\Phi_t^j(z), \Phi_t^j(\zeta))$, and $h(t) = \max\{h_1(t), \ldots, h_n(t)\}$ and notice that all these functions are defined for $t \in [0, T)$ and are absolutely continuous on this interval.

We now claim that h is not increasing.

Set

$$M = \{t \in [0,T) : \text{at least one among } h, h_1, \ldots,\ h_n \text{ is not differentiable at } t\}.$$

Since h, $h_1, \ldots, h_n$ are absolutely continuous, then M has measure 0 and, in order to prove that h is not increasing, it is enough to prove that, for any $t \in [0,T)$ outside a set of measure 0, its derivative $h'(t)$ is non-positive.

First of all we prove that $A_0 = \{t \in A : \ h_1(t) = \cdots = h_n(t) = 0\}$ is empty. Indeed, if $t \in A_0$, then this implies that $k_{\Delta^n}(\Phi_t(z), \Phi_t(\zeta)) = 0$ and hence $\Phi_t(z) = \Phi_t(\zeta)$. Now consider the Cauchy problem

$$\begin{cases} \dot{u_1} = F_1(u), \\ \vdots \\ \dot{u_n} = F_n(u), \\ u_1(0) = \Phi_t^1(z), \\ \vdots \\ u_n(0) = \Phi_t^n(z,w). \end{cases}$$

We have that both $\beta(s) = \Phi_s(\Phi_t(z)) = \Phi_{t+s}(z)$ and $\gamma(s) = \Phi_s(\Phi_t(\zeta)) = \Phi_{t+s}(\zeta)$ are solutions of this Cauchy problem, and therefore they coincide by the uniqueness of the solution. But in the first case, $t+s$ has to be strictly less than $T = I(z)$, while in the second case we can take $s = T - t$ because $I(z) < I(\zeta)$, and this is a contradiction, so that $A_0 = \emptyset$.

Now we split the set $A = [0,T) \setminus M$ into $n+1$ different pieces: $A_1, \ldots, A_n$, and B.

(1) For any $j = 1, \ldots, n$ set

$$A_j = \{t \in A : \exists \delta > 0 \text{ such that } h_l(x) \le h_j(x) \text{ for all } l \ne j \text{ and } x \in (t-\delta, t+\delta)\}.$$

That is, if $t \in A_j$, then $\rho_\Delta(\Phi_x^l(z), \Phi_x^l(\zeta)) \le \rho_\Delta(\Phi_x^j(z), \Phi_x^j(\zeta))$ for any $l \ne j$ whenever x is close to t. Notice that, by continuity, this set contains all t's in A such that $h_l(t) < h_j(t)$ for any $l \ne j$. The choice of t implies that $h(x) = h_j(x)$ for any $x \in (t-\delta, t+\delta)$ and therefore $h'(t) = h_j'(t)$. Since $h(t) = h_j(t) > 0$ we have

$$h_j'(t) = \frac{\left(1 - |\Phi_t^j(z)|^2\right)\left(1 - |\Phi_t^j(\zeta)|^2\right)}{\rho_\Delta(\Phi_t^j(z), \Phi_t^j(\zeta)) \cdot \left|1 - \overline{\Phi_t^j(z)}\Phi_t^j(\zeta)\right|^2} \cdot \mathsf{Re}\left[\frac{\Phi_t^j(z) - \Phi_t^j(\zeta)}{1 - \overline{\Phi_t^j(z)}\Phi_t^j(\zeta,\omega)} \frac{\overline{F_j(\Phi_t(z)}}{1 - |\Phi_t^j(z)|^2} - \frac{\Phi_t^j(z) - \Phi_t^j(\zeta)}{1 - \overline{\Phi_t^j(\zeta)}\Phi_t^j(z)} \frac{\overline{F_j(\Phi_t(\zeta)}}{1 - |\Phi_t^j(\zeta)|^2}\right],$$

and the condition $k_\Delta(\Phi_t^l(z), \Phi_t^l(\zeta)) \le k_\Delta(\Phi_t^j(z), \Phi_t^j(\zeta))$ is fulfilled for any $l \ne j$, that is, $(\Phi_t(z), \Phi_t(\zeta)) \in \overline{S_j}$, and we have $h'(t) = h_j'(t) \le 0$ because of (1.1).

(2) $B = A \setminus \left(\bigcup_{j=1}^n A_j\right)$. Since h, $h_1, \ldots, h_n$ are differentiable at t there exists a sequence ε_m converging to 0 such that $t + \varepsilon_m \in A$ and $h'(t) = \lim\limits_{m \to +\infty} \frac{h(t+\varepsilon_m) - h(t)}{\varepsilon_m}$. As the permutations of $\{1, \ldots, n\}$ are finite, up to a subsequence we can suppose that there exists a permutation σ of $\{1, \ldots, n\}$

such that $h_{\sigma_1}(t+\varepsilon_m) \leq h_{\sigma_2}(t+\varepsilon_m) \leq \cdots \leq h_{\sigma_n}(t+\varepsilon_m)$ for any $m \in \mathbb{N}$. Then the following chain of equalities gives us the possibility of computing $h'(t)$. Indeed we have

$$h'(t) = \lim_{m\to+\infty} \frac{h(t+\varepsilon_m) - h(t)}{\varepsilon_m} = \lim_{m\to+\infty} \frac{h_{\sigma_n}(t+\varepsilon_m) - h_{\sigma_n}(t)}{\varepsilon_m} = h'_{\sigma_n}(t),$$

where we have used that $h_1(t) = h_2(t) = \cdots = h_n(t)$ since $t \notin \bigcup_{j=1}^n A_j$. Using this fact again we have $k_\Delta\left(\Phi_t^j(z), \Phi_t^j(\zeta)\right) = k_\Delta\left(\Phi_t^{\sigma_n}(z), \Phi_t^{\sigma_n}(\zeta)\right)$ for any $j \in \{1,\ldots,n\}$. Then, arguing as in the first case, we have that $h'_{\sigma_n}(t) \leq 0$.

Summing up, we have obtained that h is not increasing. In particular we have that $h(t) \leq h(0)$ for any $t < T$, and therefore for any $j = 1,\ldots,n$,

$$L_j := \limsup_{t\to T^-} h_j(t) = \limsup_{t\to T^-} \rho_\Delta(\Phi_t^j(z), \Phi_t^j(\zeta)) < 1.$$

Take $R \in (\max\{L_1,\ldots,L_n\}, 1)$. Since $I(\zeta) > T$, we know that $\Phi_t^j(\zeta)$ converges to $\Phi_T^j(\zeta)$ for any $j = 1,\ldots,n$ when $t \to T^-$. The above inequality ensures that for any $j = 1,\ldots,n$ and t sufficiently near to T, then $\Phi_t^j(z)$ belongs to the closed disk of center $\Phi_T^j(\zeta)$ and radius R for the pseudohyperbolic metric. Since Δ is complete hyperbolic with respect to the Poincaré metric, these disks are compact in Δ, and this is a contradiction to the fact that $[0, I(z))$ is the maximal interval of definition for the Cauchy problem (C). □

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